ON AN ASYMPTOTIC CHARACTERISATION OF GRIFFITHS SEMIPOSITIVITY

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Abstract. We prove that certain possibly non-smooth Hermitian metrics are Griffiths-semipositively curved if and only if they satisfy an asymptotic extension property. This result answers a question of Deng–Ning–Wang–Zhou in the affirmative.

1. Introduction

The positivity of the curvature Θ of the Chern connection of a Hermitian holomorphic vector bundle (E, h) over a complex manifold M plays an important role in algebraic geometry through extension problems. For a line bundle, there is only one notion of positivity, namely, the curvature being a Kähler form. For a vector bundle, there are several competing inequivalent notions, of which the most natural are *Griffiths positivity* ($\langle v, \sqrt{-1}\Theta v\rangle$ is a Kähler form) and *Nakano positivity* (the bilinear form defined by $\sqrt{-1}\Theta$ on $T^{1,0}M \otimes E$ is positive-definite). A famous conjecture of Griffiths [14] asks whether *ample vector bundles* ($\mathcal{O}_E(1)$ over $\mathbb{P}(E)$ admits a positively-curved metric) admit Griffiths-positively curved metrics. The conjecture is still open. However, a considerable amount of work has been done to provide evidence in its favour [1, 4, 7, 9, 13, 16, 22, 23, 25, 29].

In the Kähler case, Demailly–Paun [8] proved a Nakai–Moizeshon–type criterion to characterise Kähler classes. Despite the assumptions and conclusions involving smooth objects, their proof used singular objects like positive currents and singular Kähler potentials. A similar phenomenon might play a role in the study of the Griffiths conjecture and hence it is fruitful to study singular Hermitian metrics on vector bundles. This topic has also been well-studied [5, 28, 3, 2, 17, 24, 27, 20] and seems to hold some surprises. For instance, even if a bundle is Griffiths-positively curved (in a certain sense), the curvature may not exist as a current (Theorem 1.3 in [28]).

In the quest for alternate characterisations of these notions of positivity, and defining similar notions for singular metrics, the following definition [11] involving the asymptotics of L^2-extension constants has proved to be a useful measure of positivity [12, 10, 11, 18, 19].

Definition 1.1. Let E be a holomorphic vector bundle over an n-dimensional complex manifold X. A singular Hermitian metric h is said to satisfy the multiple coarse L^2-extension property if the following hold.

(1) For every open subset $D \subset X$ and every holomorphic section $s : D \to E^*|_D$ that is not identically zero, the function $\ln ||s||^2_{h^*}$ is upper-semicontinuous.
(2) Consider any cover of M by relatively compact Stein trivialising coordinate neighbourhoods of the form $(\Omega'' \subset M, z, \{e_i\})$ and a subcover of Stein open subsets $\Omega' \Subset \Omega \Subset \Omega''$. Then, for every integer $m \geq 1$ there exists a constant C_m satisfying the following conditions:
   (a) Subexponential growth: $\lim_{m\to\infty} \frac{\ln C_m}{m} = 0$.
   (b) Controlled extension: If $p \in \Omega'$ and $a \in E_p$ with $||a||_h < \infty$, for every integer $m \geq 1$, there exists a holomorphic extension $f_m : \Omega \to E^{\otimes m}$ of $a^{\otimes m}$ (that is, $f_m(p) = a^{\otimes m}$) satisfying

$$\int_{\Omega'} ||f_m||^2_{h^{\otimes m}} \frac{(\sqrt{-1}\partial\bar{\partial}|z|^2)^n}{n!} \leq C_m ||a||^{2m}_{h(p)}. \tag{1.1}$$

We note that in the definition above, $a^{\otimes m} \in S^m E_p$. It turns out (Lemma 2.2) that the metric on $S^m E$ induced from E is actually the same as the metric induced by an orthogonal projection from $E^{\otimes m}$. This observation motivates the following generalisation of Definition 1.1 : A singular Hermitian metric h is said to satisfy the *general* multiple coarse L^2-extension property if it satisfies Definition 1.1 with $f_m(p)$ being *any* given element of $S^m E_p$, i.e., if $b_m \in S^m E_p$ then there exists a holomorphic section $f_m : \Omega \to S^m E$ such that $f_m(p) = b_m$ and

$$\int_{\Omega'} ||f_m||^2_{h^{\otimes m}} \frac{(\sqrt{-1}\partial\bar{\partial}|z|^2)^n}{n!} \leq C_m ||b_m||^2_{S^m h(p)}, \tag{1.2}$$

where C_m satisfies subexponential growth.

For the remainder of this paper, unless specified otherwise, an integral over a coordinate chart is understood to be an integral with the Euclidean volume form, similar to (1.1).

