SYMPLECTIC EMBEDDINGS OF 4–MANIFOLDS VIA LEFSCHETZ FIBRATIONS

DISHANT M. PANCHOLI AND FRANSICO PRESAS

ABSTRACT. In this article we study proper symplectic and iso-symplectic embeddings of 4–manifolds in 6–manifolds. We show that a closed orientable smooth 4–manifold admitting a Lefschetz fibration over $\mathbb{C}P^1$ admits a symplectic embedding in the symplectic manifold $(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{pr})$, where ω_{pr} is the product symplectic form on $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$. We also show that there exists a sub-critical Weinstein 6–manifold in which all finite type Weinstein 4–manifolds admit iso-symplectic embeddings.

1. INTRODUCTION

The study of embeddings of manifolds has a long and fascinating history. Many important techniques essential for the study of geometric topology originated from the study embeddings of manifolds. To name a few, (1) H. Whiteny's famous trick [25] was discovered by him to establish embeddings of n–dimensional manifolds in $\mathbb{R}^{2n}$, (2) Nash's C^1–isometric embedding theorem [23] – which can be regarded as precursor to the phenomenon of h–principle discovered and popularized by M. Gromov [12], and (3) Kodiara's embedding theorem [19] which characterizes which complex manifolds are projective.

In this article we discuss proper symplectic and iso-symplectic embeddings of manifolds. Recall that by a symplectic embedding of an even dimensional manifold M in a symplectic manifold (W, ω_W), we mean a proper smooth embedding of M in W such that the pull-back of the symplectic form ω_W via the embedding induces a symplectic structure on M. In case M is symplectic with symplectic form ω_M, and the pull back $f^*(\omega_W)$ via an embedding f of M in W is the form ω_M, then we say that f is an iso-symplectic embedding of (M, ω_M) in (W, ω_W).

The study of symplectic and iso-symplectic embeddings of symplectic manifolds in a given symplectic manifold was initiated by M. Gromov. Gromov showed that the question of iso-symplectic and symplectic embeddings of symplectic manifolds abides by the h-principle provided the co-dimension of the embedding is at least 4. In case, (M, ω_M) is an open manifold, Gromov showed that the iso-symplectic and symplectic embedding problem abides by the h–principle even in the case when the co-dimension of embeddings under consideration is 2. Apart form these earlier works of Gromov, a major break through in producing symplectic sub-manifolds of a closed symplectic manifold came through the works of Donaldson [8]. Donaldson, using his approximately holomorphic technique, provided many symplectic sub-manifolds of a given closed symplectic manifold. In particular, he produce closed symplectic sub-manifolds in co-dimension 2. Donaldson's technique, though very powerful and extremely useful, does not provide any insight into the question finding which manifold might occur as a co-dimension 2 symplectic sub-manifold of a given symplectic manifold.

In the present article we try to address this question. We will focus on symplectic embeddings of closed 4–manifolds in co-dimension 2, and proper symplectic as well as iso-symplectic embeddings of Weinstein manifolds in co-dimension 2.

Let us now proceed towards stating precisely statements of main results. It is well known that a large class of symplectic manifolds admit *Lefschetz fibrations*. In this article we will discuss symplectic and iso-symplectic embedding of manifolds admitting Lefschetz fibration. We will also mostly focus on embeddings of 4–manifolds in 6–dimensional symplectic manifolds occasionally pointing out the places where there is a possibility of generalisations to higher dimensions. We begin by first discussing statements regarding symplectic embeddings of closed manifolds.

1.1. Embeddings of closed manifolds.

Let us recall few notions related to Lefschetz fibrations.

Definition 1.1 (Lefschetz fibration)**.** Let M be an oriented $2n$–dimensional compact manifold, possibly with a non-empty boundary and corners. By a Lefschetz fibration on M, we mean a map $f : M \to \mathcal{S}$, where $\mathcal{S}$ is either $\mathbb{C}P^1$ or a closed unit 2–disk $\mathbb{D}^2$, which satisfies the following property.

For every x at which the map f is singular, there exists an orientation preserving parameterization $\phi : U \subset M \to \mathbb{C}^n$, and an orientation preserving parameterization $\psi : V \subset \mathcal{S} \to \mathbb{C}$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0, \cdots, 0) \in \mathbb{C}^n$.
(2) $f(x) \in V$, and $\psi(f(x)) = 0 \in \mathbb{C}$.
(3) For the map $g : \mathbb{C}^n \to \mathbb{C}$ given by $g(z_1, z_2, \cdots, z_n) = z_1^2 + z_2^2 + \cdots + z_n^2$, the following diagram commutes:

$$\begin{array}{ccc} U \subset M & \xrightarrow{\phi} & \mathbb{C}^2 \\ \downarrow{f} & & \downarrow{g} \\ V \subset \mathcal{S} & \xrightarrow{\psi} & \mathbb{C}. \end{array}$$

It is well known that given a Lefschetz fibration, a fiber containing a singular point is obtained from a nearby fiber F by collapsing an embedded n–sphere $\mathbb{S}$ to a point. The sphere $\mathbb{S}$ is called a vanishing cycle. When $\mathbb{S}$ is homologically non-trivial, we say that $\mathbb{S}$ is an essential cycle.

A Lefschetz fibration $\pi_M : M \to \Sigma$, on M which satisfies: (1) the map π_M restricted to the set of critical points is injective, (2) every vanishing cycle is essential is also known as *simplified Lefschetz fibration* [4]. Most Lefschetz fibrations that one generally encounter satisfy these properties. Further, in most cases, given a manifold M admitting a Lefschetz fibration which is not simplified, it is possible to produce another Lefschetz fibration on M which is simplified. For this reason, we will only focus on simplified Lefschetz fibration. Hence from now on, by a Lefschetz fibration, we will always mean a simplified Lefschetz fibration.

We use Lefschetz fibrations to produce embeddings, for this we need to recall the notion of Lefschetz fibration embedding. This notion was discussed in [14].

Definition 1.2. Let $\pi_M : M \to \mathcal{S}$ and $\pi_N : N \to \mathcal{S}$ be two Lefschetz fibrations. An embedding $\phi : M \hookrightarrow N$ is said to be a Lefschetz fibration embedding of M in N, provided the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\pi_M} & & \downarrow{\pi_N} \\ \mathcal{S} & \xrightarrow{Id} & \mathcal{S}. \end{array}$$

We fix the following convention:

For the ease of notations in this article article we will denote by $\mathcal{P}$ the manifold $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and by $\pi_i : \mathcal{P} \to \mathbb{C}P^1$ the projection on the i'th factor, for $i \in \{1, 2, 3\}$. On the manifold $\mathcal{P}$ there is a natural symplectic form defined as $\pi_1^*(\omega_{FS}) + \pi_2^*(\omega_{FS}) + \pi_3^*(\omega_{FS})$, where ω_{FS} is the standard Fubini-Study symplectic form on $\mathbb{C}P^1$. Let us denote this symplectic form by ω_{pr}. The map $\pi_3 : \mathcal{P} \to \mathbb{C}P^1$ is clearly trivial symplectic fibration having fiber symplectomorphic to the symplectic manifold $(\mathbb{C}P^1 \times \mathbb{C}P^1, \pi_1^*\omega_{FS} + \pi_2^*\omega_{FS})$. We will denote the symplectic manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$ with symplectic form $\pi_1^*\omega_{FS} + \pi_2^*\omega_{FS}$ by $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} + \omega_{FS})$.

Theorem 1.3. *Let M be a closed orientable 4–manifold, and let $\pi_M : M \to \mathbb{C}P^1$ be a Lefschetz fibration. There exists a Lefschetz fibration embedding of M in $(\mathcal{P}, \omega_{pr})$ which satisfies the following:*

(1) the embedding is symplectic,
(2) any smooth fiber of the fibration $\pi_M : M \to \mathbb{C}P^1$ is symplectic sub-manifold of a fiber of π_3.

Since its introduction in symplectic geometry in the seminal article [9] by S. K. Donaldson, the notion of Lefschetz fibration has become extremely important in symplectic topology. An almost immediate consequence of [9, Theorem 2] – which provides the existence of *Lefschetz pencil* on symplectic manifolds – and Theorem 1.3 is the following:

Theorem 1.4. *Let (M, ω) be a closed 4–dimensional symplectic manifold. After a finite number of blow-up there exist a symplectic embedding of the blown-up manifold $\mathcal{B}M$ in symplectic manifold $(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{pr})$, where ω_{pr} is the product symplectic form on $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$.*

1.2. **Embeddings of Weinstein manifolds.**

Next, we apply our method of studying embedding via Lefschetz fibration to the case of Weinstein manifolds. Recall that a Weinstein manifold W is a manifold admitting a triple (ω, ϕ, X), where ω is a symplectic structure on W, ϕ is a exhausting Morse function for W, and X is a complete vector field on W which is exhaustive for ϕ and Liouville for ω. The triple (ω, ϕ, X) is known as a Weinstein structure on W. In this article we will only deal with Weinstein manifolds that admit exhaustive Morse function with only finite number of critical points. Recall that such Weinstein manifolds are known as *finite type* Weinstein manifolds. Finally recall that a Weinstein manifold V^{2n} of dimension $2n$ admitting a exhaustive Morse function having critical points of index at most $\frac{n}{2} - 1$ is known as a sub-critical Weinstein manifold.

