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# Symmetry Breaking In the Double-Well Hermitian Matrix Models

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**Abstract**

We study symmetry breaking in $Z_2$ symmetric large $N$ matrix models. In the planar approximation for both the symmetric double-well $\phi^4$ model and the symmetric Penner model, we find there is an infinite family of broken symmetry solutions characterized by different sets of recursion coefficients $R_n$ and $S_n$ that all lead to identical free energies and eigenvalue densities. These solutions can be parameterized by an arbitrary angle $\theta(x)$, for each value of $x = n/N < 1$. In the double scaling limit, this class reduces to a smaller family of solutions with distinct free energies already at the torus level. For the double-well $\phi^4$ theory the double scaling string equations are parameterized by a conserved angular momentum parameter in the range $0 \leq l < \infty$ and a single arbitrary $U(1)$ phase angle.

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## 1. Introduction

We would like to highlight some unusual aspects of symmetric double-well matrix models [1]-[10] concerning spontaneous symmetry breaking and the multiplicity of solutions at the same critical point. We consider two specific models: (i) a $\phi^4$ theory with a symmetric double well and (ii) the symmetric Gaussian Penner model [11]-[13]. Both models have a $Z_2$ reflection symmetry and a standard two-band solution that respects it. However we obtain new classes of solutions that break the $Z_2$ symmetry by relaxing the initial boundary conditions on the first two recursion coefficients for the othogonal polynomials.

The single-band broken symmetry solution to the double-well $\phi^4$ theory is c=0 pure gravity, but at lower free energy there is an infinite class of two-band solutions where the tree level eigenvalue density is symmetric in the two wells. These solutions have the property that their free energy and eigenvalue density, in the planar limit, are invariant with respect to an infinite set of continuous parameters in the recursions coefficients. An analgous two-band class of broken symmetry solutions is found in the Penner model as well. In the double scaling limit for the $\phi^4$ theory the degeneracy is lifted, except for a single U(1) rotation, and a one parameter family of solutions survives satisfying the Painleve-II equation with an extra conserved "angular momentum" parameter. At this stage the physical consequences of this degeneracy of solutions are not clear. Since the tree level eigenvalue density is the same for all these solutions it is possible that some of them could tunnel into each other with an instanton action that is lower order in $N$. If so, this would be of particular interest in the context of the Gaussian Penner model, where one of the broken symmetry solutions discussed is related to the $c = 1$ string at twice the self-dual radius.

The organization of this paper is as follows: In section 2, we give a brief overview of symmetry breaking and the formalism for the orthogonal polynomial method. In section 3, we consider the consequences of relaxing the boundary condition on the recursion coefficients for the symmetric $\phi^4$ potential and the Gaussian Penner model, and classify all two-cut solutions in the planar ($N \to \infty$) limit. In particular we exhibit the class of solutions that have the same tree level eigenvalue density and free energy. We discuss some correlators that distinguish between the various solutions and a numerical approach to investigating finite N solutions. Section 4 discusses the double-scaling limit of the free energy for both models emphasizing that one gets an expanded class of double-scaling solutions. The discussion and conclusions are given in section 5.

## 2. Overview of Problem

It has been suggested that the singular behavior of the tree level eigenvalue density near the edge of the cuts determines the critical behavior of the matrix model in the following sense [14][5][9]: One identifies the polynomials in the matrix variable $\phi$ or scaling operators $O_n$ ($O_n$'s are traces of the polynomials) which produce a particular kind of singularity (labeled by $n$) near the edge of the cut or cuts. Knowing the $O_n$'s, one then considers $Z(t) = \int d\phi \exp[-\sum t_n O_n(\phi)]$ and shows that $Z(t)$ satisfies a certain hierarchy of equations. This hierarchy depends, then, only on the kind of ensemble of matrices considered (hermitian, antihermitian, unitary, etc.), the class of singularities of the eigenvalue density allowed (e.g., single-cut density with multiple zeroes coalescing at the edge, two cuts colliding and sandwiching zeroes in between, etc.), and any symmetry of the potential that restricts the class of eigenvalue densities considered (e.g., restriction to even perturbations in $Z_2$ symmetric 2-cut models) [10].

*Orthogonal Polynomials*

It is generally believed for potentials $V(\phi) = t_n O_n$ that are bounded from below that the orthogonal polynomial method [15] uniquely fixes the solution for the free energy, correlators, etc. Here we will show that orthogonal polynomial method actually allows one to construct a whole class of closely related solutions. To clarify this further, we remind the reader of the precise condition for a unique solution for the free energy in terms of orthogonal polynomials. Consider the partition function, $Z_N = \int d\phi \;\; e^{-NtrV(\phi)}$, where $\phi$ is an $N \times N$ hermitian matrix and $V(\phi)$ is a real potential. The integral is expressed in terms of a set of orthogonal polynomials $P_n(x)$, $\int dx \; P_n(x) \; P_m(x) \; e^{-NV(x)} = h_n \delta_{nm}$, normalized by the convention that the leading term for $P_n(x)$ is $x^n$, $P_n(x) = x^n + c_1 x^{(n-1)} + .....$ (Note that this convention sets $P_0(x) = 1$ or $h_0 = \int dx \; exp[-NV(x)]$.) Since these orthogonal polynomials can be iteratively determined by Gram-Schmidt orthogonalization, the exact free energy, $F_N = \log Z_N = \sum_{n=0}^{N-1} \log h_n$, as well as all thermodynamic averages are *uniquely* determined.