Remark 1.2. The definition given in [11] differs slightly from Definition 1.1 in two (minor) aspects. Firstly, the definition in [11] is for general Finsler metrics. Secondly, the controlled extension property in our definition requires control over the extension (to a set Ω) on a smaller set Ω'.

We recall that a singular Hermitian metric h is said to be Griffiths-semipositively curved if whenever u is a local holomorphic section of E^*, the function $|u|^2_{h^*}$ is a plurisubharmonic (psh) function [28]. (It turns out that this definition is equivalent to $\ln |u|^2_{h^*}$ being psh.) In [12] (Theorem 6.4), it was proved that multiple coarse L^2-extension implies Griffiths semipositivity. Since the proof is local, it is easily seen to apply even to our definition. A question was raised as to whether it completely characterises Griffiths semipositivity. We answer that question in the affirmative. Slightly more strongly:

Theorem 1.3. *Let h be a singular Griffiths semipositively curved Hermitian metric on a holomorphic vector bundle E over a complex Hermitian manifold (X, ω). Let h_0 be a fixed* *smooth background metric on E. If* $\ln\det(hh_0^{-1})$ *is bounded on compact sets, then* (E, h) *satisfies the general multiple coarse* L^2*-extension property. Moreover, one can choose a uniform extension constant* $C = C_m$ *that is independent of* m*.*

Remark 1.4. If h is continuous, it trivially meets the requirements of Theorem 1.3. If instead, $0 \leq \sqrt{-1}\bar{\partial}\partial \ln(\det(h)) \leq C\omega$, then $\ln(\det(h))$ and $-\ln(\det(h))$ are quasi-psh and hence satisfy the hypotheses of Theorem 1.3.

Remark 1.5. The main point of Theorem 1.3 is that, while typically Nakano semipositivity produces extension theorems, this theorem merely needs Griffiths semipositivity.

It is interesting to know whether our result can be improved to general singular Hermitian metrics. However, seeing that $\det(h)$ seems to play an important role, we are pessimistic about such a result. There are other positivity notions for vector bundles like MA-positivity, for instance [26]. It might be fruitful to explore a similar extension/estimate-type characterisation for such notions as well.

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2. Proof

We first prove Theorem 1.3 for smooth Hermitian metrics.

Proposition 2.1. *Let* h_0 *be a smooth background metric. If* h *is a smooth Hermitian metric, then Griffiths semipositivity implies the general multiple coarse extension property. Moreover, if* h_ν *is a family of such smooth Hermitian metrics and there exists a constant* K *such that* $\frac{1}{K}\det(h_0) \leq \det(h_\nu) \leq K\det(h_0)$*, then the extension constants* C_m *depend only on* K, h_0*, and the chosen coordinate trivialising charts.*

Proof. Consider a point $p \in \Omega' \Subset \Omega \Subset \Omega''$ and $b_m \in S^m E_p$. We want to extend b_m to a holomorphic section in Ω with L^2-estimates in Ω'. Let $\epsilon_m > 0$ be a sequence of real numbers. The new metric $\tilde{h}_{\epsilon_m} := he^{-\epsilon_m |z|^2}$ is strictly Griffiths-positive on Ω.

We recall that the symmetric group S_m acts on $E^{\otimes m}$ which decomposes into irreducible representations. The metric $\tilde{h}_{\epsilon_m}$ induces an S_m-invariant metric on $E^{\otimes m}$. The fixed point set is $S^m E$ and hence we decompose $E^{\otimes m} = S^m E \oplus V$. Note that V is stable under the action of S_m.

Lemma 2.2. *The subbundle* $S^m E$ *is orthogonal in the induced metric to* V*.*

Proof. Indeed, suppose $x \in S^m E_q$, $y \in V_q$ (for some q). Then $m!\langle x, y\rangle = \sum_{g\in S_m}\langle g\cdot x, y\rangle = \sum_{g\in S_m}\langle x, g^{-1}\cdot y\rangle = \langle x, y_0\rangle$ where $y_0 = \sum_{g\in S_m} g^{-1}\cdot y$ is in V_q as well as in the fixed-point set $S^m E_q$. Hence, $y_0 = 0$ and so is $\langle x, y\rangle$. □

Endow Ω with the Euclidean metric. Since E is trivial over Ω, we pretend that $\det(E)$ is a trivial bundle. Let r be the rank of E. At this point, suppose h_ν is a family of smooth Hermitian metrics as in the statement of the proposition, and let $\epsilon_m = \frac{1}{m}$. We drop the subscript ν for the remainder of the proof.

A result of Demailly–Skoda [9] shows that if (E, h) is Griffiths-positively curved, then $E \otimes \det(E)$ with the induced metric is Nakano-positively curved. This result was generalised in Theorem 7.2 of [21] which states that the induced metric on $S^mE\otimes\det(E)$ is Nakano-positively curved for all $m \geq 1$.