Our main result related to iso-symplectic embeddings of Weinstein manifolds is the following:

Theorem 1.5. *Let $\mathcal{L}(-3)$ denote the complex line bundle over $\mathbb{C}P^1$ with Chern class -3. The manifold $L(-3) \times \mathbb{C}$ admits a Weinstein structure (ω_U, X, ϕ) which satisfies the following property.*

If (V, ϕ, X) is a Weinstein manifold of dimension 4 satisfying the property that ϕ has only finite number of critical points, then there exists an iso-symplectic embedding of (V, X, ϕ) in $(L(-3) \times \mathbb{C}, \omega_U, \phi, X)$.

We would like to point out that the universal model for iso-symplectic embeddings that we have constructed is not unique. A large class of Stein manifolds can serve as universal model. See the remark after the proof of Theorem 1.5 at the end of Section 6. Furthermore, since any Stein manifold has an underlying Weinstein structure, Theorem 1.5 provides iso-symplectic embeddings of Stein manifolds which are not holomorphic embeddings.

It was pointed out to us by Prof. Yakov Eliashberg that a complete h–principle for iso-symplectic embedding of Weinstein manifold in co-dimension 2 is well known provided the target manifold is flexible. This follows form the work discussed in [6]. However, producing an embedding of Weinstein manifold which is formally iso-symplectic is generally not very easy. We circumvent this problem by producing explicit iso-symplectic embeddings.

Let us have few words regarding the arrangement of this article. Essential preliminaries related to Lefschetz fibration and mapping class groups are collected in Section 2. In Section 3 we discuss main ideas involved in proofs of Theorem 1.3 and Theorem 1.5. The notion of *flexible embeddings* of surfaces is discussed in Section 4 while sections 5 and 6 deal with proof of Theorem 1.3 and Theorem 1.5 respectively.

2. PRELIMINARIES

We start this section by recalling some results related to Lefschetz fibrations.

2.1. **Lefschetz fibration.**

In this sub-section we recall two theorems about Lefschetz fibrations. First of which is due S. K. Donaldson [9, Theorem:2] and J.Amorś, V. Muñoz and F. Presas [1], which roughly says that every symplectic manifold, after finite number of blow-up admits a Lefschetz fibration structure. The second one is due to R. Gompf which –in certain sense – establishes the converse of Donaldson's result providing the existence of symplectic structures on Lefschetz fibrations.

Theorem 2.1 (S. Donaldson [9], J. Amorós, V. Muñoz, and F.Presas [1])**.** *Let (M, ω) be a closed symplectic manifold. After finite number of blow-ups of M, there exists a Lefschetz fibration structure on the blown-up manifold $\widetilde{M}$ which satisfies the property that regular fiber of this fibration is a symplectic sub-manifold of $\widetilde{M}$. Furthermore, we can always find a Lefschetz fibration on $\widetilde{M}$ which is simplified.*

We would like to mention here that Theorem 2.1 is actually not stated in the form mentioned here but it is an easy consequence of Theorem 2 of [9], and [1, Theorem 1.3]. Theorem 2 of [9] provides us with the existence of Lefschetz pencils, while [1, Theorem 1.3] shows that given a pair of vanishing cycles c_1 and c_2 associated to a symplectic Lefschetz fibration, there exists a symplectic isotopy of the symplectic manifold which send c_1 to c_2. This implies that even if one vanishing cycle is essential all vanishing cycles are essential. This, in particular, implies that pencil induced by large degree line bundles satisfy the property that all its vanishing cycles are essential.

Finally let us state a result due to R. Gompf [16, Theorem:10.2.18]:

Theorem 2.2 (R.Gompf). *Let X^{2n} be a smooth manifold that admits a Lefschetz fibration $\pi : X \to \mathcal{S}$ with fiber F. Then X admits a symplectic structure on ω for which the fiber F is symplectic if and only if $[F]$ in $H^2(X, \mathbb{Z})$ is non-zero. Furthermore, if $\{e_1, e_2, \cdots e_n\}$ is a finite collection of sections of the Lefschetz fibration, then the symplectic structure can be assumed to be such that each one of these sections is symplectic.*

We would like to remark that we are not assuming that X is a closed manifold in the statement of Theorem 2.2. This will be the case provided $\mathcal{S}$ is $\mathbb{C}P^1$. In case, $\mathcal{S}$ is $\mathbb{D}^2$, then X is a manifold with boundary and corners.

2.2. **Review of symplectic mapping class group.**

One of the main ingredient for establishing symplectic embeddings is the notion of flexible embedding of surfaces. In order to get flexible embeddings of surfaces, we use some results about mapping class groups of surfaces. Let us review these.

Throughout this article by the mapping class group of an orientable surface (Σ, ω) we mean the group of symplectic form preserving diffeomorphism of Σ up to a symplectic isotopy. In case, the boundary of Σ is non-empty, then the mapping class group consists of all symplectomorphisms which are identity when restricted to the boundary of Σ up to isotopies that are identity when restricted to the boundary of Σ.. Furthermore, since the symplectic isotopy class of a particular symplectomorphism is the only thing that is relevant for this article, the word symplectomorphism will always mean the isotopy class of the symplectomorphism.

It follows from the works of M. Dehn [7] and C. Lickorish [20] that the mapping class group of a closed orientable surface is generated by Dehn twists [5, Section 3.1.1]. S. Humphries extended their work to established the most economical set of generators for the mapping class group of an orientable genus g surface. He showed that the mapping class group is generated by Dehn twists along the curves a_i, $i = 1$ to g, b_j, $j = 1$ to g_1, c_1, and c_2 as depicted in Figure 2.2 provided, Σ is a closed orientable surface of genus $g, g \geq 2$.

Since we are working with surfaces together with a symplectic form ω, Σ naturally comes equipped with an orientation. When we say a positive (left handed) Dehn twist [5, Section 3.1.1], we always mean a positive Dehn twist with respect to this orientation. The general term Dehn twist refers to either a positive or a negative Dehn twist.

For a surface $(\Sigma, \partial\Sigma)$ having a unique boundary component as depicted in Figure 2.2, the mapping class group is generated by the same set of Humphries generators together with Dehn twist along boundary parallel curve d as depicted in Figure 2.2.

Having collected necessary preliminaries, we proceed now to provide proofs of main results. To make our ideas accessible, in the next section, we outline the main ideas involved in the proofs.

3. Main ideas involved in proofs of Theorem 1.3 and Theorem 1.5

.

Let us first discuss ideas involved in establishing Theorem 1.3. Given any Lefschetz fibration $\pi_M : M \to \mathbb{C}P^1$, we know that removing finite number of singular fibers, we get a fiber bundle over punctured $\mathbb{C}P^1$. We also know that the monodromy around each of this puncture is a positive Dehn twist.

Let F denote a fixed smooth fiber of the Lefschetz fibration $\pi_M : M \to \mathbb{C}P^1$. We observe the following: if there exists a symplectic manifold (W, ω_W) and a symplectic embedding $i : F \hookrightarrow W$ such that for every symplectomorphism $\psi : F \to F$ the embeddings $\psi \circ i$ and i are symplectically isotopic, then the fiber bundle
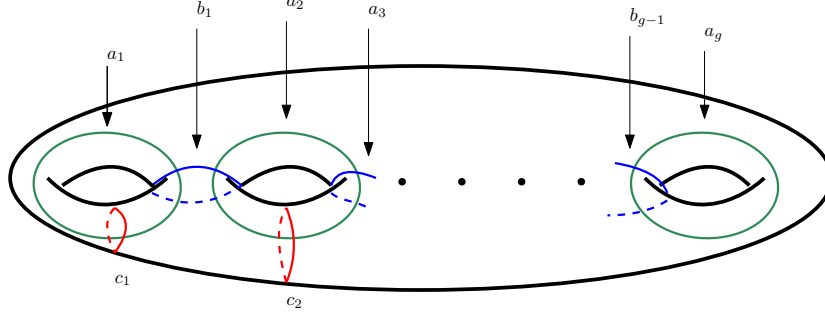
FIGURE 1. The picture depicts the set of Humphries generators for the mapping class group of a closed orientable surface of genus g. The curves a_i's are depicted in green, curves b_j's are depicted in blue, curves c_1 and c_2 are depicted in red.
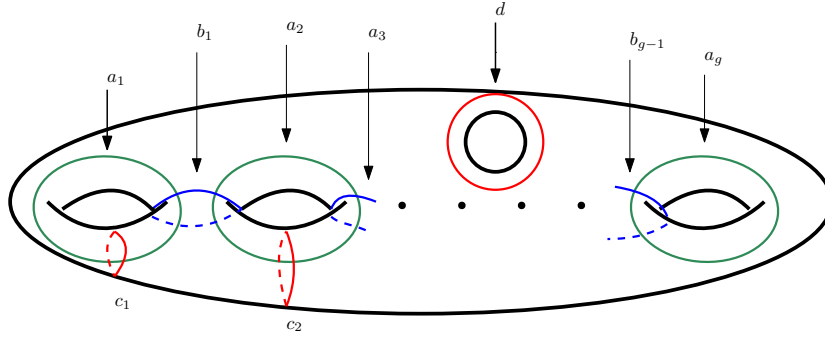