Instead of actually finding these orthogonal polynomials, one in practice uses recursion relations for the coefficients $R_n$ and $S_n$ in the expression, $xP_n(x) = P_{n+1}(x) + S_n P_n(x) + R_n P_{n-1}(x)$. Once the $R_n$'s are known, the free energy can be found by using the fact that $R_n = h_n/h_{n-1}$. It is convenient to introduce a self-dual orthonormal basis $|n\rangle$, where $\langle x|n\rangle = P_n(x)/\sqrt{h_n}$, and $|x\rangle$ are eigenvectors of the operator $\hat{\phi}$ with eigenvalues $x$, satisfying the normalization $\langle x'|x\rangle = exp(NV(x)) \; \delta(x' - x)$. Matrix elements of the operator $\hat{\phi}$ in this new orthonormal basis are directly related to the recursion coefficients

by $\langle m|\hat{\phi}|n\rangle = \sqrt{R_m}\delta_{m,n+1} + S_n\delta_{m,n} + \sqrt{R_n}\delta_{m,n-1}$. In terms of $\hat{\phi}$, the recursion relations for $R_n$ and $S_n$ can be expressed in operator notation,

$$\begin{aligned} n/N &= \sqrt{R_n}\langle n-1|V'(\hat{\phi})|n\rangle \\ 0 &= \langle n|V'(\hat{\phi})|n\rangle. \end{aligned} \tag{2.1}$$

Once initial conditions are specified, a *unique* solution for $R_n$ and hence the normalization for each orthogonal polynomial, $h_n$, can be found by iteration. For example, in the case of the $\phi^4$ model, Eq. (2.1) is a pair of coupled two-term recursion equations for $R_n$ and $S_n$ which require four inital conditions: the numerical values

$$\begin{aligned} S_0 &= \langle 0|\hat{\phi}|0\rangle = {h_0}^{-1}\int dx x e^{-NV(x)} \\ R_1 &= \langle 0|\hat{\phi}^2|0\rangle - \langle 0|\hat{\phi}|0\rangle^2 = {h_0}^{-1}\int dx x^2 e^{-NV(x)} - S_0^2, \end{aligned} \tag{2.2}$$

and the trivial values $R_0 = 0$ and $S_{-1} = 0$, which are independent of the potential. Given these values, all other coefficients are given iteratively as rational function of $S_0$ and $R_1$. (A more elegant formulation would introduce a single sequence of coeffients $C_k \propto \langle 0|\hat{\phi}^k|0\rangle + ....$ with a single four-term recursion relation $C_k = F_k(C_{k-1}, C_{k-2}, C_{k-3}, C_{k-4})$, where $C_k = R_{k/2}$ for k even and $C_k = S_{(k-1)/2}$ for k odd.)

*Symmetry Breaking*

This formalism should make it clear that there is *no* ambiguity in defining the matrix models for all *finite* $N$, assuming of course that the potential is *bounded from below* and that the integrals defining the inner product are finite. However typical of all statistical mechanical problems this does not imply that we know the correct way to take the thermodynamic (or in this instance large $N$) limit. To understand this potential source of ambiguity in using the recursion relations at large $N$, consider the double-well potential, $V(\phi) = \sigma\phi + \frac{1}{2}\mu\ \phi^2 + \frac{1}{4}g\ \phi^4$, with $\mu < 0$, $g > 0$ and a small symmetry breaking term $\sigma\phi$. To investigate symmetry breaking it is useful to study the effect of interchanging the limits $N \to \infty$ and $\sigma \to 0\pm$ on the values of $S_0 = \langle 0|\hat{\phi}|0\rangle$. By $Z_2$ symmetry, if we take $\sigma \to 0\pm$ followed by $N \to \infty$, we must get $S_0 = 0$, whereas if we take $N \to \infty$ followed by $\sigma \to 0\pm$ we have

$$S_0 = \pm\sqrt{-\mu/g}, \tag{2.3}$$

as can be demonstrated by using steepest descent at the stationary minima of the potential $V(x)$. Similarly, to complete the necessary boundary conditions, one can show that $R_1 = \langle 0|\hat{\phi}^2|0\rangle - S_0^2$ takes on $R_1 = -\mu/g$ and $R_1 = 0$ for these two limits respectively. Now we can in principle use the recursion relations to obtain both symmetric and broken symmetry solutions. In general terms, this is just the familiar feature of spontaneous

symmtry breaking. For example the Ising model with no external field on a finite lattice must have $\langle s_i \rangle = 0$ by $Z_2$ symmetry, but in the large volume limit (at temperatures below the Curie point) the relevant (*i.e.* stable) solution is a broken vacua with non-zero values for $\langle s_i \rangle = \pm m$ obtained by applying an infinitesimal magnetic field.

As we will see shortly, a more general possibility at *infinite* $N$ is to characterize the "vacuum" state for the double-well $\phi^4$ model by a mixing angle $\theta_0$,

$$|0, \theta_0\rangle = cos(\theta_0/2)\ |0, +\rangle + sin(\theta_0/2)\ |0, -\rangle \tag{2.4}$$

where $|0, \pm\rangle$ are orthonormal and $\hat{\phi}|0, \pm\rangle \simeq \pm\sqrt{-\mu/g}|0, \pm\rangle$. As a consequence for the mixed state the boundary conditions becomes

$$S_0 = \sqrt{-\mu/g}\ cos(\theta_0), \tag{2.5}$$

instead of (2.3), with the constraint,

$$R_1 = -\mu/g - S_0^2 \geq 0. \tag{2.6}$$

The goal of this paper is to understand how the solutions of the double-well matrix models depend on the initial boundary conditions, *e.g.*, for the $\phi^4$ model, on the first two moments, $S_0$ and $R_1$. In the large $N$ (or planar limit) we find a large class of solutions consistent with the broken symmetry boundary condition in addition to the (meta stable) pure gravity solution at higher free energy. Indeed we will show that the above qualitative discussion is a rigorous consequence of the planar solutions in the two-band ansatz. More generally however, we feel that the lack of a full understanding of the effects of this boundary condition represents an important gap in our ability to fully determine and solve the string equations resulting from matrix models.

## 3. Multiple Solutions In Matrix Models : Tree Level Analysis

In this section we establish the existence of multiple solutions in two models: (i) the double well $\phi^4$ model and (ii) Gaussian Penner model.

