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The place of an identity of Ramanujan in prime number theory

R. Balasubramanian and K. Ramachandra, F.A.Sc.

School of Mathematics, Tata Institute of Fundamental Research, Bombay 400005

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1. Introduction

The object of this note is to point out the place of the following formula (2) due to S. Ramanujan[1] (p. 135) in prime number theory. (Though our final results are not new there are some new lemmas which may be of independent interest in themselves). Ramanujan noticed that the innocent identity

$$\sum_{n=0}^{\infty} (1 + Z_1 + Z_1^2 + \ldots + Z_1^n) \quad (1 + Z_2 + Z_2^2 + \ldots + Z_2^n) q^n$$

$$= \frac{(1 - Z_1 Z_2 q^2)}{(1 - q)(1 - Z_1 q)(1 - Z_2 q)(1 - Z_1 Z_2 q)} \tag{1}$$

valid under obvious conditions could be used to establish

$$\sum_{n=1}^{\infty} \sigma_a(n)\, \sigma_b(n)\, n^{-s} = \frac{\zeta(s)\, \zeta(s-a)\, \zeta(s-b)\, \zeta(s-a-b)}{\zeta(2s-a-b)} \tag{2}$$

where a and b are complex numbers Res $>$ max $(1,\ 1 + Re\, a,\ 1 + Re\, b,\ 1 + Re(a+b))$, and $\sigma_a(n) = \sum_{d \mid n} d^a$ and a similar definition for $\sigma_b(n)$. It was A.E. Ingham[2] who first noticed that $\zeta(1 + it) \neq 0$ is a simple consequence of (2). He also generalised (2) to L series by using (1). This is a very simple matter but its consequence $L(1 + it, \chi) \neq 0$ pointed out by Ingham is really very striking. In the traditional notation these imply already $\pi(x) \sim x/\log x$ and for fixed k, l with $(k, l) = 1$, $\pi(x, k, l) \sim x/\phi(k) \log x$. However, Ingham's method of deduction of results like $\zeta(1 + it) \neq 0$ depended upon Landau's theorem on the singularity of Dirichlet series with positive coefficients and was not therefore capable of giving results like $|\zeta(1 + it)| > (\log(|t| + 3))^{-3}$ (This is clear from his remarks on lines 4–10 from the top on page 109 in his paper [2]). We are going to prove results of this kind by some now lemmas. In fact without using ideas like $3 + 4 \cos\theta + \cos 2\theta \geqslant 0$ we are going to prove

THEOREM 1. *Uniformly for* $1 \leqslant k \leqslant e^{A(\log x)^{1/12}}$ ($A > 0$ *arbitrary positive constant*) *and for* $(l, k) = 1$ *there holds,*

$$\vartheta(x, k, l) = \sum_{\substack{p \equiv l \ (\mathrm{mod}\ k) \\ p \leqslant x}} \log p = \frac{1}{\phi(k)} \left(x - \frac{x^{\beta}}{\beta} \chi(t) \right)$$

$$+ 0 \left(x e^{-CA^{-11} (\log x)^{1/12}} \right) \tag{3}$$

where C *is a positive constant independent of* A *and* β *is the maximum of the real zeros of all* L *functions to the modulus* k. *If* $\beta = 0$ *we just omit this term. Also*

$$\vartheta(x) = \sum_{p \leqslant x} \log p = x + 0 \left(x e^{-C(\log x)^{1/6}} \right) \tag{4}$$

In (3) χ *is the character corresponding to the* L *series of which* β *is a zero.*

Remark.—From this we can pass on to $\pi(x, k, l)$ and $\pi(x)$ and we leave these to the reader.