FIGURE 2. The figure depicts Humphries generators for the mapping class group of a surface with single boundary component and a surface having exactly one puncture. The mapping class group of punctured surface is generated by Dehn twists along a_i, b_j, c_1, and c_2, while to generate the mapping class group of the surface with one boundary component we need an additional Dehn twist along the curve d.

over punctured $\mathbb{C}P^1$ admits a symplectic embedding in $W \times \mathbb{C}P^1$. An embedding satisfying this property is termed as *symplectically flexible embeddings*. We refer to Section 4 for a precise definition of symplectically flexible embeddings.

We observe that $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} + \omega_{FS})$ is the symplectic manifold in which every genus g surface admits a symplectically flexible embedding. The existence of symplectically flexible embeddings is discussed in Lemma 4.4. As remarked earlier, Lemma 4.4 implies that there is fiber preserving symplectic embedding of the Lefschetz fibration $\pi_M : M \to \mathbb{C}P^1$ restricted to the complement of singular fibers in the trivial fibration $\pi_3 : \mathcal{P} \to \mathbb{C}P^1$, where recall that $\mathcal{P} = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with product symplectic form and π_3 is the projection on the third factor.

In order to extend the fiber preserving embedding constructed in previous paragraph to an embedding of M we turn to local model of Lefschetz critical point. We observe that there exist a symplectic embedding ϕ of $\mathbb{C}^2$ in $\mathbb{C}^3$ given by $\phi(z_1, z_2) = (z_1, z_2, 0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\phi} & \mathbb{C}^3 \\ \downarrow{f} & & \downarrow{g} \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C}, \end{array}$$

where $f(z_1, z_2, z_3) = z_1 z_2 + z_3$ and $g = f \circ \phi$. Observe that this implies we can get embed a neighborhood of Lefschetz singular point in a standard symplectic ball such that trivial fibration of this ball to a disk induces

the given Lefschetz fibration on the neighborhood of Lefschetz singularity. This observation allows us to extend the embeddings obtained at the end of first step to neighborhoods of singular fibers. Though we have not written down final argument in this explicit format, the main idea behind proofs is this. We refer to [14] for smooth embeddings of all 4–manifolds in $\mathbb{C}P^3$ using similar ideas. In fact, our main observation was that most of the ideas discussed in [14] can be adopted in symplectic setting.

Let us end this section be briefly discussing main ideas in the proof of Theorem 1.5. Theorem 1.5 follows relatively easily from Theorem 6.7 where we discuss Stein Lefschetz fibration embedding in a fixed Stein domain.

The ideas involved in producing Stein Lefschetz fibration embeddings are similar to the one discussed earlier for the proof of Theorem 1.3. It is clear that one needs to produce a Stein manifold W which admits flexible embeddings of Stein 1–manifolds. We observe that the Stein manifold $\mathbb{D}\mathcal{L}(-3)$ has the property that every Stein 1–manifold which is biholomorphic to a once punctured Riemann surface, admit a proper symplectically flexible embedding in $\mathbb{D}\mathcal{L}(-3)$. See Lemma 4.6 for a precise statement regarding flexible embeddings in $\mathbb{D}\mathcal{L}(-3)$. Rest of the proof to establish Theorem 6.7 follows essentially the same logic.

4. Flexible embeddings of surfaces

We begin by recalling few definitions regarding flexible embeddings of surfaces from [14].

Definition 4.1. Let Σ be an orientable surface. Let (M, ω) be a symplectic manifold. Let $\Psi : \Sigma_g \hookrightarrow (M, \omega)$ be a symplectic embedding. Let f be an element of $\mathcal{MCG}(\Sigma, \Psi^*(\omega) = \Omega)$. We say that the element f is conjugate via the embedding Ψ provided the following properties are satisfied:

(1) There exits a 1–parametric family ϕ_t of symplectomorphisms of (M, ω) such that $\phi_0 = id$ and $\phi_1(\Psi(\Sigma)) = \Sigma$.
(2) $\Psi^{-1} \circ \phi_1 \circ \Psi = f$.

Definition 4.2 (Symplectically flexible embedding). A symplectic embedding Ψ of an orientable surface Σ in a symplectic manifold (M, ω) is said to be symplectically flexible provided every $f \in \mathcal{MCG}(\Sigma, \Psi^*(\omega))$ is conjugated by a symplectic isotopy of (M, ω).

For the sake of brevity, we will refer to a symplectically flexible embedding just by the term *flexible embedding*. Recall that the term flexible embedding is used in [15] for smooth embedding satisfying properties similar to the define in Definition 4.2. Since in this article, we will only be dealing with symplectically flexible embeddings, we are going to take slight liberty and refer them as flexible embeddings in the rest of the article unless stated otherwise explicitely.

Consider a pencil $\pi : M^4 \setminus B \to \mathbb{C}P^1$ on a closed symplectic manifold (M, ω). Let c be a critical point for the pencil and let ν be a vanishing cycle corresponding to the critical point c. Our first lemma is regarding the existence of a symplectic isotopy of M which conjugates the element τ_ν, where τ_ν denotes the positive Dehn twist along the curve ν.

Lemma 4.3. *Let $p : M \setminus B \to \mathbb{C}P^1$, c, ν, and τ_ν be as in the previous paragraph. Let F be a fiber of pencil with $\nu \subset F$. Then there exists a 1-parametric family, $\Psi_t, t \in [0, 1]$, of symplectomorphisms of M and a 4–ball embedded in M which satisfies the following:*

(1) *The 4–ball is symplectomorphic to standard symplectic 4–ball of some radius r.*
(2) *For each t, Ψ_t is the identity outside B^4 and Ψ_0 is the identity.*
(3) *The family Ψ_t conjugates τ_ν.*

Proof. In order to prove this result, let us first consider an abstract Weinstein Lefschetz fibration $\pi_B : B^4 \to \mathbb{D}^2$, where $\mathbb{D}^2$ is the closed disk of radius δ around 0 in $\mathbb{C}$, which satisfies the following properties:

(1) The fibration has a unique Lefschetz critical point on the fiber over 0, and each fiber symplectomorphic to a Weinstein domain $DT^*\mathbb{S}^1$, where $DT^*\mathbb{S}^1$ denotes a disk bundle associated to the co-tangent bundle T^*S^1.
(2) The fibrations is such that the monodromy associated to the fiber bundle $\pi_B : B^4 \setminus \pi_B^{-1}(\{0\}) \to \mathbb{D}^2 \setminus \{0\}$ is a Dehn twist which is supported away from a neighborhood of $\partial DT^*\mathbb{S}^1$.

Consider a vector-field V on $\mathbb{D}$ such that the flow associated to this vector field keeps a small collar of the boundary $\partial \mathbb{D}$ fixed and rotates the circle of radius less than or equal to $\frac{1}{4}\delta$ by 2π while rotating the circle of radius $\frac{1}{2}\delta$ by π as depicted in Figure 4. The lift of this vector field via a symplectic connection on the fibration produces a flow that conjugates the monodromy Dehn twist on the fiber $\pi_B^{-1}\{(\frac{1}{2}, 0)\}$ Let us denote this flow by $\widetilde{\phi}_t$.

Next, construct a properly embedding of an annulus A in B which satisfies the following:

(1) $A = \widetilde{D}T^*\mathbb{S}^1 \subset DT^*\mathbb{S}^1$, where $\widetilde{D}$ is disk of smaller radius than the radius of the disk D, and the disk $\widetilde{D}$ is chosen such that the monodromy Dehn twist associated to the fibration $\pi_B : B \to \mathbb{D}$ is the identity when restricted to $DT^*\mathbb{S}^1 \setminus \widetilde{D}T^*\mathbb{S}^1$.
(2) $\partial A = \pi_B^{-1}(\{0\}) \cap \partial B^4$.
(3) The embedding is symplectic.