### *3.1. The Double Well $\phi^4$ potential*

For the double well potential ( $V(\phi) = \sigma\phi + \frac{1}{2}\mu\ \phi^2 + \frac{1}{4}g\ \phi^4$), Eq. (2.2) reduces to the recursion relations

$$\begin{aligned} \frac{n}{N} &= R_n[\mu + g(R_{n+1} + R_n + R_{n-1} + S_n^2 + S_{n-1}^2 + S_{n-1}S_n)] \\ 0 &= \sigma + \mu S_n + g[R_{n+1}(S_{n+1} + 2S_n) + R_n(2S_n + S_{n-1}) + S_n^3]. \end{aligned} \tag{3.1}$$

We shall first illustrate our procedure by considering a symmetry-breaking solution under a period-one ansatz for both the R's and S's: $R_n \to R(\frac{n}{N})$, $S_n \to S(\frac{n}{N}) \neq 0$. Ignoring the $1/N$ corrections, Eq. (3.1) leads to two relations, which allow us to solve for $R(x)$ and $S(x)$:

$$\begin{aligned} R(x) &= (1/15g)[-\mu - \sqrt{\mu^2 - 15gx}] \\ S(x) &= \pm\sqrt{-\mu/g - 6R(x)}. \end{aligned} \tag{3.2}$$

With $\mu < 0$ and $g > 0$, one has $R(0) = 0$, $S(0) = \pm\sqrt{-\mu/g}$, consistent with our discussion on symmetry breaking in Sec. 2, and $R(x)$ monotonically increasing for $0 < x < \mu^2/15g$.

The generating function $F(z) \equiv \frac{1}{N}\langle Tr\frac{1}{z-\phi}\rangle$ for a period-one ansatz at the tree level can be expressed in terms of $R(x)$ and $S(x)$ as follows:

$$F(z) = \int_0^1 dx \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{z - \phi(x,\theta)} = \int_0^1 dx \frac{1}{\sqrt{[z - S(x)]^2 - 4R(x)}}, \tag{3.3}$$

where $\phi(x,\theta) = S(x) + \sqrt{R(x)}(e^{i\theta} + e^{-i\theta})$. One can verify by explicit calculation using (3.2) that the eigenvalues lie in an interval $[z_-, z_+]$, where $z_\pm \equiv S(1) \pm 2\sqrt{R(1)}$, with a single-band eigenvalue density, $\rho(z)$, in agreement with the result of Shimamune (Ref. [1]) obtained by using the Schwinger-Dyson equation. On the line $\mu = -\sqrt{15g}$, $R(x)$ develops a square-root type singularity at $x = 1$, leading to a Painleve-I equation in the double scaling limit, appropriate for the $c = 0$ 2D gravity solution.

Since all the eigenvalues are concentrated in a single well, this symmetry-breaking solution does not correspond to a configuration with the lowest free energy, but is a subdominant solution. In the double scaling limit it is unstable against the tunnelling of a single eigenvalue into the other well, exactly like the subdominant solution for the pure gravity in the $\phi^6$ model. We now turn to solutions which have eigenvalues in both wells, which include a class of solutions for which $\rho(z)$ is exactly $Z_2$-symmetric (when $\sigma = 0$) at the tree level, but which in general break this symmetry at higher order in $1/N$.

Let us consider a period-two ansatz for both the R's and S's

$$\begin{aligned} R_n &= A(\frac{n}{N}) \ , \quad S_n = C(\frac{n}{N}) \quad \text{for n = even,} \\ R_n &= B(\frac{n}{N}) \ , \quad S_n = D(\frac{n}{N}) \quad \text{for n = odd.} \end{aligned} \tag{3.4}$$

Taking $A$, $B$, $C$ and $D$ to be continuous, and ignoring the $1/N$ corrections in the recursion equations (3.1), we obtain four tree level recursion relations,

$$2x = \mu_{eff}(A+B) + g(A^2 + 4AB + B^2), \tag{3.5a}$$

$$0 = (A-B)(A+B+\frac{\mu_{eff}}{g}), \tag{3.5b}$$

$$0 = 2\sigma + \mu(C+D) + g[3(A+B)(C+D) + C^3 + D^3], \tag{3.5c}$$

$$0 = (C-D)(A+B+\frac{\mu_{eff}}{g}), \tag{3.5d}$$

where $\mu_{eff} \equiv \mu + g(C^2 + CD + D^2)$.

Since we have already considered the single-band (pure gravity case), we can assume that either $A \neq B$ or $C \neq D$. To carry out an exhaustive analysis of the full set of solutions to these equations, it is useful to note that only three out of the four equations are independent. The three independent equations take the simple form,

$$A + B - CD = -\frac{\mu}{g} - (C+D)^2 \ , \ AB = \frac{x}{g} \ , \quad \text{and } V'[-(C+D)] = 0. \tag{3.6}$$

Note that there is no condition on $C - D$ which can be independently chosen for every value of $x$. The first two equations allow one to express the explicit solution for $A$ and $B$ in the familiar form[2],

$$A = \frac{1}{2g}(-\mu_{eff} \pm \sqrt{\mu_{eff}^2 - 4gx} \ ) \qquad B = \frac{1}{2g}(-\mu_{eff} \mp \sqrt{\mu_{eff}^2 - 4gx} \ ). \tag{3.7}$$

In general for the double-well potential (for $\mu < 0$ and small enough $\sigma$), there are three $x$-independent real solutions to $V'[-(C+D)] = 0$. With $C + D$ fixed, it follows that the

first combination $A+B-CD$ in (3.6) is also fixed. Consequently, a "circular" constraint on $A-B$ and $C-D$ can be found

$$(A-B)^2+\frac{4x}{g}=\frac{1}{16}[(C-D)^2+\frac{4\mu}{g}+3(C+D)^2]^2. \tag{3.8}$$

This constraint can be represented by contours for each fixed value of $x$, $0\leq x\leq 1$, in a two-dimensional plane with $A-B$ as the vertical axis and $C-D$ the horizonal axis. In fig. 1, we exhibit them for the class of solutions where $\sigma=0$ and $C+D=0$. The external contour, corresponding to $x=0$, is precisely the constraint suggested in the qualitative discusion of symmtry breaking of Sec. 2. To see this one must consider carefully the proper definitions at the boundary, namely $A-B=R_0-R_1=-R_1$ and $C=-D=S_0$ at $x=0$, which yields the contraint Eq. (2.6) in the form,

$$R_1^2=(S_0^2+\mu/g)^2.$$

For $x=x_{cr}\equiv\mu^2/4g$, the contour shrinks to a point, $A-B=C-D=0$, about which a double scaling limit can be taken [16].