2. RAMANUJAN'S FORMULA FOR L-SERIES

Let a and b be complex numbers, k a positive integer, and $Re\ s > \max(1, 1 + Re\ a, 1 + Re\ b, 1 + Re(a + b))$. Further for any character χ mod k let $\sigma_{a,\chi}(n) = \sum_{d \mid n} \chi(d) d^a$ and a similar meaning for $\sigma_{b,\chi}(n)$. Then we have by a trivial application of (1),

THEOREM 2. *If* χ_1 *and* χ_2 *are two characters mod* k, *then,*

$$\sum_{n=1}^{\infty} \frac{\sigma_{a,\chi_1}(n)\, \sigma_{b,\chi_2}(n)}{n^s}$$

$$= \frac{\zeta(s)\, L(s - a, \chi_1)\, L(s - b, \chi_2)\, L(s - a - b, \chi_1\chi_2)}{L(2s - a - b, \chi_1\chi_2)}$$

In particular when $\chi_1 = \chi = \bar{\chi}_2$ *and* $b = \bar{a}$ *we have*

$$\sum_{n=1}^{\infty} \frac{|\sigma_{a\chi}(n)|^2}{n^s}$$

$$= \frac{\zeta(s)\, \zeta(s - a - \bar{a})\, L(s - a, \chi)\, L(s - \bar{a}, \chi)}{\zeta(2s - a - \bar{a})} \prod_{p \mid k} \frac{1 - p^{-s + a + \bar{a}}}{1 - p^{-2s + a + \bar{a}}}$$

Thus if the real numbers a_n *are defined by*

$$F(s) = \zeta(s)\,\zeta(s-a-\bar{a})\,L(s-a,\chi)\,L(s-\bar{a},\bar{\chi}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

then

$$a_n \geqslant |\sigma_{a,\chi}(n)|^2, \quad \textit{where } \sigma_{a,\chi}(n) = \sum_{d|n} \chi(d)\, d^a$$

Also if a_n' *are defined by* $\zeta(s)\,F(s) = \sum_{n=1}^{\infty} a_n' n^{-s}$ *then* $a_n' \geqslant 1$ *for all n.*

3. PROOF FOR $L(1+it,\chi) \neq 0$.

A quick proof that $L(1+it,\chi) \neq 0$. *From now on we assume* $0 \leqslant a + \bar{a} \leqslant 1$. One can prove that the right hand side of

$$\sum_{n=1}^{\infty} \frac{n^2 a_n e^{-n^2/X^2}}{X^2 n^{a+\bar{a}}} = \frac{1}{2\pi i} \int_{Re\, W=4} F(a+\bar{a}+2W-2)\, X^{2W-2}\, \Gamma(W)\, dW \tag{5}$$

is the sum of the residues of the integrand on the right at its poles with an error $0(e^{-X^{(1/10)}})$ provided that $N = (X^{(1/10)}) \geqslant k(|a|+30)$. This can be done by moving the line of integration to the line $Re\, W = -N - \frac{1}{2}$ and using the estimate[3] (p. 340).

$$L(Z,\chi) = 0\left(\left|\frac{kZ}{2\pi}\right|^{k/2 - Re\, Z}\right) \tag{6}$$

valid uniformly for $Re\, Z \leqslant -10$.

[Note that Γ function is not essential for our main purposes but we use it only for convenience. Also error terms like $0(e^{-X^{(1/10)}})$ are given only for curiosity and that functional equations, etc., are dispensible]. The possible poles of the integrand are $W = 3/2$ $(W = 3 - a - \bar{a})/2$, $W = (3-\bar{a})/2$ and $W = (3-\bar{a})/2$ and $W = (3 \sim a)/2$ $W = (3-\bar{a})/2$ $W = 3 - \bar{a}/2$. The last two poles do not exist when χ is non-principal. The sum of the residues is (assuming $a = i\alpha$, α real):

$$\begin{aligned} &\frac{1}{4}\frac{d}{dW}\Big(L(\bar{a}+2W-2,\bar{\chi})\,L(a+2W-2,\chi)\; X^{2W-2}\,\Gamma(W)\Big)_{W=3/2} \\ &+ \gamma L(1+\bar{a},\chi)\,L(1+a,\bar{\chi})\, x\,\Gamma(3/2) + \tfrac{1}{2}\delta_\chi\,(\zeta(1+\bar{a})\,\zeta(1-a) \\ &\quad L(1+\bar{a}-a,\chi)\, X^{1-a}\,\Gamma((3-a/2) + \zeta(1+a)\,\zeta(1-\bar{a}) \\ &\quad L(1+a-\bar{a},\bar{\chi})\, X^{1-\bar{a}}\,\Gamma(3-\bar{a}/2)) \end{aligned} \tag{7}$$

where $\delta_\chi = 1$ *if* χ *is principal and zero otherwise.* (*when* $\delta_\chi = 1$ *it is assumed that* $a \neq 0$). Here γ is the Euler's constant.