Observe that the flow $\widetilde{\phi}_t$ conjugates the Dehn twist on A, This implies by enlarging the vertical boundary of the Weinstein fibration $\pi_B : B^4 \to \mathbb{D}$, we get that there exists a Weinstein fibration $\pi : \widetilde{B} \to \mathbb{D}^2$, and an embedding of an annlus $\widetilde{A}$ in $\widetilde{B}$ which satisfies the following:

(1) $A \subset \widetilde{A}$.
(2) There exist a family $\widetilde{\phi}_t$ of diffeomorphism which conjugates a Dehn twist on $\widetilde{A}$
(3) For $\widetilde{\phi}_t = \phi_t$, when restricted to $B \subset \widetilde{B}$.

Till now, we have produce the flow $\widetilde{\phi}_t$ which need not be a symplectic flow, however, $\widetilde{\phi}_t^{-1}(\{\frac{1}{2}, 0)\}$ is a 1–parametric family of symplectic sub-manifolds of B^4 all of them have common boundary embedded in ∂B^4. Hence in light of [2][Proposition:4] by D. Auroux, we get that there exits a flow ϕ_t which preserve the symplectic structure associated to the Weinstein fibration – i.e., ϕ_t is a symplectomorphism for each t – $\pi_B : B \to \mathbb{D}$, and agrees with $\widetilde{\phi}_t$ near the boundary of B^4. Clearly, ϕ_t Observe that the pair $(\widetilde{A}, \phi_t)$ satisfy the following properties:

(1) $\widetilde{A}$ is a symplectically embedded annulus in B^4 symplectomorphic to a unit disk bundle associated to $T^*\mathbb{S}^1$.
(2) ϕ_t is a 1–parametric family of symplectomorphism of B^4 each identity close to ∂B^4 and which conjugates the Dehn twist on $\widetilde{A}$.

Now, given $p : M \setminus B \to \mathbb{C}P^1, c, \nu$, and τ_ν, without loss of generality we can assume that τ_ν is supported in a small neighborhood of ν. This implies that there exists a neighborhood $\mathcal{N}$ of the critical point c symplectomorphic to some abstract Weinstein fibration of the type $\pi_B : B \to \mathbb{D}^2$. This clearly implies that there exits a family $\widetilde{\Psi}_t$ of symplectomorphisms of supported in $\mathcal{N}$ which conjugates positive Dehn twist on an annulus $\mathcal{A}$ embedded in $\mathcal{N}$ which satisfies the property that ∂A is pair of circles on the singular fiber $\pi_M^{-1}(\{c\})$.

Finally, consider the symplectic sub-manifold $\widetilde{F}$ of M which is the union of $\mathcal{A}$ with $\pi_M^{-1}(\{c\}) \setminus \mathcal{N}$. Clearly, this is a symplectically embedded sub-manifold symplectically isotopic to any smooth fiber F of the pencil $\pi_M : M \setminus B \to \mathbb{C}P^1$. Hence, we get that given a fiber F and ν, on the fiber F there exists a family Ψ_t of symplectomorphisms of M which conjugates τ_ν. Hence the proof.

$\square$

4.1. **Flexible embeddings in $\mathbb{C}P^1 \times \mathbb{C}P^1$.**

Lemma 4.4. *Consider the symplectic manifold $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} + \omega_{FS})$. For every g positive there exists a flexible symplectic embedding of genus g surface in $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} + \omega_{FS})$.*

Proof. Recall that $\mathbb{C}P^1 \times \mathbb{C}P^1$ admits holomorphic Lefschetz pencil having its smooth fiber a symplectically embedded 2–torus $\mathbb{T}^2$, and the base locus consisting of 8 points. Furthermore, the 2–torus is the zero of a generic section of the line bundle $\mathcal{O}(2) \otimes \mathcal{O}(2)$, and the vanishing cycles – up to Hurewicz moves – consist of curves a and b as depicted in Figure 4.1 having relation $(\tau_a \cdot \tau_b)^6 = Id$. Let us denote this pencil by $\pi_{(2,2)} : \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus B_{(2,2)} \to \mathbb{C}P^1$, where $B_{(2,2)}$ denotes the base locus for the pencil consisting of 8 points.
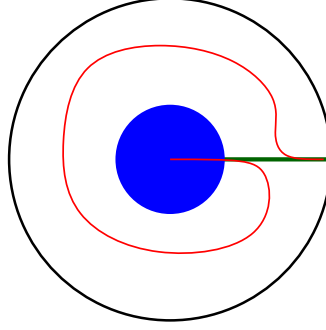
FIGURE 3. The picture depict the flow assoicated to the vector field whose lift via symplectic connection produces the flow on B^4 which induces positive Dehn twist on the fiber over $\frac{1}{2}\delta + i0$. The vector field is such that the associated flow rotates the blue disk by 2π while keeping the boundary fixed. The red curve depicts the time 1 image of the black curve under the flow.

Since the homology class of this torus is $(2,2)$, we know that any such torus is symplectically isotopic to a symplectically embedded torus constructed via the following procedure.

Choose a pair of points N, S in $\mathbb{C}P^1$. Next, consider two vertical $\mathbb{C}P^1$ given by $\{N\} \times \mathbb{C}P^1$ and $\{S\} \times \mathbb{C}P^1$. Next, consider two horizontal $\mathbb{C}P^1$'s given by $\mathbb{C}P^1 \times \{N\}$ and $\mathbb{C}P^1 \times \{S\}$. Now perform ambient Gromov sum at the points $\{(N,N),(N,S)(S,N),(S,S)\}$. Let us denote this torus by $\mathbb{T}_{(2,2)}$.

We claim that $\mathbb{T}_{(2,2)}$ is flexible in $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} + \omega_{FS})$. This is because, the pencil $\pi_{(2,2)}$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$ with fiber $\mathbb{T}_{(2,2)}$ has vanishing cycles isotopic to the curves a and b as depicted in the Figure 4.1. Hence according to Lemma 4.3 we can conjugate Dehn twists τ_a and τ_b via family symplectomorphisms. Furthermore, each one of this family is supported in a small ball which contains a tubular neighborhood of the vanishing cycle. We know that τ_a and τ_b generate the mapping class group of any torus, and hence we get that $\mathbb{T}_{(2,2)}$ is flexible in $\mathbb{C}P^1 \times \mathbb{C}P^1$.
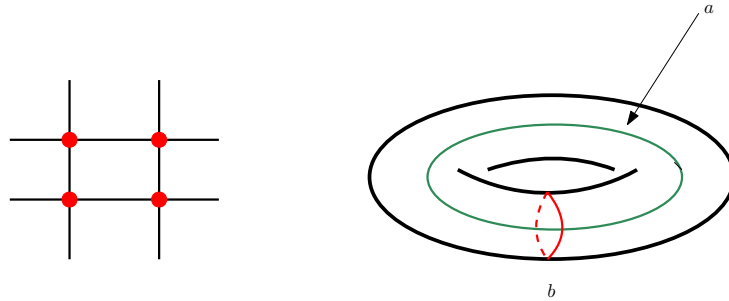

FIGURE 4.   The left of the figure is the schematic description of the embedded curve $\mathbb{T}_{(2,2)}$. Two vertical lines correspond to sphere $\{N\} \times \mathbb{C}P^1$ and $\{S\} \times \mathbb{C}P^1$, while the two horizontal lines depict $\mathbb{C}P^1 \times \{N\}$ and $\mathbb{C}P^1 \times \{S\}$. The red disk depicts the Gromov sum along the point of intersection. On the right of the figure the picture depicts the curve $\mathbb{T}_{(2,2)}$ obtained as a result Gromov sum of two vertical $\mathbb{C}P^1$'s with two horizontal $\mathbb{C}P^1$'s together with vanishing cycles for the pencil $\pi_{(2,2)}$.

Till now we have shown that there exists a flexible embedding of tours in $\mathbb{C}P^1 \times \mathbb{C}P^1$. We now show how to produce a symplectic embedding of a surface of genus g for any $g > 0$. Consider $g - 1$ distinct points

$P_1, \cdots, P_{g-1}$ on $\mathbb{C}P^1$ such that $\{P_1, \cdots, P_{g_1}\} \cap \{N, S\} = \emptyset$. Furthermore, we choose the points so that non of the point in the collections $\{(N, P_i)\}$ or $\{(S, P_i)\}$ lie on curves λ and μ embedded in $\mathbb{T}_{(2,2)} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$.