The different solutions to (3.6) can be parametrized by curves in this plane traversing from the $x=0$ to the $x=1$ contours (see for example fig. 4(b)). For instance, a solution can be specified, at each value of $x$, by the polar coordinate, $\theta(x)$, for the intersection of the curve with the contour (3.8). Conversely, once $\theta(x)$ is chosen for every $x$, a unique solution to Eq. (3.6), $(A(x),B(x),C(x),D(x))$, is obtained [17]. In analogy with (2.4), each state, $|n\rangle$, $0\leq n/N\leq 1$, in the large $N$ limit could be thought of as a linear conbination of "left-" and "right" states, specified by an arbitrary density function, $\theta(x)$. The choice of $\theta(x)$ represents the ambiguity of solution at the tree level. (In fact the "orbit" need not even be continuous. However, if the orbit is discontinuous, derivatives of $A,B,C,D$ are large and cannot be ignored, as is assumed in the tree level analysis.)

An interesting feature of these broken symmetry solutions is that, within the class specified by one of the three values of $C+D$, they all have the *same* tree level eigenvalue density and free energy. To see this consider the generating function $F(z)\equiv\frac{1}{N}\langle Tr\frac{1}{(z-\phi)}\rangle$ for a *general* period-two ansatz at tree level[18]

$$F(z)=1/2\int_0^1 dx\frac{(2z-(C+D))}{\sqrt{[z^2-z(C+D)-(A+B-CD)]^2-4AB}}. \tag{3.9}$$

Notice that Eq. (3.9) involves precisely the three combinations, Eq. (3.6), which are fixed by the four recursion relations. Therefore, once the value of $C+D$ is chosen, one gets the

same generating function $F(z)$. It follows that the tree level eigenvalue density $\rho(x)$ and free energy are the **same** for all solutions within a class, *i.e.*, are independent of the choice of $\theta(x)$.

For the most part we now shall restrict further discusions to the case $\sigma = 0$ and $C + D = 0$. All the solutions in this class give rise to the same eigenvalue density $\rho(z)$. This class of solutions is continuously deformable into the limiting case $A(x) \neq B(x)$, $C(x) = D(x) = 0$, which is just the standard *symmetric* solution for the two-band solution, with $\mu_{eff} = \mu$. At the other extreme there is a *maximally asymmetric* two-band solution satisfying the condition $A(x) = B(x)$, $C(x) = -D(x) \neq 0$, with

$$A(x) = B(x) = R(x) = \sqrt{x/g}, \qquad C(x) = -D(x) = \pm \left[ |\mu|/g - \sqrt{4x/g} \right]^{1/2}. \quad (3.10)$$

These values of $C$ and $D$ form the turning points at which our numerical solutions change the sign of $A - B$. (See fig. 1.) The symmetric solution corresponds to the choice $\theta(x) = \pm\pi/2$ and the maximally asymmetric solution to the choice $\theta(x) = 0, \pi$. Our expanded class of solutions includes ones where the branch of the square root singularity ($\pm$) in Eq. (3.7) is exchanged between $A$ and $B$ as the trajectory rotates in the $A - B$ versus $C - D$ plane, as we note in the discussion of our numerical results for finite $N$ (see Sec 3.4 and fig. 4 (c)). This corresponds to $\theta(x)$ winding around the circle a number of times as $x$ goes from 0 to 1. This is a precursor of the angular momentum variable of the double scaling solutions. The rigid implementation of $Z_2$ symmetry and the boundary conditions on the recursion coefficients would have yielded only the $\theta(x) = \pm\pi/2$ solution. All other solutions correspond to a breaking of the $Z_2$ symmetry.

### *3.2. Gaussian Penner Model*

The second example, we would like to consider, is the Gaussian Penner model. The potential for a general Penner model is $V(\phi) = V_0(\phi) - t \ \ log\ \phi$, where $V_0$ is a polynomial[12]. If $V_0(\phi) = \phi$ the model is the linear Penner model [19][20], if $V_0(\phi) = \mu\ \phi^2/2$ the model is the Gaussian Penner model [12][13], where we interpret the $\log \phi$ term as $\frac{1}{2} \log \phi^2$. In fig. 2 we display the Gaussian Penner potential for different values of t. Consider first the situation where $t > 0$. (The region $t < 0$ is reached by analytic continuation.[11][12]) As the potential is a double well, the period-two ansatz may be applied here also.

The recursion relations (Eq. (2.2)) for a general Penner model reduce to

$$\frac{n}{N} = \sqrt{R_n}\langle n-1|V_0'(\hat{\phi})|n\rangle - t\sqrt{R_n}\langle n-1|\hat{\phi}^{-1}|n\rangle \quad (3.11a)$$

$$0 = \langle n|V_0'(\hat{\phi})|n\rangle - t\langle n|\hat{\phi}^{-1}|n\rangle. \quad (3.11b)$$

Let us denote $W_n = \sqrt{R_n}\langle n-1|V_0'(\hat{\phi})|n\rangle$ and $Y_n = \langle n|V_0'(\hat{\phi})|n\rangle$ for later notational convenience. For the Gaussian Penner model, $W_n = \mu R_n$ and $Y_n = \mu S_n$.

Eqs. $(3.11a, b)$ are unusual since they involve matrix elements of $\hat{\phi}^{-1}$. For $t > 0$, they can be solved in the spherical limit by a procedure similar to that used for deriving the generating function $F(z)$, Eq. (3.9), under a period-two ansatz [21]. By considering $n$ even and odd, Eqs. $(3.11a, b)$ should normally lead to four conditions. Just like the $\phi^4$ model, only three are independent, and they can be cast in the following form:

$$C + D = 0, \tag{3.12a}$$

$$A + B - CD = \frac{2x+t}{\mu}, \tag{3.12b}$$

$$AB = \frac{x(x+t)}{\mu^2}. \tag{3.12c}$$

For the symmetric solution where $C = D = 0$, one finds

$$A(x) = \frac{x}{\mu}, \qquad B(x) = \frac{x+t}{\mu}. \tag{3.13}$$

For the maximally asymmetric solution, on the other hand, one has

$$A(x) = B(x) = \frac{1}{\mu}\sqrt{x(x+t)}, \qquad C(x)^2 = \frac{1}{\mu}\left[(2x+t) - 2\sqrt{x(x+t)}\right]. \tag{3.14}$$

Observe that Eqs. $(3.12a-c)$ are precisely the necessary combinations which are needed in Eq. (3.9) for determining the generating function of our symmetric Gaussian Penner model in the spherical limit leading to symmetric two-band structure. Therefore, for this class of solutions and in particular for the symmetric and maximally asymmetric solutions, the eigenvalue density and free energy are again identical at tree level [22].