From these remarks it follows immediately that

$$L(1+it,\ \chi) \neq 0 \tag{8}$$

for all real t and all characters χ. It is apparent now that in order to obtain lower bounds for $L(1+it, \chi)$ we should have 'good' lower bounds for $\sum_{X \leqslant n \leqslant 2X} a_n$ or (what is roughly the same) for

$$S = S(\alpha,\ \chi,\ X) = \sum_{X \leqslant p \leqslant 2X} |1 + \chi(p) p^{i\alpha}|^2 \tag{9}$$

and this we propose to solve in a somewhat satisfactory way.

Before leaving this section we note down two curiosities as a

Remark.---Let $1-\rho$ be a zero of $L(s, \chi)$. Then for $X \geqslant \left(k(|a|+30)\right)^{10}$, we have by putting $a = \rho$ (Note that $0 < \rho + \bar{\rho} \leqslant 1$)

$$\frac{1}{X^2}\sum_{n=1}^{\infty} n^{2-\rho-\bar{\rho}}\, a_n e^{-n^2/x^2}$$

$$= \tfrac{1}{2}\,\zeta(1+\rho+\bar{\rho})\,|L(1+\bar{\rho},\ \chi)|^2\,\Gamma(3/2)X + 0\left(\exp(-X^{1/10})\right) \tag{10}$$

and

$$\sum_{n=1}^{\infty} n^{-\rho-\bar{\rho}}\, a_n\, e^{-n^2/x^2}$$

$$= \tfrac{1}{2}\,\zeta(1+\rho+\bar{\rho})\,|L(1+\bar{\rho}, \chi)|^2\,\Gamma(\tfrac{1}{2})\,X + 0\left(\exp.(-X^{1/10})\right) \tag{11}$$

4. Some Remarks

Starting with

$$\frac{1}{X^2}\sum_{n=1}^{\infty} a'_n\, n^2 e^{-\left(\frac{n}{x}\right)^2}$$

$$= \frac{1}{2\pi i}\int \zeta(W)\,F(2W-2)\,X^{2W-2}\,\Gamma(W)\,dw\ (a = i\alpha,\ \alpha \text{ real}) \tag{12}$$

($|\alpha| \geqslant |$ in case χ is principal) and proceeding as before we can prove that either $|L(1+i\alpha,\ \chi)| \gg \left(\log[k(|\alpha|+3)]\right)^{-3}$ or $|L'(1+i\alpha,\ \chi)| \gg 1$ (the implied constants being absolute). To prove this we have only to note that $a_n' \geqslant 1$ for all n. Since always $L'(1 + i\alpha, \chi) = 0(\{\log [k(|\alpha| + 3)]\}^2)$ we have, in case the first alternative holds, a zero free region of the type

$$\sigma \geqslant 1 - \frac{C_1}{(\log [k(|t| + 3)])^5}$$

($C_1 > 0$ is an absolute constant) for $L(s, \chi)$. To meet the second case we prove

LEMMA 1.—*Let* $|L'(1 + i\alpha, \chi)| \geqslant C_2 > 0$ (*where* $C_2 > 0$ *is an absolute* constant) *and let* $s_0 = 1 + i\alpha$. *Then in* $|s - s_0| \leqslant d$ *where* $d \gg (\log [k(|\alpha| + 3)])^{-3}$ *there is at most one zero of* $L(s, \chi)$ *and if it exists it must be a simple zero.*

Proof.—Let $L(s, \chi) = b_0 + b_2(s - s_0) + b_2(s - s_0)^2 + \ldots$ be the Taylor expansion about $s = s_0$, where for $n \geqslant 0$,

$$b_n = \frac{1}{2\pi i} \int \frac{L(s)}{(s - s_0)^{n+1}} ds$$

(the integration path being the circle $|s - s_0| = (\{\log [k(|\alpha| + 3)]\}^{-1})$ and so $b_n = 0(\{\log [k(|\alpha| + 3)]\}^{n+1})$ uniformly in all the parameters. Let s_1 and s_2 be two zeros of $L(s, \chi)$ in $|s - s_0| \leqslant d_1$. If $s_1 = s_2$ (*i.e.*, s_1 is a double zero then we differentiate the Taylor expansion and then put $s = s_1$.