Next, consider the vertical $\mathbb{C}P^1 \times \{P_i\}$ which intersect the torus $\mathbb{T}_{(2,2)}$ in a pair of points (N, P_i) and (S, P_i). For each i perform ambient Gromov sum to produce an embedding of genus g surface. We claim that this embedded surface is flexible. Let us denote this embedded surface by Σ_g

In order to establish the claim, notice that Σ_g for each g admits an embedding of a torus isotopic to $\mathbb{T}_{(2,2)}$ minus a pair of disk. This torus consist of the tours obtained by taking ambient Gromov sum of $\mathbb{C}P^1 \times \{P_i\}$, $\mathbb{C}P^1 \times \{P_{i+1}\}$ for $i = 0, \cdots, g+1$, where $P_0 = N$ and $P_{g+1} = S$ with $\{N\} \times CP^1$ and $\{S\} \times \mathbb{C}P^1$ along four points of intersections. Let a_i and b_i be curves of this torus which corresponds to vanishing cycles for the pencil $\pi_{(2,2)}$. Hence, by Lemma 4.3 for the embedded surface Σ_g, Dehn twists $\{\tau_{a_i}, \tau_{b_i}\}, i = 1, \cdots, g$ can be conjugated in $\mathbb{C}P^1 \times \mathbb{C}P^1$ via families of symplectomorphisms each family having support in a small ball. Furthermore, notice that since the relation for the pencil among Dehn twists is

$$(\tau_{a_i} \cdot \tau_{b_i})^6 = 1,$$

we get that for the curve c depicted in Figure 4.1 the Dehn twist τ_c also can be conjugated. Now we know form Theorem [5] that the mapping class group of genus g surface is generated by Dehn twists along curves a_i, b_j, c_1 and c_2 and the collection of curves on which we can perform Dehn twist in the embedded surfaces contains these curves. Hence we have the lemma.
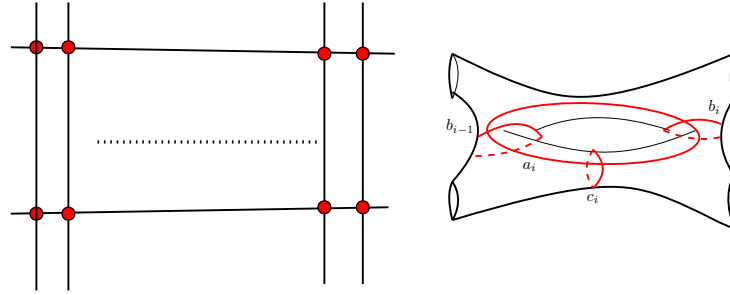
□

FIGURE 5. The figure depicts a flexibly embedded surface in $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is constructed out of a flexibly embedded torus $\mathbb{T}_{(2,2)}$. As usual horizontal lines depict $\mathbb{C}P^1 \times \{p\}$, where p is a point in $\mathbb{C}P^1$, while vertical lies depict $\{q\} \times \mathbb{C}P^1$, where $q \in \mathbb{C}P^1$. The red dot at the intersection depicts that we have performed ambient symplectic Gompf sum in a neighborhood of an intersection point of a horizontal $\mathbb{C}P^1$ with a vertical $\mathbb{C}P^1$.

Remark 4.5. It is almost immediate that Lemma 4.3 has a natural generalization to higher dimensions, where we work with Dehn-Seidel twists instead of Dehn twists.

4.2. **Flexible embedding of surface in $\mathcal{L}(-3)$.**

Let $\mathcal{L}(-3)$ denote the Stein domain corresponding to the complex disk bundle over $\mathbb{C}P^1$ having first Chern class -3. Any disk bundle associated to the line bundle $\mathcal{L}(-3)$ will be denoted by $\mathbb{D}\mathcal{L}(-3)$. The purpose of this section is to show that given a compact orientable surface of genus g having one puncture there exits a symplectic embedding of the surface in $\mathcal{L}(-3)$. More precisely, we have the following:

Lemma 4.6. *Let Σ be a once punctured surface of genus g with $g \geq 2$. There exists a proper symplectic embedding of $f_\Sigma : \Sigma \hookrightarrow \mathcal{L}(-3)$ which satisfies the following properties:*

(1) *There exists a plurisubharmonic proper and exhausting function F_Σ on $\mathcal{L}(-3) \to [0, \infty)$ such that for sufficiently large M F_Σ has no critical value c with $c \geq M$ and for all such c, $f_\Sigma(\Sigma) \cap F_\Sigma^{-1}(M, \infty)$ is a properly embedded annuls in $f(\Sigma)$.*

(2) *The embedding is flexible.*

Proof. We know that $\mathcal{L}(-3)$ admits a Stein Lefschetz fibration having fiber a genus g surface and vanishing cycles consisting of curves a_i, b_j, c_1, and c_2 depicted in Figure 2.2. Let $\widetilde{f_\Sigma}$ be the embedding which send $\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is a surface with one boundary component, to a smooth fiber of this Lefschetz fibration. Clearly this embedding is symplectic. By adding an appropriate to the Stein Lefschetz fibration, it is easy to see that there exists an embedding f_Σ of Σ in the Stein manifold $\mathcal{L}(-3)$ which satisfies the first property in the statement. Hence in order to establish the theorem, we just need to show that the embedding is flexible. We now establish this.

First of all note that we can assume that the monodromy associated to any of the vanishing cycles is supported in a small neighborhood of the vanishing cycle. Hence by an argument similar to the one used in the proof of Lemma 4.3, we get that every Dehn twist along vanishing cycles a_i, b_j, c_1, and c_2 can be conjugated. Since the mapping class group of Σ is generated by Dehn twist along a_i, b_j, c_1, and c_2 the lemma follows. $\square$

We would like to remark that Lemma 4.6 is implicitly proved in the article [10] by J. Etnyre and Y. Lekili. The motivation for studying $\mathcal{L}(-3)$ and flexible surfaces in them for them is related to their construction of universal 5–manifold in which all contact 3–manifolds embed.

5. Proof of Theorem 1.3

Let us now establish Theorem 1.3. The proof is divided in four steps. In the first step we observe that given a Lefschetz fibration $\pi_M : M \to \mathbb{C}P^1$, there is a Lefschetz fibration embedding of a small tubular neighborhood $\mathcal{N}$ of any singular fiber containing a unique critical point such the embedding restricted to any smooth fiber contained in $\mathcal{N}$ is flexible. We denote any such embedding by L_i, where i is indexed over the set of critical points of the fibration $\pi_M : M \to \mathbb{C}P^1$.

In the second step we produce embeddings of small tubular neighborhoods of singular fibers using the first step in such way that no two embeddings intersect, and any pair of smooth fibers are ambiently symplectically isotopic in $\mathbb{C}P^1 \times \mathbb{C}P^1$. In the third step, we use flexibility of the embedded fibers to produce an embedding of the Lefschetz fibration $\pi_M : M \to \mathbb{C}P^1$ restricted to the inverse image of a disk embedded in $\mathbb{C}P^1$ which contains all critical values.

In the final step, the triviality of π_1 of the group of symplectic form preserving diffeomorphisms isotopic to the identity of any surface of genus $g, g \geq 2$ is used to conclude that we have a symplectic Lefschetz fibration embedding of M in (P, ω_{pr}) as claimed.

Proof of Theorem 1.3.
We are given a Lefschetz fibration $\pi : M \to \mathbb{C}P^1$. Let $x_1, x_2, \cdots, x_n$ be the set of critical points of the fibration. Let x_i be the critical point of M corresponding to the vanishing cycle c_i. Denote by F_{x_i} the fiber which contains the critical point x_i and by $\mathcal{N}(F_{x_i})$ a neighborhood of F_{x_i} containing the unique critical point x_i. The neighborhood $\mathcal{N}(F_{x_i})$ is obtained by taking the inverse image under π_M of a small disk $\mathbb{D}_{p_i}$ of radius $\epsilon(p_i)$, which contains only $\pi(x_i) = p_i$ as critical value. The proof is divided in three steps. The first step deals with embeddings of $\mathcal{N}(F_{x_i})$.

Step-1:

In this step we prove that there exists an embedding ϕ_i of a small tubular neighborhood $\mathcal{N}(F_{x_i})$ of $F_{x_i}, i \in \{1, 2, \cdots, l\}$ in $V \times \Sigma$ such that the following diagram commutes:

$$(1) \qquad \begin{array}{ccc} \mathcal{N}(F_{x_i}) & \overset{\phi_i}{\lhook\joinrel\longrightarrow} & \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{D}_{p_i} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} \\ \pi(\mathcal{N}(F_{x_1})) & \lhook\joinrel\longrightarrow & \mathbb{D}_{p_i}. \end{array}$$

According to Lemma 4.4 there exists an embedding $H : \Sigma_g \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ which is symplectic and flexible. Recall that this is a $(2, g + 1)$–curve embedded in $\mathbb{C}P^1 \times \mathbb{C}P^1$. We will denote this curve by Σ_g.