### *3.3. Correlation Functions that distinguish between symmetric and asymmetric ansatz solutions*

Given that the tree level generating function and hence eigenvalue density and free energy are the same, one might ask if there are other correlation functions that distinguish between the various solutions. It is normally assumed that after taking the period-two

ansatz the large N limit is smooth for all correlators. However, consider the correlator $\langle Tr\phi Tr\phi\rangle_c$. In terms of recursion coefficients,

$$\langle Tr\phi Tr\phi\rangle_c = R_N. \tag{3.15}$$

In the symmetric solution, since the $R_N$ alternate between $A_N$ and $B_N$ as $N$ goes from odd to even, this correlator at large $N$ depends on whether $\infty$ is approached through odd or even $N$. In one case it is $A(1)$, in the other $B(1)$. For the $\phi^4$ model, they differ even at tree level by $\sqrt{\mu^2 - 4g}/g$ which is singular at criticality, (see Eq. (3.7)). On the other hand in the maximally asymmetric solution $S_N$ is period two but $R_N$ is of period one, (see below Eq. (3.10)) hence this correlator has no discontinuity between odd and even. This is an example of a correlator that distinguishes between the two solutions. A similar difference between odd and even $N$ is known to exist in the context of unitary matrix models [23].

Another example is $\langle Tr\phi Tr\phi Tr\phi\rangle_c = R_N\left(S_{N-1} - S_N\right)$. In the symmetric solution $S_N = 0$ and this vanishes. But in the maximally asymmetric solution since $S_n$ is period two, (and $C = -D$), $S_{N-1} - S_N$ changes sign as one goes from odd to even $N$. In particular, $\langle Tr\phi Tr\phi Tr\phi\rangle_c = (|\mu|/g^2 - 2(1/g)^{3/2})^{1/2}$ for $N$ odd and its negative for $N$ even.

Another characterization of the difference between the solutions is the following: If one truncates the infinite dimensional matrix $\langle n|\hat{\phi}|m\rangle$ to an $N \times N$ matrix corresponding to the subspace of the first $N$ orthogonal polynomials, the eigenvalues of this $N \times N$ matrix are a good approximation to the saddle point configuration of the eigenvalues at large $N$. Since its matrix elements are given in terms of $R's$ and $S's$, we can determine the eigenvalues numerically from a given solution of the recursion coefficients. In fig. 3, we show the locations of the eigenvalues so obtained for the tree level symmetric and maximally asymmetric solutions for $N = 24$, 25. For $N = 24$, half the eigenvalues are in one well and half in the other, for both solutions. However, for $N = 25$, (odd $N$), there is a striking difference between the two solutions. For the asymmetric solution, there is one extra eigenvalue located in one of the two wells, (the well selected depends upon the sign of $S_0$), whereas for the symmetric solution, this extra eigenvalue in the the center (on top of the barrier), thus preserving the symmetry between both the wells.

### *3.4. Numerical Approach to Finite N Solutions*

Another approach to understanding the role of these multiple solutions is to pursue a numerical study of finite $N$ solutions and attempt to take $N$ large enough to see a cross

over to the large $N$ (or double scaling) regimes. Although we will postpone a detailed analysis of our results, there are several general features which can help to understand the present discussion. The recursion relations for the double well potential follow from the variation of an effective action, [24]

$$\begin{aligned} V_{eff}(R_n, S_n) = \sum_{n=0}^{\infty} \{ -\frac{n}{N} log(R_n) + \mu R_n + \frac{g}{2}(R_n^2 + 2R_n R_{n+1}) \\ + \sigma S_n + \frac{\mu}{2} S_n^2 + \frac{g}{4} S_n^4 \\ + g R_n (S_n^2 + S_{n-1}^2 + S_{n-1} S_n) \}, \end{aligned} \tag{3.16}$$

with the defintions, $S_{-1} = 0$ and $R_0 = 0$. Morover we must take $g > 0$, if the effective action (like the actual potential) is to be bounded from below. This formalism provides a natural way to investigate our set of multiple solutions, by removing the boundary conditions on $S_0$ and $R_1$ and replace them with the asymptotic condition that $R_n$ and $S_n$ are smooth functions as $n \to \infty$. This is simply the one band ansatz in the extreme limit of $n/N$ very large. Therefore it is again interesting to ask what is the full set of local minima.

Earlier work on symmmetric solutions for the degenerate three-well potential have observed the recursion coefficients with very complicated, "chaotic looking" behavior, when calculated by a numerically method logically equivalent to minimizing an effective potential[25]. We also have observed complicated behavior for two degenerate wells when we allow symmetry breaking terms ($S_n \neq 0$), which we have been able to relate to the existence of our multiple solutions in the planar limit. As an illustration consider the solution presented in fig. 4. However due to the degeneracy of multiple solutions at $N_\infty$, great care must be taken with the minization procedures.

For example we have minimized $V_{eff}$ for the double-well potential with $N = 512$, $\mu = -2$, $g = 1$ and $\sigma = 0.1$, starting from a random distribution of 2048 coefficients for $R_n$ and $S_n$. Using a variety of minimizaton procedures on the CM-5 at Boston University and the NeXT station at CERN, we see that after only several 100 iterations the curves conform roughly with the large $N$ constraints but they can have a great variety of coutours in the $A - B$ vs $C - D$ plane. However if we go further for another 100,000 iterations, a smooth spiral curve (see fig. 4 (c)) begins forming near critical $x$ ($x_{cr} = 1$), with the large $N$ constraints improving to about 1% as might be expected in a transition region from one of our $N = \infty$ solution to a particular double scaling solution with non-zero orbital quantum number $l \neq 0$. After 100,000 iterations the value of $V_{eff}/N$ departs from its theoretical $N = \infty$ value by 0.0034. The final results on questions as to the stability of non-zero orbital solutions, the possibility of residual degeneracies at finite $N$ and especially the existence of choatic regimes require accurate and non-trivial compuatational power. Further details on this as well as a study of higher $1/N$ corrections will be presented in a future publication[26].