In the other case we put $s = s_1$ and $s = s_2$ and subtract the resulting equalities. We then cancel out the factor $s_1 - s_2$ from each term and get

$$0 = b_1 + \sum_{n=2}^{\infty} 0(b_n (nd_1^{\,n-1}))$$

$$= b_1 + 0(\{\log [k(|\alpha| + 3)]\}^3 d_1 \sum_{n=2}^{\infty} n \{d_1 \log [k(|\alpha| + 3)]\}^{n-2}.)$$

This proves the required result.

5. SOME LEMMAS OF BALASUBRAMANIAN

LEMMA 2.—*Let* $k = 1$ *and* $|\alpha| \gg 1$, $X \geqslant (|\alpha| + C_3)^2$ ($C_3 > 0$ *an absolute* constant). *Let* $S_1 = \sum_{X \leqslant p \leqslant 2X} |1 + p^{i\alpha}|^2$ (*a special case of* (9)).

Then

$$S_1 \gg \frac{X}{\log X}.$$

REMARKS.—In $0 \leqslant |\alpha| \leqslant 1$ it is true that $S_1 \gg \frac{X}{(\log X)^3}$ and this is best possible. As a corollary to Lemma 2 we can deduce easily [using (5) and (7)] that $|\zeta(1+it)| \gg (\log(|t|+3))^{-3}$.

Proof.—Let $\alpha > 0$. The angle $\alpha \log p$ changes in the interval of summation by at most $\alpha \log 2$ so that the complex number $p^{i\alpha}$ makes

$\leqslant R_1 = \left[\frac{\alpha \log 2}{2\pi}\right] + 1$ revolutions on the unit circle. In each revolution

let us agree to omit N_1 primes corresponding to which $\alpha \log p$ is close to $\pi \pmod{2\pi}$ on either side of it by an angle at most equal to θ radians where

$0 < \theta < \frac{\pi}{4}$. Plainly when $\alpha \log p$ does not lie in this angle,

$|1 + p^{i\alpha}|^2 \geqslant 2\left|\sin\frac{\theta}{2}\right|^2 \geqslant \frac{4\theta^2}{\pi^2}$. Consider now the number M_1 of integers n

for which $\alpha \log n$ lies in this angle (during one revolution). It consists of at most $M_1' + 1$ consecutive integers where M'_1 is the smallest positive integer with

$$\frac{\alpha M_1'}{2X} \geqslant \theta \text{ i.e., } M_1 \leqslant 2M_1' + 2 = 2\left[\frac{2X\theta}{\alpha}\right] + 4.$$

Assuming now that $\left(0 < \theta \leqslant \min\left(\frac{\alpha}{8}, \frac{\pi}{8}\right)\right.$ and using the theorem $\pi(x+h)$

$-\pi(x) = 0\left(\frac{h+3}{\log(h+3)}\right)$ valid uniformly (in $x \geqslant 1$, $h \geqslant 0$[4] (p. 44) or a

more convenient reference is the recent book of Halberstam and Richert[5] Actually in the latter book we find a sharpening of Brun-Titchmarsh Theorem (p. 107) in the form

$$\pi(x, k, l) - \pi(x-y, k, l) \leqslant \frac{y}{\phi(k) \log\sqrt{\frac{y}{k}}}\left(1 + \frac{4}{\log\sqrt{\frac{y}{k}}}\right) \cdots$$

valid in $1 \leqslant k < y \leqslant x$, $(k, l) = 1$. This can be proved by the Sieve method of Selberg we have

$$N_1 = 0\left\{\left(\frac{X\theta}{\alpha} + 1\right)\left[\log\left(\frac{4X\theta}{\alpha} + 10\right)\right]^{-1}\right\}$$

These remarks in combination with the well known inequality $\pi(2X) - \pi(X) \geqslant$

$\frac{X}{8 \log X}$ $(X \geqslant 10000)$ due to Chebyshev[1] (p. 209) lead to the lemma.

by a proper choice of θ.

LEMMA 3. *Let* $\alpha \neq 0$, $0 < \theta \leqslant \min\left(\frac{\pi}{8}, \frac{|\alpha|}{8}\right)$, $X \geqslant 10000\,(k\,|\,\alpha\,| + k)^2$.