Since any two essential cycles on the fiber of π can be conjugated by an isotopy of M, we can assume that the fiber of π over a point p, where p is a point $\mathbb{C}P^1$ which lies on the boundary of $\mathbb{D}_{p_1}$ admits a symplectic embedding in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \{p\}$ such that its image is the flexibly embedded $(2, g+1)$-curve in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \{p\}$ and the vanishing cycle c_1 is mapped a fixed vanishing cycle $\mathcal{C}$ for the pencil $\pi_{(2,n)} : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1$ whose smooth fiber is Σ_g. Let us denote the vanishing cycle by ν. Consider the blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ at base locus B corresponding to the pencil of $\mathbb{C}P^1 \times \mathbb{C}P^1$ for which Σ_g is a smooth fiber. Let us denote the blown-up manifold by $\widetilde{\mathbb{C}P^1 \times \mathbb{C}P^1}$. Let $\pi_B : \widetilde{\mathbb{C}P^1 \times \mathbb{C}P^1} \to \mathbb{C}P^1$ be the corresponding Lefschetz fibration. Let $T : \widetilde{\mathbb{C}P^1 \times \mathbb{C}P^1} \to \mathbb{C}P^1 \times \mathbb{C}P^1$ be the blow-up projection. We know that in $\widetilde{\mathbb{C}P^1 \times \mathbb{C}P^1}$ there exits a diagonal embedding of $\mathcal{N}(F_{x_i})$ in $\widetilde{\mathbb{C}P^1 \times \mathbb{C}P^1} \times \mathbb{D}_{p_1} \to \mathbb{D}_{p_1}$ given by diagonal embedding consisting of $x \to (x, \pi_B(x))$ followed by a translation to ensure that the projected disk via π_B is mapped to the disk D_{p_1}. Call this embedding $\phi = (\phi_1, \phi_2)$. Given ϕ we can define the required embedding of $\mathcal{N}(F_{x_i})$ in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{D}_{p_1}$ by sending x to $(T(\phi_1(x)), \pi_B(x))$. Let us denote this embedding by L_1

Step-2:

Following step-1, produce embeddings L_i of $\mathcal{N}(F_{x_i})$ for each i. Observe that the embedding is produced such that the vanishing cycle c_i is mapped to the curve $\mathcal{C}$, which is a fixed vanishing cycle for the pencil $\pi_{(2,g+1)} : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1$. In addition, we have that any embedding L_i restricted to a smooth fiber is a flexible embedding embedding of the fiber in $\mathbb{C}P^1 \times \mathbb{C}P^1$, and fibers $L_i(F_{u_i})$ and $L_j(F_{u_j})$, where $u_i \in \mathbb{D}_{p_i}$ is a regular value, are symplectically isotopic in $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Step-3:

For each i let Z_i be a point on the boundary of $\mathbb{D}_{p_i}$, and let F_{z_i} denote the embedded fiber over Z_i. Fix the point $Z = Z_1 \in \mathbb{C}P^1$ which does not lie on any of the disk $\mathbb{D}_{p_i}$ when $i \neq 1$. For each i let $\overline{ZZ_i}$ be a smooth embedded path joining Z_i to Z. Furthermore, let us assume that $\overline{ZZ_i}$ and $\overline{ZZ_j}$ intersect only at Z when $i \neq j$.

Let L be a fixed embedding of $(2, g + 1)$–curve Σ_g in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \{Z\}$ obtained by the restriction of the embedding L_1 to the fiber over $Z = Z_1$. The triviality of the fiber bundle $\mathcal{P} \to \mathbb{C}P^1$ implies that there exist an embedding $\widetilde{L_i}$ of Σ_g in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \{z_i\}$ obtained by identifying the fiber at Z_i with fiber at Z via this given trivialization such that images of Σ_g under embedding coming from $L_i|\pi^{-1}\{z_i\}$ and $\widetilde{L_i}$ coincide.

Observe that the flexibility of the embedding with image $\widetilde{L_i}$ in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \{z_i\}$ implies that these two embeddings are isotopic. Hence there exist an embedding of $\Sigma_g \times [0, 1]$ along the path $\overline{ZZ_i}$ such that $\Sigma_g \times \{0\}$ is the embedding L_i restricted to $\pi^{-1}(Z_i)$ while $\Sigma \times \{1\}$ is embedding L.

Observe that till now we have produce an embedding of $M \setminus \pi^{-1}(\cup \mathbb{D}_{p_i} \cup \overline{zz_i})$ in the manifold $\mathcal{P}$. Consider a small neighborhood of $\cup \mathbb{D}_{p_i} \cup \overline{zz_i}$ in $\mathbb{C}P^1$. It is clear that this neighborhood is a disk $\mathbb{D}$ in $\mathbb{C}P^1$ and there exits a Lefschetz fibration embedding of $M \setminus \pi_M^{-1}(\mathbb{D})$ in $\mathcal{P}$.

Step-4:

Since M is a closed manifold. We know that the product of τ_{c_i} is the identity. This implies that $\pi^{-1}(\partial \mathbb{D})$ is an embedding of $\mathbb{S}^1 \times \Sigma_g$

Since the genus of smooth fiber is at least 2, and since the fundamental group of any fixed symplectic form preserving diffeomorphisms of a surface of genus g is trivial, we can assume that M is obtained by gluing $M \setminus \pi_M^{-1}(\mathbb{D})$ with its complement via the identity map.

It is now clear that the constructed embedding can be assumed to agree with the embedding of $\Sigma \times \partial \mathbb{D}^2$ given by $(x, \theta) \to (L(x), \theta), \theta \in \partial \mathbb{D}$. Let us denote this embedding of $M \setminus \pi_M^{-1}(\mathbb{D})$ by $\widetilde{\Psi}$. Finally, we define the required embedding of M by the formula:

$$\Psi(x) = \begin{cases} \widetilde{\Psi}(x), & \text{if } x \in M \setminus \pi_M^{-1}(\mathbb{D}); \\ (L(x), x) & \text{otherwise.} \end{cases}$$

Hence the theorem.

$\square$

5.1. **Proof of Theorem 1.4.**

For the sake of completeness, let us discuss how to prove Theorem 1.4 assuming Theorem 1.3.

Proof of Theorem 1.4. Let (M, ω) be a given symplectic manifold. It follows from Theorem 2.1 that after finite number of blow-ups there exists a (simplified) Lefschetz fibration on the blown-up manifold $\widetilde{M}$. It is clear that the manifold $\widetilde{M}$ satisfies the hypothesis of Theorem 1.4. Hence applying Theorem 1.4, we get the required embedding of $\mathcal{B}M$ in $(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{pr})$. $\square$

6. Proof of Theorem 1.5

.

In order to establish Theorem **??**, we need the notion of Weinstein Lefschetz fibration and the existence of Weinstein Lefschetz fibration. We will follow the treatment discussed in the article [15] by E. Giroux and J. Pardon. For this we need need the definition of Stein domains and associated Lefschetz fibration. Readers familiar with Stein domains and symplectic geometry associated with them can skip this introductory discussion.

6.1. **Stein domains and Stein Lefschetz fibrations.**

Let us begin by recalling the definition of a Stein domain.

Definition 6.1 (Stein domain)**.** A Stein domain is a compact complex manifold $(V, \partial V)$ with boundary together with a smooth Morse function ϕ which satisfies the following:
  (1) ϕ is J–convex, i.e., $-d(d\phi \circ J)(v, Jv) > 0$ for $v \neq 0$, where J is the almost complex structure associated to the complex structure on V.
  (2) ∂V is a regular level set for the function ϕ.

Next, we define Stein Lefschetz fibration.

Definition 6.2. Let V be a smooth manifolds with corners and let $\pi : V \to \mathbb{D}^2$ be Lefschetz fibration. We say that this Lefschetz fibration is a Stein Lefschetz fibration provided the following properties are satisfied:
  (1) There exists a complex structure on V such that the map π is holomorphic.
  (2) There exists a J–convex function $\phi : V \to \mathbb{R}$ with $\partial_h V = \{\phi = 0\}$.

The most important property of a Stein Lefschetz fibration on V is that V can be smoothened out to obtain a Stein domain which is unique up to a deformation in the following sense.

Let V_1 and V_2 denote two Stein domains obtained as a result of smoothing of a Stein Lefschetz fibration $\pi : V \to \mathbb{D}$, then there exits a 1–parametric family $V_t, t \in [1, 2]$ of Stein domains connecting V_1 and V_2. Hence, form now on by V^{sm} we will mean a Stein domain obtained after smoothing of the total space V of a Stein Lefschetz fibration $\pi : V \to \mathbb{D}^2$.

The converse of this was established by Giroux and Pardon [15, Theorem:1.5].

Theorem 6.3 (Theorem:1.5 [15])**.** *Let V be a Stein domain. There exists a Stein Lefschetz fibration $\pi : V' \to \mathbb{D}^2$ such that V'^{sm} is a deformation of V. Furthermore, we can always provide a fibration for which (1) the boundary of the fiber is connected, (2) every vanishing cycle is essential, and (3) the genus of any smooth fiber is at least 3.*

It follows from the definition that the 2–form $-d(d\phi \circ J)$ is a symplectic form on $(V, \partial V)$. Hence, Stein domains are natural examples of exact symplectic manifold. the symplectic geometry of Stein domains is naturally captured by the associated Weinstein domain structure. We now discuss this.