## 4. The Double Scaling Limit

### *4.1. Double Well $\phi^4$ Potential*

The double scaling equations for the $\phi^4$ model have been discussed by a number of authors [4,2,5-10]. The steps involved in the double scaling analysis of symmetric breaking solutions for a $Z_2$ *symmetric* potential are the same as that for solutions of a general *asymmetric* potential [7-10], since in both cases one includes both $R_n$ and $S_n$ in the analysis. One sets $x = 1 - \epsilon^2 t$, (recall $x = n/N$) and $\epsilon = N^{-1/3}$. For the symmetric solution, $C_n = D_n = 0$, while $A_n$ and $B_n$ are

$$A_n = a_0 + \epsilon(f_e(t) + f_o(t)) + \epsilon^2(r_e(t) + r_o(t)) + ..... \tag{4.1a}$$

$$B_n = a_o + \epsilon(f_e(t) - f_o(t)) + \epsilon^2(r_e(t) - r_o(t)) + ..... \tag{4.1b}$$

On substituting this symmetric double scaling ansatz into the recursion relations (3.1) and equating terms with powers $\epsilon^0, \epsilon^1, \epsilon^2, and \epsilon^3$, we get eight equations, two of these are used up by $a_o$ ($\epsilon^0$ equations), the tree level result. (Note $a_o = -\mu/(2g)$; in what follows, we adopt the convention where $\mu = -2$ and $g = 1$). That leaves us with six unknowns and six equations hence all the unknowns can be determined. Most of them are zero (e.g. $f_e = r_o = 0$....), while the others are determined in terms of $f_o(t) = f(t)$, *e.g.*, $r_e = (f^2 - t)/4$. The function $f(t)$ satisfies the Painleve-II equation

$$f'' - \frac{1}{4}\ f^3 + \frac{1}{2}\ ft = 0. \tag{4.2}$$

The suseptibility $\chi \sim \frac{\partial^2 \Gamma}{\partial \mu^2} \sim f^2/2 - r_e = (f^2 + t)/4$.

For the maximally asymmetric solution, the double scaling ansatz for $C_n$ and $D_n$ are

$$C_n = \epsilon g(t) + \epsilon^2 ..... \tag{4.3a}$$

$$D_n = -\epsilon g(t) + \epsilon^2 ..... \tag{4.3b}$$

and $A_n = a_0 + \epsilon^2 r_e(t) + ....., B_n = a_o + \epsilon^2 r_e(t) + \cdots$. Substituting this maximally asymmetric double scaling ansatz into the recursion relations and equating powers of $\epsilon$ we get, $r_e = -(g^2 + t)/4$, etc., and

$$g'' - \frac{1}{4}\ g^3 + \frac{1}{2}\ gt = 0, \tag{4.4}$$

the same as Eq. (4.2), with $g$ replacing $f$. Under this ansatz, the suseptibility $\chi$ can be expressed as $(g^2+t)/4$.

We next consider the general symmetry breaking solutions, where $A-B\neq 0$ and $C-D\neq 0$ in the planar limit. Substituting the double scaling ansatz, Eqs. (4.1) and (4.3) into the recursion relations and equating powers of $\epsilon$ we get $r_e=(f^2-g^2-t)/4$, etc., and the following coupled equations [7-10]

$$f''-f(g^2+f^2)/4+ft/2=0, \tag{4.5a}$$
$$g''-g(g^2+f^2)/4+gt/2=0. \tag{4.5b}$$

The suseptibility is now given by $\chi=(f^2+g^2+t)/4$.

While Eqs. (4.5) have been obtained previously in the context of asymmetric potentials, we would like to emphasize that they describe the multiple (and in general symmetry breaking) solutions that exist even for a $Z_2$-symmetric potential. To see the symmetry breaking nature of these solutions more explicitly and to make contact with the tree level discussion in the previous section, introduce a two-dimensional vector $\vec{r}=(g,f)$, in terms of which the coupled equations can be written as $\ddot{\vec{r}}-(1/4)(r^2-2t)\vec{r}=0$. We can next make a change of coordinates

$$f=r\ sin\theta(t), \qquad\qquad g=r\ cos\theta(t), \tag{4.6}$$

so that $\chi\sim(r^2+t)/4$ and the coupled equations become

$$\ddot{r}-\frac{1}{4}\ r^3+\frac{1}{2}\ rt-\frac{l^2}{r^3}=0, \tag{4.7a}$$
$$r^2\dot{\theta}=l. \tag{4.7b}$$

Note that since $A-B\propto f$ and $C-D\propto g$ in the double scaling limit, the variable $\theta$ is the same as that introduced in the previous section. The constant $l$ in Eq. (4.7$b$) is the "angular momentum", and it is a constant of the motion due to the $U(1)$ invariance of Eqs. (4.5). Note that for $l=0$ equation (4.7$a$) is just the Painleve-II equation in the $r$ coordinate. Thus in the $l=0$ sector of this model, we reproduce the same double scaling results for both the symmetric and maximally asymmetric solutions. But for the $l\neq 0$ sector the double scaling equation is different; hence the behavior of the system in this sector for the multiple solutions is different. Iterating Eq. (4.7$a$) at large $t$, one finds

$$\chi=\frac{3t}{4}-(\frac{1+l^2}{4})t^{-2}+\cdots \tag{4.8}$$

At the tree level we had a degeneracy of solutions parametrized by the function $\theta(x)$, whose value could be independently chosen for $x\epsilon[0,1]$. The double scaling analysis based on the (4.1 )and (4.3 )tells us that only a two parameter family of these solutions, labelled by $l$ and one global rotation angle $\theta_0$, survives in the double scaling limit. ($\theta(x)$ is no longer any function of $x$, but constrained such that $r^2\dot{\theta} = l$ is a constant.) Solutions labelled by different values of $l$ give rise to the same susceptibility at tree level (the first term in (4.8) is $l$-independent) as expected, but differ at higher orders.

