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Quantized symplectic oscillator algebras of rank one

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Abstract

A quantized symplectic oscillator algebra of rank 1 is a PBW deformation of the smash product of the quantum plane with $U_q(\mathfrak{sl}_2)$. We study its representation theory, and in particular, its category $\mathcal{O}$.

Keywords: Symplectic reflection algebras; Quantization; Category $\mathcal{O}$

1. Introduction

Let V be a finite-dimensional complex vector space equipped with a nondegenerate skew-symmetric bilinear form. In [8, Section 4], Etingof, Gan and Ginzburg introduced the family of infinitesimal Hecke algebras $\mathscr{H}_\beta$ associated to $\mathfrak{sp}(V)$. The algebras $\mathscr{H}_\beta$ are PBW deformations of $\mathbb{C}[V] \rtimes U(\mathfrak{sp}(V))$. On the one hand, they are similar to the symplectic reflection algebras introduced by Etingof and Ginzburg in [7] (and by Crawley-Boevey and Holland in [4] when $\dim V$ is 2). On the other hand, they are also similar to universal enveloping algebras of Lie algebras. In the case when $\dim V$ is 2, the algebra $\mathscr{H}_\beta$ was also called a symplectic oscillator algebra in [12] (see [8, Example 4.12]); we shall refer to the $\mathscr{H}_\beta$ in this case as the symplectic oscillator algebras of rank 1.

The representation theory of the symplectic oscillator algebras of rank 1 was studied by Khare in [12]. In our present paper, we show that the main results of [12] can naturally be q-deformed. One of our main results is that, in the q-deformed setting, there exist PBW deformations whose

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finite-dimensional representations are completely reducible. (The same proof can also be adapted to the original setting in [12].)

Fix a ground field $\Bbbk$, with char $\Bbbk \neq 2$, and an element $q \in \Bbbk^\times$ such that $q^2 \neq 1$. Since the quantum plane $\Bbbk_q[X, Y] := \Bbbk\langle X, Y\rangle/(XY - qYX)$ is a module-algebra over the Hopf algebra $U_q(\mathfrak{sl}_2)$, one can form the smash product algebra $\Bbbk_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$, cf. [13]. Our main object of study is a deformation of this algebra, defined for each element C_0 in the center of $U_q(\mathfrak{sl}_2)$ as follows.

Definition 1.1. The *quantized symplectic oscillator algebra of rank* 1 is the algebra A generated over $\Bbbk$ by the elements E, F, K, K^{-1}, X, Y with defining relations

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \tag{1.2}$$

$$EX = qXE, \qquad EY = X + q^{-1}YE, \tag{1.3}$$

$$FX = YK^{-1} + XF, \qquad FY = YF, \tag{1.4}$$

$$KXK^{-1} = qX, \qquad KYK^{-1} = q^{-1}Y, \tag{1.5}$$

$$qYX - XY = C_0. \tag{1.6}$$

The PBW Theorem for A is the statement that the set of elements $F^aY^bK^cX^dE^e$ (for $a, b, d, e \in \mathbb{Z}_{\geqslant 0}$, $c \in \mathbb{Z}$) form a basis for A. We will prove this in Section 2. Let us make some comments on Definition 1.1.

Remark 1.7.

(1) Observe that the subalgebra of A generated by E, F, K and K^{-1} is isomorphic to $U_q(\mathfrak{sl}_2)$. When $C_0 = 0$, the algebra A is $\Bbbk_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$.

(2) In [12], the (deformed) symplectic oscillator algebra H_f is defined, for each polynomial $f \in \Bbbk[t]$, to be the quotient of $T(V) \rtimes U(\mathfrak{sp}(V))$ by the relations $[y, x] = \omega(x, y)(1 + f(\Delta))$ for all $x, y \in V$, where Δ is the Casimir element in $U(\mathfrak{sp}(V))$. The PBW Theorem for H_f was proved in [12, Theorem 9] when $\dim V = 2$. However, it is not true in general when $\dim V > 2$. The formula for obtaining PBW deformations of $\mathbb{C}[V] \rtimes U(\mathfrak{sp}(V))$ is given in [8, Theorem 4.2].
For the rest of this paper, H_f will always mean the case $\dim V = 2$.

(3) The symplectic oscillator algebra when $\dim V = 2$ is analogous to the algebra

$$\frac{\Bbbk\langle X, Y\rangle \rtimes \Bbbk[\Gamma]}{(YX - XY - \zeta)}$$

where Γ is a finite subgroup of $SL(V)$ and ζ is an element in the center of $\Bbbk[\Gamma]$, introduced and studied by Crawley-Boevey and Holland in [4].