By the Ohsawa–Takegoshi theorem for vector bundles, there exists a universal constant C (whose optimal value can be computed [15]) and an extension f_m of b_m such that

$$\int_\Omega \|f_m\|^2_{S^m\tilde{h}_{\epsilon_m}} \det(\tilde{h}_{\epsilon_m}) \leq C\|b_m\|^2_{S^m\tilde{h}_{\epsilon_m}(p)} \det(\tilde{h}_{\epsilon_m}). \tag{2.1}$$

By Lemma 2.2, $\|b\|_{S^mh} = \|b\|_{h^{\otimes m}}$ if $b \in S^mE$. Note that

$$\|f_m\|^2_{S^m\tilde{h}_{\epsilon_m}} \det(\tilde{h}_{\epsilon_m}) = \|f_m\|^2_{S^mh} e^{-m\epsilon_m|z|^2} \det(h) e^{-r\epsilon_m|z|^2} = \|f_m\|^2_{S^mh} e^{-\left(1+\frac{r}{m}\right)|z|^2} \det(h).$$

Rewriting (2.1),

$$\begin{aligned}\frac{\inf_{\overline{\Omega}} \det(h_0) e^{-(r+1)|z|^2}}{K} \int_\Omega \|f_m\|^2_{S^mh} &\leq \int_\Omega \|f_m\|^2_{S^mh} e^{-\left(1+\frac{r}{m}\right)|z|^2} \det(h) \\ &\leq C\|b_m\|^2_{S^mh(p)} e^{-\left(1+\frac{r}{m}\right)|z(p)|^2} K \sup_{\overline{\Omega}} \det(h_0).\end{aligned}$$

Hence, we are done. □

Now we prove Proposition 2.1 in greater generality. Let $p \in \Omega'$ and $b_m \in S^mE_p$. Proposition 6.2 of [28] implies that the duals of the convolutions of the dual metric with mollifiers ρ_ν, i.e., $h_\nu = ((h^*) * \rho_\nu)^*$ of h (where $0 < \nu \leq 1$), increase to h pointwise as $\nu \to 0^+$ and are Griffiths-semipositively curved. Choose $\nu \leq \nu_0$ to be small enough that the convolutions are defined on $\Omega_{2\delta}$, where $\Omega_{2\delta} \Subset \Omega''$ is a 2δ-neighbourhood of Ω for some fixed small $\delta > 0$. By Theorem 7.2 of [21], $S^mE \otimes \det(E)$ equipped with the metric induced from h_ν is Nakano-semipositively curved.

Since $0 < \nu \leq \nu_0$, by the monotonicity of h_ν, we see that $\frac{1}{L}\ln(\det(h_0)) \leq \ln(\det(h_{\nu_0})) \leq \ln(\det(h_\nu)) \leq \ln(\det(h)) \leq L\ln(\det(h_0))$ for some constant L independent of ν. Hence we may use Proposition 2.1 to conclude that there exist extensions $f_{m,\nu}$ of b_m on $\Omega_{2\delta}$ such that

$$\int_{\Omega_\delta} \|f_{m,\nu}\|^2_{h_{\nu_0}^{\otimes m}} \leq \int_{\Omega_\delta} \|f_{m,\nu}\|^2_{h_\nu^{\otimes m}} \leq C\|b_m\|^2_{h_\nu^{\otimes m}(p)} \leq C\|b_m\|^2_{h^{\otimes m}(p)}$$

where C is a constant independent of m and ν. Henceforth, all such constants will be denoted by C. Therefore, $f_{m,\nu}$ is uniformly bounded (independent of ν) in $L^2(\Omega_\delta)$. The sub-mean value property shows that it is pointwise bounded in $\Omega_{\delta/2}$. Cauchy's estimates show that $\|f_{m,\nu}\|_{C^3(\Omega_{\delta/3})}$ is bounded above by some K_m (uniformly in ν). Thus, by the Arzela–Ascoli theorem, there is a sequence $\nu_i \to 0$ such that $f_{m,\nu_i} \to v_m$

in $C^2(\Omega_{\delta/4})$. The limit v_m is a holomorphic extension of b_m over Ω. Fixing ν_0, by uniform convergence of f_{m,ν_i}, we see that

$$\int_{\Omega'} ||v_m||^2_{h^{\otimes m}_{\nu_0}} \leq C||b_m||^2_{h^{\otimes m}(p)}.$$

By the monotone convergence theorem,

$$\int_{\Omega'} ||v_m||^2_{h^{\otimes m}} \leq C||b_m||^2_{h^{\otimes m}(p)}.$$

Thus, the general multiple coarse L^2-extension property is met and the proof of Theorem 1.3 is complete. □

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