A relevent question that arises is: Just as the symmetric solution ($S_n = 0$) is the "natural" solution that follows from the symmetric potential $V(x)$, ("natural" in that it respects the $Z_2$ symmetry of $V(x)$, and the recursion coefficients are specified by initial conditions given by the integrals discussed in section 2), is there a perturbed potential of which the symmetry breaking solutions are natural solutions? This is presently being investigated. At this point we remark that the perturbation cannot be a rigid translation of the potential, which induces linear and cubic terms. Such a perturbation has been discussed in Ref. [10]. It can be easily seen that the natural solution to the shifted potential $V_b(x) \equiv V(x-b)$ is given by the same $R_n$ as for $V(x)$, with $S_n = b = \text{const}$.

Should one decide to introduce a small explicit symmetry-breaking term, $\sigma = N^{-2/3}\tilde{\sigma}$, as was done in Ref. [7], the vector equation above remains $U(1)$ invariant by simply adding a "constant magnetic field" term of the form $B_0\dot{\vec{r}} \times \hat{z}$, where $B_0 \propto \tilde{\sigma}$ and $\hat{z}$ denotes a unit normal to this two-dimensional plane. One can thus again reduce it to a single radial equation with the solution depending on a "generalized angular mumentum", $l = r^2\dot{\theta} + B_0 r^2/2 = \text{constant}$,

$$\ddot{r} - \frac{1}{4}\, r^3 + \frac{1}{2}\, rt - \frac{(l - \frac{B_0 r^2}{2})^2}{r^3} = 0. \tag{4.9}$$

*4.2. The Gaussian Penner Model*

The double scaling solutions for the Gaussian Penner model have been discussed in Refs. [12] and [13]. We reproduce the proofs below for completeness (also simplifying them somewhat). The critical point is $t = -1$. By strictly enforcing $Z_2$ symmetry, this model can be solved exactly first at $t > 0$, so that the criticality at $t = -1$ can be exhibited explicitly. Note that $\langle n|\hat{\phi}^{-1}|n\rangle = 0$ by $Z_2$ symmetry, Eq. (3.11$b$) thus reduces to $S_n = 0$, the symmetric ansatz. Eq. (3.11$a$) can also be solved exactly for all n. Since $W_n = \mu R_n$ and since $\langle n-1|\hat{\phi}^{-1}|n\rangle = 0$ for n even, $\langle n-1|\hat{\phi}^{-1}|n\rangle = \frac{1}{\sqrt{R_n}}$ for n odd, it follows that, for n even, $R_n = n/\mu N$, and for n odd, $R_n = (n+tN)/\mu N$[12].

Since we know the exact result for the $R_n's$ for the symmetric solution, the exact free energy may be obtained

$$\Gamma = \sum_{k=1}^{N/2-1} k\ log\left[(2k+\mu+1)(2k+\mu-1)\right], \tag{4.10}$$

where $t = -1 + \frac{\mu}{N}$. On expanding the free energy in powers of $\mu$ we get

$$\Gamma = \frac{1}{4}\ \mu^2\ log\ \mu + \frac{1}{12}\ log\ \mu + .... \tag{4.11}$$

The coefficient of the second $\log \mu$, $\chi_1 = 1/12$, comes from the torus contribution, indicates that this free energy *cannot* be identified with the Legendre transform of the free energy of the c=1 string at self dual or twice the self dual radius [12].

It has been stressed in Ref. [12] that the exact solution to the Gaussian Penner model is characterized by the fact that $B(x)$ has a *linear* zero at $x = 1$ when $t = -1$ while $A(x)$ is non-zero there, (see Eq. (3.13)). This same feature also holds in general for symmetry breaking solutions where $C(x) = -D(x) \neq 0$. However, the maximally asymmetric solution provides an exception to this rule. When $A(x) = B(x)$, one has $A(x) = B(x) \sim (x+t)^{1/2}$, (see Eq. (3.14)). That is, both $A(x)$ and $B(x)$ have square-root type behavior near $x = 1$ in the spherical limit at $t = -1$. Since it is the behavior of $R_n$ near $x = 1$ which determines the criticality of the model, it follows that the resulting double scaling limit for the maximally asymmetric ansatz could be non-generic.

Let us next concentrate on the maximally asymmetric solution. We note first that, with $S_n \neq 0$, Eqs. (3.11$a$) and (3.11$b$) can be re-written as [11][12]

$$W_n + W_{n+1} + S_n Y_n = \frac{2n+1+Nt}{N}, \tag{4.12a}$$

$$S_N[W_{n+1} - W_n - \frac{1}{N}] = R_n Y_{n-1} - R_{n+1} Y_{n+1}, \tag{4.12b}$$

so that matrix elements of $\hat{\phi}$ would not appear explicitly. For the maximally asymmetric solution the double scaling solutions may be found as follows. With $x = 1 - z/N$, $t = -1 + \mu/N$, $\epsilon = 1/\sqrt{N}$, $A_n, C_n,$ and $D_n$ can be expanded as [13]

$$\begin{aligned} A_n &= 1/\sqrt{N}\ \rho(z) + ... \\ C_n &= 1 + 1/\sqrt{N}\ \sigma(z) + 1/N\ \sigma_1(z) + .... \\ D_n &= -\left(1 + 1/\sqrt{N}\ \sigma(z) + 1/N\ \sigma_1(z) + ....\right). \end{aligned} \tag{4.13}$$

On substituting into the recursion relations (4.12 )and equating equal powers of $N$, we can determine $\sigma(z)$ in terms of $\rho(z)$ and $\sigma_1(z)$ in terms of $\sigma(z)$. The equation for $\rho$ is $\rho(z)\rho(z-1) = \mu - 1/2 + z$ and the double scaling result for $R_n$ is $R_n \sim \Gamma\left(\frac{1}{2}(N-n+\mu+3/2)\right)/\Gamma\left(\frac{1}{2}(N-n+\mu+1/2)\right)$. The double scaled free energy is $\Gamma = \sum_{k=1}^{N/2-1} k\ log\left[(2k+\mu+1/2)(2k+\mu-1/2)\right]$ plus$\mu$ independent terms. On expanding in powers of $\mu$ one gets

$$\Gamma = \frac{1}{4}\ \mu^2\ log\ \mu - \frac{5}{48}\ log\ \mu..... \tag{4.14}$$

The coefficient $\chi_1 = -5/48$ confirms that this criticality corresponds to that for the free energy of the c=1 string at twice the self dual radius as conjectured in [20]. We note that although the tree level free energy gave identical results for the symmetric and maximally asymmetric ansatz (see remarks below Eq. (3.14)), the double scaled free energies (4.11) and (4.14) are quite different.