The algebra A is very similar to quantized universal enveloping algebras of semisimple Lie algebras in many ways. For example, we can construct Verma modules using the PBW Theorem, define highest weight modules, and study its category $\mathcal{O}$. The main results of the paper are the following:

- Necessary and sufficient conditions for a simple highest weight module to be finite-dimensional (Theorem 4.1).
- A description of the Verma modules of A using the Verma modules of its subalgebra $U_q(\mathfrak{sl}_2)$ (Theorems 6.1 and 7.1).
- A block decomposition for $\mathcal{O}$, and a proof that $\mathcal{O}$ is a highest weight category (Corollary 8.11 and Proposition 8.13).
- Necessary and sufficient conditions for the finite-dimensional representations to be completely reducible (Theorem 10.1).
- A proof that the center of A is trivial when $C_0 \neq 0$ (Theorem 11.1).

Note that since the center of A is trivial when $C_0 \neq 0$, the original approach in [3] for the decomposition of $\mathcal{O}$ does not work for our algebra.

Organization of the paper

We prove the PBW Theorem in Section 2. Then we study in Section 3 the actions of the "raising" operators E and X on highest weight modules. Thenceforth we assume that q is not a root-of-unity. In Section 4, we determine necessary and sufficient conditions for a simple highest weight module to be finite-dimensional. In Section 5, we determine conditions for existence of maximal vectors in Verma modules. Beyond this point, we assume that $C_0 \neq 0$. Then we study those Verma modules in Section 6 whose highest weights are not of the form $\pm q^n$, where $n \in \mathbb{Z}$. In Section 7 we study Verma modules whose highest weights are of the form $\pm q^n$ where $n \in \mathbb{Z}$. We obtain, in the following section, a decomposition of the category $\mathcal{O}$ into blocks, each of which is a highest weight category. In Section 9, we show that various ways of decomposing $\mathcal{O}$ into blocks are actually equivalent. A characterization of all cases when complete reducibility holds is the content of Section 10. The proof of the complete reducibility in this section makes use of our results obtained in the earlier sections, in particular, the decomposition of $\mathcal{O}$. In Section 11, we prove that the center of A is trivial if $C_0 \neq 0$. The next section contains some more results about the Verma modules. Finally, we explain how to take the classical limit $q \to 1$ to obtain the algebra H_f and its highest weight modules in Section 13.

2. PBW Theorem

The relations (1.3), (1.4) and (1.5) imply that $qYX - XY$ commutes with E, F, K and K^{-1}. However, C_0 does not necessarily commute with X, Y.

Theorem 2.1. *The set of elements $F^aY^bK^cX^dE^e$, where $a, b, d, e \in \mathbb{Z}_{\geqslant 0}$, $c \in \mathbb{Z}$, is a basis for A.*

Proof. We shall use the Diamond Lemma; see [1, Theorem 1.2] or [2].

To be precise, we write K^{-1} as L, so

$$KL = LK = 1. \tag{2.2}$$

We now define a semigroup partial ordering $\leqslant$ on the set W of words in the generators E, F, K, L, X, Y. First, define the lexicographic ordering $\leqslant_{\text{lex}}$ on W by ordering the generators as F, Y, L, K, X, E. For each word $w \in W$, let $n(w)$ be the total number of times X and Y appear in w. Now, given two words w and u, we define $w \leqslant u$ if

- $n(w) < n(u)$, or
- $n(w) = n(u)$ and $\mathrm{length}(w) < \mathrm{length}(u)$, or
- $n(w) = n(u)$, $\mathrm{length}(w) = \mathrm{length}(u)$ and $w \leqslant_{\mathrm{lex}} u$.

This is a semigroup partial ordering which satisfies the descending chain condition and is also compatible with the reduction system given by our relations (1.2)–(1.6) and (2.2). We have to check that the ambiguities are resolvable, which we do below. The Diamond Lemma then implies that the irreducible words

$$\left\{F^aY^bL^cX^dE^e \,\middle|\, a,b,d,e \in \mathbb{Z}_{\geqslant 0},\ c \in \mathbb{Z}_{>0}\right\}$$
$$\cup \left\{F^aY^bK^cX^dE^e \,\middle|\, a,b,c,d,e \in \mathbb{Z}_{\geqslant 0}\right\}$$

form a basis for A.