6.2. **Weinstein domains and Weinstein Lefschetz fibration.**

Definition 6.4 (Weinstein domain)**.** A Weinstein domain consists of is a compact exact symplectic manifold with boundary $(W, \omega = d\lambda)$, where λ is a 1-form, admitting a Morse function $\phi : W \to \mathbb{R}^+$ which satisfy the following:

(1) ∂W is a level set of a regular value of the function ϕ.
(2) There exists a vector-field X_λ defined as $i_{X_\lambda}\omega = \lambda$ which is gradient like for the Morse function ϕ.

Definition 6.5 (Weinstein Lefschetz fibrations)**.** Let (V, J) be a smooth manifolds with corners and let $\pi : V \to \mathbb{D}^2$ be Lefschetz fibration. We say that this Lefschetz fibration is a Weinstein Lefschetz fibration provided the following properties are satisfied:

(1) there exists a J–convex function $\phi : V \to \mathbb{R}$ such that $\partial_h V = \{\phi = 0\}$, where $\partial_h V = \cup_{p \in \mathbb{D}^2}\partial(\pi^{-1}|[p])$,
(2) If $p \in \mathbb{D}^2$ is a regular value then on $\pi^{-1}\{p\}$ the pair (ϕ, J) induces a Weinstein structure on $\pi^{-1}\{p\}$.

It follows from Theorem 6.3 and [15, Theorem 1.10] that every 4–dimensional Weinstein domain admits a Weinstein Lefschetz fibration as defined in Definition 6.5.

We now define the universal Weinstein manifold $\mathcal{U}$ mentioned in the statement of Theorem 1.5.

Definition 6.6 (Definition and basic properties of the universal Weinstein 3–manifold $\mathcal{U}$)**.**

(1) Let $\mathcal{L}(-3)$ denote the complex disk bundle of -3 over S^2. It is well known that this bundle admits a natural Stein structure. See [10] and [24] for a precise description of this Stein structure.
(2) Let $\mathbb{D}\mathcal{L}(-3)$ denote any disk bundle associated to the line bundle $\mathcal{L}(-3)$. Observe that $\mathbb{D}\mathcal{L}(-3)$ are Stein domains obtained by considering the inverse images under plurisubharmonic Morse functions of the type $(\infty, a]$ which have no critical value in the interval $[a, \infty)$.
(3) The product of a Stein manifold with $\mathbb{C}$ and the product of a Stein domain with the open unit disk is a Stein domain. In particular $\mathbb{D}\mathcal{L}(-3) \times \mathbb{D}^2$ is a Stein domain after suitable smoothing of corners, while $\mathcal{L}(-3) \times \mathbb{C}$ is Stein manifold.

The Weinstein manifold structure associated to the Stein manifold $\mathcal{L}(-3) \times \mathbb{C}$ with stein structure as discussed above will be called the universal Weinstein manifold $\mathcal{U}$.

Let us now state the result regarding embeddings of Weinstein domains via Lefschetz fibrations.

Theorem 6.7. *Let $\pi_V : V^4 \to \mathbb{D}^2$ be any (simplified) Weinstein fibration. There exists a symplectic Lefschetz fibration embedding of $\pi_V : V^4 \to \mathbb{D}^2$ in the trivial Weinstein fibration $\pi_2 : \mathbb{D}\mathcal{L}(-3) \times \mathbb{D}^2 \to \mathbb{D}^2$ such that $\partial_h V \subset \partial\mathbb{D}\mathcal{L}(-3) \times \mathbb{D}^2$ and $\partial_v V \subset \mathbb{D}\mathcal{L}(-3) \times \partial\mathbb{D}^2$. Furthermore, by choosing the radius of the disk bundle$\mathbb{D}\mathcal{L}(-3)$ with respect to an auxiliary metric correctly there exist an iso-symplectic Weinstein fibration embedding of V^4 in $\mathbb{D}\mathcal{L}(-3) \times \mathbb{D}^2$.*

We know that the mapping class group of a once punctured surface is generated by product of Dehn twists along the curves a_i, b_j, c_1 and c_2. For a technical reason –which will become clear while going through the proof of Theorem 6.7 – we need to show that every Dehn twist along a homologically essential simple closed curve on a surface with one boundary component is also isotopic to the product of Dehn twist along curves $a_1, b_1, \cdots, a_{g-1}, b_{g-1}, a_g, c_1$, and c_2 We establish this in the following:

Proposition 6.8. *Let $(\Sigma_g, \partial\Sigma_g)$ be a surface with one boundary component. Let ϕ be an element of the mapping class group that brings a non-separating simple closed curve c to a_i, for some i, then there exists an element $\widetilde{\psi}$ of the mapping class group of Σ_g can be expressed as product of Dehn twists $\tau_{a_i}, \tau_{a_i}^{-1}$ $\tau_{b_i}, \tau_{b_i}^{-1}$, $\tau_{c_1}, \tau_{c_1}^{-1}$, τ_{c_2}, and $\tau_{c_2}^{-1}$, and which satisfies the property that $\phi(c) = \widetilde{\phi}(c)$.*

Proof. We know that the mapping class group of a genus g surface with one boundary component is generated by Dehn twists $\tau_{a_i}, \tau_{a_i}^{-1}, \tau_{b_i}, \tau_{b_i}^{-1}, \tau_{c_1}, \tau_{c_1}^{-1}, \tau_{c_2}, \tau_{c_2}^{-1}, \tau_d$, and τ_d^{-1} Since d does not intersect with a_i, b_j, c_1 and c_2, we can assume that when expressing ϕ as a product of positive and negative Dehn twists along curves a_i, b_j, c_1, c_2 and d, positive and negative Dehn twists along d appear at the very beginning of the product. More precisely, ϕ can be expressed as:

$$\tau_d^k \cdot \tau_d^{-l} \cdot \psi,$$

where ψ is a product of positive and negative Dehn twists along curves a_i, b_j, c_1 and c_2.

Now define $\widetilde{\phi} = \tau_d^l \circ \tau_d^{-k} \circ \phi$. Since c can be assumed to be disjoint from d, we get that $\widetilde{\phi}(c) = \phi(c)$ as claimed.

$\square$

Proof of Theorem 6.7. We first discuss the case of symplectic embedding of Stein Lefschetz fibration. Consider the given (simplified) Stein Lefschetz fibration $\pi : V^4 \to \mathbb{D}^2$. Suppose the regular fiber is a genus $g, g \geq 3$, surface with one boundary component.

Next recall [24] that the Stein domain $\mathbb{D}\mathcal{L}(-3)$ admits a genus g Stein Lefschetz fibration $\pi_g : \mathbb{D}\mathcal{L}(-3) \to \mathbb{D}^2$ for every g having vanishing cycles $c_1, c_2, a_1, b_1, \cdots a_{g-1}, b_{g-1}, a_g$.

Fix an identification of a smooth fiber of $\pi : V \to \mathbb{D}^2$ with a fiber of $\pi_g : \mathbb{D}\mathcal{L}(-3)$. Let us denote by Σ the fiber of $\pi : V \to \mathbb{D}^2$ and f it's identification with a smooth fiber of π_g. Next, we claim that we can produce this identification such that a vanishing cycle on the fiber of $\pi : V \to \mathbb{D}$ is mapped to a_1 under this identification.

To begin with pick an identification $\widetilde{f} : \Sigma \to \pi_g^{-1}(p) \subset \mathbb{D}\mathcal{L}(-3)$ for some regular value p of π_g. Next observe that any vanishing cycle under this identification goes to an essential simple closed curve on the fiber of π_g as the fibration $\pi : V \to \mathbb{D}$ is simplified. Now, fix a vanishing cycle c on Σ and consider the essential cycle $\widetilde{f}(c)$ on $\pi_g^{-1}(p)$. Next we note, Proposition 6.8 implies that there exists a symplectomorphism $\phi : f(\Sigma) \to f(\Sigma)$ which send the vanishing cycle $f(c)$ to a_1 which is identify when restricted to $\partial f(\Sigma)$ is the identity. Clearly $\phi \circ \widetilde{f}$ is the required identification f.

Next let $c_1, c_2, \cdots, c_n$ be critical points, let $d_i = \pi(c_i)$ be critical values, and let $\nu_1, \nu_2, \cdots, \nu_n$ be the corresponding vanishing cycles of the fibration $\pi : V \to \mathbb{D}$. It is clear that by an argument similar to the one used in the proof of 1.3 we get for each $i, i = 1$ to n, an embedding ψ_i of $\pi^{-1}(\mathbb{D}_i)$, where D_i is a small disk containing the point d_i, in $\mathcal{D}(E)(-3) \times \mathbb{D}^2$ which satisfy the following:

(1) Whenever $i \neq j$ $\psi_i(\pi^{-1}(\mathbb{D}_i)) \cap psi_j(\pi^{-1}(\mathbb{D}_j)) = \emptyset$,
(2) the following diagram Commutes:

(2)
$$
\begin{array}{ccc}
\pi^{-1}(\mathbb{D}_i) & \xrightarrow{\psi_i} & \mathbb{D}\mathcal{L}(-3) \times \mathbb{D}_{p_i} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_2} \\
\mathbb{D}_i & \longrightarrow & \mathbb{D}.
\end{array}
$$

(3) The vanishing cycle ν_i is mapped to the vanishing cycle a_i in $\mathbb{D}\mathcal{L}(-3) \times \{t_i\}$ for $\nu_i \in \pi^{-1}(u_i)$, for some u_i in $\partial \mathbb{D}_i$.