Then the sum S *defined by* (9) *satisfies*

$$S \geqslant \frac{\theta^2}{400}\left\{\frac{X}{8\log X} - \phi(k) - \frac{C_4\left(\theta X + \frac{\theta X}{|\alpha|} + k(|\alpha| + 1)\right)}{\log\left(\frac{\theta X}{k\,|\alpha|} + 3\right)}\right\}$$

where C_4 *is an absolute positive constant.*

COROLLARY. *For* $|t| > k^{-20}$ *we have*

$$|L(1 + it, \chi)| \gg \min(1, t^2)\,\{\log[k(|t| + 3)]\}^{-3}.$$

Proof.—The number of revolutions $R \leqslant \alpha + 1$ (we have assumed $\alpha > 0$ without loss of generality). We now consider one revolution r and denote the total number of primes in this revolution by $\pi_r(X)$. Let us now fix an arithmetic progression $l \pmod k$, $(l, k) = 1$, $1 \leqslant l \leqslant k$. Let $\pi_r(X, k, l)$ denote the total number of primes of this arithmetic progression. We estimate the number N^* of these primes which lie in an angle 2θ (θ on either side of a certain angle corresponding to $(-\bar{\chi}(l))$. Very roughly the integers M_2 and M_2' corresponding to M_1 and M_1' have to satisfy

$$\frac{k\alpha M_2'}{2X} > \theta,\ M_2 \leqslant 2(M_2' + 1) \leqslant 2\left[\frac{2X\theta}{k\alpha}\right] + 4.$$

Imposing $0 < \theta < \min\left(\frac{\pi}{8}, \frac{\alpha}{8}\right)$ and using the estimate for $\pi(x, k, l) - \pi(x - y, k, l)$ quoted above we find that

$$N^* = 0\left\{\left(\frac{\theta X}{\alpha} + k\right)\left[\phi(k)\log\left(\frac{X\theta}{k\alpha} + 3\right)\right]^{-1}\right\},$$

and so

$$S' = \sum_{p,\ |1 + \chi(p)p^{i\alpha}| \geqslant \frac{\theta}{20}} 1$$

$$\geqslant \sum_r \sum_l \left\{\pi_r(X, k, l) - \frac{C_4}{\phi(k)}\left(\frac{\theta X}{\alpha} + k\right)\left[\log\left(\frac{\theta X}{k\alpha} + 3\right)\right]^{-1}\right\}$$

$$\geqslant \frac{X}{8\log X} - \phi(k) - C_4(\alpha + 1)\left(\frac{\theta X}{a} + k\right)\left[\log\left(\frac{\theta X}{k\alpha} + 3\right)\right]^{-1}$$

and this proves the lemma.

LEMMA 4. *Let now* $a = i\alpha$, $0 \leqslant |\alpha| \leqslant 1$ *and* χ *a character which takes at least one value which is not real. Let* S_2 *denote this special case of the sum* (9). *Then if* X *exceeds a certain large (but fixed) power of* 2k, *we have*

$$S_2 \gg \frac{X}{\log X}.$$

COROLLARY. *If* χ *is a character which takes at least one value which is not real then*

$$|L(1+it, \chi)| \gg \{\log [k(|t|+3)]\}^{-3}$$

Proof.—We can assume that α is positive and bounded above by any small positive constant. The values $\chi(p)$ for various $p > k$ form a cyclic group of order d for some d/k. That is, its values are $e^{2\pi im/d}$. The angle $\alpha \log p$ varies by at most $\phi = \alpha \log 2$ radians which is small if α is small. We now consider $1 + e^{2\pi im/d}\, p^{i\alpha}$. We avoid those m for which $e^{2\pi im/d}\, p^{i\alpha}$ is approximately-1, the approximation error being not more than θ. The number of m's to be avoided is given by $\left|\frac{m_1}{d} - \frac{m_2}{d}\right| \leqslant 2\theta + \phi$ and so not more than $(2\theta + \theta)\, d + 1$. The number of arithmetic progressions corresponding to this many m's is at most $\big((2\theta + \phi)\, d + 1\big)\, \phi(k)\, d^{-1}$ and so

$$S_2 \gg \theta^2 \left(\pi(2X) - \pi(X) - \left(2\theta + \phi + \frac{1}{d}\right) \frac{(2+\epsilon)\,(X + 10^8 k^2)}{\log (X + 10^8 k^2)}\right)$$

for all $k \geqslant k_0$ (k_0 depending only on ϵ). This proves the lemma since θ and ϕ are arbitrary and $d \geqslant 3$.