## 5. Discussion

The existence of multiple double scaling solutions at the same critical point, which share the same tree level behavior, is an unusual and previously unnoticed feature. We have exhibited this behaviour in two completely different models-the double-well $\phi^4$ potential and the Gaussian Penner model. Both these models possess $Z_2$ symmetry, and a charateristic feature of the class of solutions we consider is the fact that only one of the solutions respects this symmetry to all orders, all other solutions break this symmetry. Although the physical consequences of symmetry breaking in matrix models are not yet fully understood, it is useful to consider the analogy of the multiplicity of solutions here to the property of coexisting ferromagnetic phases below the Curie point. In the Ising model there is an infinite set of mixed phases with identical free energy per unit volume in the infinite volume limit. The domain walls that characterize the mixed phases give rise to lower order contributions in the free energy expanded around the infinite volume limit. The "bulk" contribution in the multiple solutions is the same (the tree level eigenvalue density is the same) but they differ by amounts suppressed by powers of $1/N$, like "surface" contributions.

A significant difference between the Gaussian Penner model and the symmetric $\phi^4$ model is that in the Gaussian Penner model there does not seem to be an angular momentum parameter $l$ characterizing the double scaling solutions. Further, the double scaled free energies of the symmetric and maximally asymmetric solutions already differ, unlike in the $\phi^4$ case where both of these were $l = 0$ solutions with the same double scaled free energy. In spite of these differences, however, both models display the same general phenomenon, namely, that the enlarged class of symmetry breaking solutions produces the *same* free energy at *tree level*, and contain solutions that produce *different* free energies at *higher orders*. We expect this to be a generic feature of multi-cut matrix models. The existence of multiple solutions is related to the fact that when the potential has more than one minima, the number of smooth functions required to represent the recursion coefficients exceeds the number of constraints obtained from the recursion relations. Thus multiple solutions will exist even when the potential has no symmetry as we noted for $\sigma \neq 0$ in the $\phi^4$ model.

It would be interesting to know whether and how these solutions can tunnel into each other. This would be particularly interesting for the Gaussian Penner model where the maximally asymmetric solution corresponds to the $c = 1$ string compactified at twice the self-dual radius. We are now studying in greater detail higher order terms beyond the planar approximations as well as numerical solutions at finite N to the effective potential, Eq. (3.16), to determine more precisely the relationship between multiple solutions of the planar versus the double scaling limit.

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[17] An alternative parametrization is, $\frac{1}{4}(C-D)^2 = (-\mu/g)\ cos^2\theta(x)$ and $(A-B)^2 = (-\mu/g)^2\ sin^4\theta(x) - 4x/g$, from which both the $x = 0$ and $x = 1$ contours can be readily obtained. In this parametrization, $\theta(x)$ is restricted to $-\theta_{max} \leq \theta(x) \leq \theta_max(0)$ where $sin^4\theta_{max} = 4gx/\mu^2$ and $\theta_{max}$ goes to $\pi/2$ at $x_{cr}$. Another convenient parametrization which turns all contours into concentric circles is, $Y = r(x)sin\theta(x) \equiv A - B$, $X = r(x)cos\theta(x) \equiv \text{sign}(cos\theta)[(-\mu/2g)(C-D)^2 - (C-D)^4/16]^{-1/2}$, where $r^2 = (\mu/g)^2(x_{cr} - x)$. In this parametrization, $\theta(x)$ is unrestricted.

[18] This representation for a period-two structure was first derived in Ref. [2] under the symmetric ansatz where $C = D = 0$. With $C \neq 0 \neq D$, in analogy to Eq. (3.3), one can introduce a $2 \times 2$ matrix $\Phi_{\alpha,\beta}(x,\theta)$, where $\alpha, \beta$ take on values 0 and 1, (for even and odd respectively), i.e., $\Phi_{0,0} = C(x)$, $\Phi_{1,1} = D(x)$, $\Phi_{0,1} = \sqrt{A}e^{i\theta} + \sqrt{B}e^{-i\theta}$, and $\Phi_{1,0} = \sqrt{A}e^{-i\theta} + \sqrt{B}e^{i\theta}$. It follows that $F(z) = 1/2 \int_0^1 dx \int_0^{2\pi} \frac{d\theta}{2\pi} \sum_\alpha [z - \Phi(x,\theta)]^{-1}_{\alpha,\alpha}$. The $\theta$-integration can easily be carried out.

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[22] This also resolves th contradiction between Refs. [12] and [13]. These papers contain two different solutions to the Gaussian Penner model at the same critical point. However, from the viewpoint of the present work, this should not be regarded as a contradiction, but rather a special case of a general phenomenon in multi-cut matrix models – the existence of multiple solutions. The solutions obtained in Refs. [12] and [13] are the symmetric and maximally asymmetric solutions respectively.

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## Figure Captions

Fig. 1. Constraint on $A - B$ versus $C - D$ for $x = 0$, 0.5, .75, .875, .95, with $\mu = -2$ and $g = 1$.

Fig. 2. The Gaussian Penner potential for $t < 0$, $t = 0$ and $t > 0$.

Fig. 3. Eigenvalue distributions for even and odd $N$: (a) $N = 24$, symmetric solution; (b) $N = 24$, asymmetric solution; (c) $N = 25$, symmetric solution; (d) $N = 25$, asymmetric solution.

Fig. 4. Graphs of recursion coefficients for a sponteneously broken solution of the double-well potential. (a) The $R_n$ and (b) the $S_n$ coefficients after 100,000 minimizaton steps from a random start. (c) Orbit in the $A - B$ vs $C - D$ plane.