Next, observe that embeddings ψ_1 and ψ_i for each $i, i = 2, \cdots, n$ gives rise to two embeddings of Σ in $\mathbb{D}\mathcal{L}(-3)$ via their restrictions to $\pi^{-1}(\{u_1\})$ and $\pi^{-1}(\{u_i\})$ respectively. Observe that the first embedding is such that the vanishing cycle ν_1 is mapped to a_1 while the second embedding is such that ν_i is mapped to a_1. Observe again that Proposition 6.8 and Lemma 4.6 imply that these two embeddings are isotopic via a family of symplectomorphims of $\mathbb{D}\mathcal{L}(-3)$ each of which is identity when restricted to $\partial\mathcal{D}E(-3)$.) This implies we can produce an embedding of $V \setminus \pi^{-1}(\mathcal{N})$, where $\mathcal{N}$ is a regular neighborhood of D_i union with $\overline{u_1 u_i}$ which is fiber preserving. Since $\mathcal{N}$ is a disk embedding in $\mathbb{D}$, extending the embedding restricted to $\pi^{-1}(\partial \mathcal{N})$ fiber wise via identity to V, we get the required embedding of V in $\mathbb{D}\mathcal{L}(-3) \times \mathbb{D}$.

It remains to show that we can upgrade the symplectic embedding of iso-symplectic embedding in a trivial Stein fibration of the form $\mathbb{D}\mathcal{L}(-3) \times \mathbb{D}^2$. In light of Theorem [Theorem:3.1][11] due to Gompf, it is sufficient to produce an iso-symplectic identification of fiber of the fiber of the fibration $\pi_V : V \to \mathbb{D}^2$ with the fiber of $\mathbb{D}\mathcal{L}(-3)$ for some disk bundle $\mathbb{D}\mathcal{L}(-3)$. Furthermore, notice that any two symplectic form on a compact surface having same volume are symplectomorphic implies that we need an identification such that the induced volume via this identification agrees with the given volume of a smooth fiber of the fibration

$\pi_V : V \to \mathbb{D}$. But by adjusting the size of the disk bundle with respect to a fixed auxiliary metric, this is always possible. Hence, we get the required iso-symplectic embedding.

$\square$

Proof of Theorem 1.5.

Given a Stein 2–manifold V admitting a plurisubharmonic exhaustive Morse function $v : V \to \mathbb{R}$ with only finite number of critical points, let $t_0 \in \mathbb{R}$ be such that there are no critical values that belong to the interval $[t_0, \infty)$. Observe that if we iso-symplectically and properly embed the Stein domain $V^{t_0} = v^{-1}(-\infty, t_0]$ in the Stein domain $\mathcal{U}^\beta$, then we can properly and iso-symplectcially embed V in $\mathcal{U}$. Hence our task is to establish this.

In order to establish this we first apply Theorem 6.3 to produce a Stein Lefschetz fibration of $\pi : V^{t_0} \to \mathbb{D}^2$ such that the fibers of the Stein fibration are connected. Next, we apply [22, Proposition:1.5] to get a new Stein Lefschetz fibration with connected boundary, connected fiber, and the fibration having every vanishing cycle essential. That is, we produce a simplified Lefschetz fibration on V^{t_0}.

Hence, by Theorem 6.7 there exits a symplectic Lefschetz fibration embedding of $\pi : V^{t_0} \to \mathbb{D}$ in the Stein trivial Stein Lefschetz fibration associated to $\mathcal{U}^\beta$ as $\mathcal{U}$ is just $\mathcal{L}(-3) \times \mathbb{C}$.

Finally, observe that applying the smoothing of corner operation on the Stein Lefschetz fibration associated to $\mathcal{U}^\beta$ – due to the facts that (1) the smoothing operation induces a smoothing operation on V, and (2) the smooth is operation is canonical up to deformation equivalence – we get the required symplectic embedding of V^{t_0} in $\mathcal{U}^\beta$.

$\square$

In the end, we would like to remark that if W is a Weinstein manifold such that W is obtained from the Stein domain $\mathcal{U}^\beta := f^{-1}((-\infty, \beta]) \subset \mathcal{U}$ by attaching finite number of Weinstein 2-handles along the contact boundary of $\mathcal{U}^\beta$ as described in [6], then W is also universal. This is because, for the dimension reasons, the core of any attached Weinstein handle can always be assumed to be disjoint form ∂V^{t_0} embedded in $\partial \mathcal{U}^\beta$.

REFERENCES

1. Amorós, Jaume; Muñoz, Vicente; Presas, Francisco, *Generic behavior of asymptotically holomorphic Lefschetz pencils.* J. Symplectic Geom. 2 (2004), no. 3, 377—392.
2. D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds,* Geom. Funct. Anal., Vol. 7, (1997) 971 – 995.
3. D. Auroux and L. Katzarkov, *A degree doubling formula for braid monodromy and Lefschetz pencil*, Pure and applied mathematics quarterly, Vol. 4, (2008) 237–318.
4. I. Baykur and S. Osamu, *Simplified broken Lefschetz fibrations and trisections of 4-manifolds,* Proc. Natl. Acad. Sci. USA 115, Vol. 43 (2018), 10894 – 10900.
5. F. Benson and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical series, Vol. 49, Princeton University Press, (2012).
6. K. Ceiliback and Y. Eliashberg, *From Stein to Weinstein and back*, Colloquium Publication, Vol .59. AMS.
7. M. Dehn, *Die Gruppe der Abbildungsklassen.* Acta Math., 69(1), (1938), 135–206.
8. S. K. Donaldson, *Symplectic submanifolds and almost–complex geometry.* Journal of Differential Geometry, Vol. 44, (1996), 666–705.
9. S. K. Donaldson, *Lefschetz pencils on symplectic manifolds.* Journal of Differential geometry, Vol. 55 (1999), 205–236.
10. J. Etnyre and Y. Lekili, *Embedding of all contact 3–manifolds in a fixed contact 5–manifold*, Journal of LMS, Vol. 99(1), (2018).
11. R. Gompf, *Topology of Symplectic manifolds*, Turk. Journal of Math., Vol. 25, (2001), 43–59.
12. M. Gromov, *Partial differential relations.* A series of modern surveys in mathematics. Springer. (1986).
13. D. Gay and R. Kirby, *Constructing Lefschetz type fibration on 4–manifolds*, Geometry and Topology, Vol. 11(4), (2005), 2075–2115.
14. A. Ghanwat and D. M. Pancholi,*Embeddings of 4–manifolds in $\mathbb{C}P^3$.* arXiv:2002.11299v2 [math.GT]
15. E. Giroux and J. Pardon, *Existence of Lefschetz fibrations on Stein and Weinstein domains*, Geometry and Topology, Vol. 21(2), 963–997.
16. R. Gompf and A. I. Stipsicz, *4–manifolds and Kirby calculus.* Graduate Studies in Mathematics, Vol. 20, American Mathematical Society, Providence, RI, (1999).
17. Humpries, *Generators of the mapping class group.* Topology of Low-dimensional manifolds, Proceedings of the Second Sussex Conference, (1977), 44–47.

18. S. Hirose and A. Yasuhara, *Surfaces in 4-manifolds and their mapping class groups.* Topology, Vol. 47(1) (2008), 41–50.

19. K. Kodaria, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Annals of Mathematics, Vol. 60 (1), 28–48.

20. W.B.R Lickorish, *A representation of orientable combinatorial 3–manifolds.* Ann. of Math., Vol. 76(2), (1962), 531–540.

21. W.B.R Lickorish, *A finite set of generators for the homeotopy group of a 2–manifold.* Proc. Cambridge Philos. Soc., Vol.60, (1964), 769–778.

22. A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of B^4*, Invent. Math. Vol.143 (2), (2001), 325–348.

23. J. Nash, The Imbedding Problem for Riemannian Manifolds, *Annals of Mathematics*, Vol. 63 (1), 20–63, (1956).

24. B. Ozbagci and A. I. Stipsicz, *Surgery on contact 3–manifolds and Stein surfaces,* Vol. 13 of Bolyai Society Mathematical Studies. Springer-Verlag, Berlin, 2004.

25. H. Whitney, *The self-intersections of a smooth n–manifold in 2n–space,* Ann. of Math, vol. 45(2), (1944), 200–246.

The Institute of Mathematical Sciences, No. 8, CIT Campus, Taramani,, Chennai 600113, India
Email address: `dishant@math.tifr.res.in`

ICMAT, Madrid, Spain
Email address: `fpresas@icmat.es`