6. APPLICATION TO PRIME NUMBER THEORY

The results of sections 3, 4 and 5 can be summarized as:

LEMMA 5. *Let* $k \geqslant 1$ *be a positive integer. Then the region*

$$\sigma \geqslant 1 - \frac{C_5}{(\log (k|t| + 3k))^5}, \quad |t| \geqslant 1$$

is free from the zeros of $L(s, \chi)$ *for all characters* χ *mod k, where* C_5 *is an absolute positive constant. The region*

$$\sigma \geqslant 1 - \frac{C_5}{(\log (k|t| + 3k))^5}, \quad |t| < 1$$

may contain zeros ρ_χ *of* L (s, χ). *But whenever this happens* χ *is a real character mod* k and ρ_χ *are all simple zeros of* L (s, χ). *Also in the region*

$$\sigma \geqslant 1 - \frac{C_5}{(\log (k \mid t \mid + 3k))^5}$$

we have

$$-\frac{L'(s, \chi_0)}{L(s, \chi_0)} - \frac{1}{s-1} = 0\left([\log (k \mid t \mid + 3k)]^{50}\right)$$

where χ_0 *is the principal character and for each real non-principal character* χ

$$-\frac{L'(s, \chi)}{L(s, \chi)} + \sum_{\rho_\chi} \frac{1}{s-\rho_\chi} = 0\left([\log (k \mid t \mid + 3k)]^{50}\right).$$

Proof.—We have only to prove the 0-estimates. These follow by applying first maximum modulus principle and then Borel-Caratheodory theorem (*see* pages 174–175 of Titchmarsh's book[7]).

From Lemma 5 follows (for the usual method of deduction *see* pages 53–54 of Titchmarsh's book[8], Prachar's book[4] pages 60–62).

LEMMA 6. *We have uniformly for* $1 \leqslant k \leqslant e^{A (\log x)^{1/6}}$, (k, l) = 1,

$$\vartheta (x, k, l) = \frac{1}{\phi(k)}\left(x - \sum_{\chi} \chi(l) \sum_{\rho_\chi} \frac{x^{\rho_\chi}}{\rho_\chi}\right) + 0\left(xe^{-CA^{-5} (\log x)^{1/6}}\right),$$

where the sum over χ *is over all the real characters and the sum over* ρ_χ *is over the possible simple zeros of the corresponding* L*-series in the region*

$$\sigma \geqslant 1 - \frac{C_5}{(\log (k \mid t \mid + 3k))^5}, \mid t \mid \leqslant 1.$$

7. FINAL DEDUCTION OF THEOREM 1

The second part of the theorem follows easily from Lemma 6. In[5] it was proved that if $3 \leqslant k_1 \leqslant k_2$ and χ_2 and χ_1 are two non-principal real characters mod k_1 and k_2 such that the character $\chi_1\chi_2$ mod k_1k_2 is again non-principal and $L(1, \chi_1) < C_6 (\log k_1)^{-1}$, then

$$L(1, \chi_2) > \frac{C_7 \log k_1}{(\log k_2)^2} \exp\left(-C_8 \frac{\log k_2}{\log k_1}\right)$$

(C_6, C_7, C_8 are some effective positive constants). From this and corollary to Lemma 3 it follows that the region

$$\sigma \geqslant 1 - \frac{C_9}{(\log(k|t| + 3k))^{11}}, \ |t| \leqslant 1$$

contains at most one zero of $\Pi L(s, \chi)$ (the product being over all characters χ mod k) and this zero if it exists lies on the real axis and is a simple zero of this function. From this remark follows Theorem 1. (It must be mentioned that all the O-constants and C, $C_1, \ldots, C_9$ are effective positive constants). Finally we remark that by the usual sophistications it is possible to get the most up to date zero free regions by Ramanujan's identities.

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