Here are the details of the verification. Let us first write down our reduction system:

$$EK \to q^{-2}KE, \qquad KF \to q^{-2}FK, \qquad LK \to 1, \qquad KL \to 1,$$
$$EF \to FE + (K-L)/\big(q-q^{-1}\big), \qquad EX \to qXE, \qquad EY \to X + q^{-1}YE,$$
$$XF \to FX - YL, \qquad YF \to FY, \qquad XY \to qYX - C_0,$$
$$EL \to q^2LE, \qquad LF \to q^2FL, \qquad XK \to q^{-1}KX,$$
$$KY \to q^{-1}YK, \qquad XL \to qLX, \qquad LY \to qYL.$$

Observe that there is no inclusion ambiguity and all overlap ambiguities appear in words of length 3. Moreover, if X and Y do not appear in a word, then it is reduction unique by the PBW Theorem for $U_q(\mathfrak{sl}_2)$. Thus, the words which we have to check are:

$$LYF, KYF, XYF, EYF, EXF, XLF, XKF, KLY,$$
$$XLY, ELY, XKY, EKY, EXY, XKL, EXL, EXK.$$

We now show that all these ambiguities are resolvable:

$$L(YF) \to (LF)Y \to q^2F(LY) \to q^3FYL,$$
$$(LY)F \to qY(LF) \to q^3(YF)L \to q^3FYL,$$
$$K(YF) \to (KF)Y \to q^{-2}F(KY) \to q^{-3}FYK,$$
$$(KY)F \to q^{-1}Y(KF) \to q^{-3}(YF)K \to q^{-3}FYK,$$
$$X(YF) \to (XF)Y \to F(XY) - Y(LY) \to qFYX - FC_0 - qYYL,$$
$$(XY)F \to qY(XF) - C_0F \to q(YF)X - q(YY)L - C_0F \to qFYX - qYYL - C_0F,$$
$$E(YF) \to (EF)Y \to F(EY) + (KY - LY)/\big(q-q^{-1}\big)$$
$$\to FX + q^{-1}FYE + \big(q^{-1}YK - qYL\big)\big/\big(q-q^{-1}\big),$$

$$
\begin{aligned}
(EY)F &\to XF + q^{-1}Y(EF) \\
&\to FX - YL + q^{-1}(YF)E + \big(q^{-1}YK - q^{-1}YL\big)/\big(q - q^{-1}\big) \\
&\to FX - YL + q^{-1}FYE + \big(q^{-1}YK - q^{-1}YL\big)/\big(q - q^{-1}\big), \\
E(XF) &\to (EF)X - (EY)L \\
&\to F(EX) + (KX - LX)/\big(q - q^{-1}\big) - XL - q^{-1}Y(EL) \\
&\to qFXE + (KX - LX)/\big(q - q^{-1}\big) - qLX - qYLE, \\
(EX)F &\to qX(EF) \to q(XF)E + (qXK - qXL)/\big(q - q^{-1}\big) \\
&\to qFXE - qYLE + \big(KX - q^2LX\big)/\big(q - q^{-1}\big),
\end{aligned}
$$

$$X(LF) \to q^2(XF)L \to q^2F(XL) - q^2YLL \to q^3FLX - q^2YLL,$$

$$(XL)F \to qL(XF) \to q(LF)X - q(LY)L \to q^3FLX - q^2YLL,$$

$$X(KF) \to q^{-2}(XF)K \to q^{-2}F(XK) - q^{-2}YLK \to q^{-3}FKX - q^{-2}Y(LK),$$

$$(XK)F \to q^{-1}K(XF) \to q^{-1}(KF)X - q^{-1}(KY)L \to q^{-3}FKX - q^{-2}YKL,$$

$$K(LY) \to q(KY)L \to Y(KL),$$

$$(KL)Y \to Y,$$

$$X(LY) \to q(XY)L \to q^2Y(XL) - qC_0L \to q^3YLX - qC_0L,$$

$$(XL)Y \to qL(XY) \to q^2(LY)X - qLC_0 \to q^3YLX - qLC_0,$$

$$E(LY) \to q(EY)L \to qXL + Y(EL) \to q^2LX + q^2YLE,$$

$$(EL)Y \to q^2L(EY) \to q^2LX + q(LY)E \to q^2LX + q^2YLE,$$

$$X(KY) \to q^{-1}(XY)K \to Y(XK) - q^{-1}C_0K \to q^{-1}YKX - q^{-1}C_0K,$$

$$(XK)Y \to q^{-1}K(XY) \to (KY)X - q^{-1}KC_0 \to q^{-1}YKX - q^{-1}KC_0,$$

$$E(KY) \to q^{-1}(EY)K \to q^{-1}XK + q^{-2}Y(EK) \to q^{-2}KX + q^{-4}YKE,$$

$$(EK)Y \to q^{-2}K(EY) \to q^{-2}KX + q^{-3}(KY)E \to q^{-2}KX + q^{-4}YKE,$$

$$E(XY) \to q(EY)X - EC_0 \to qXX + Y(EX) - EC_0 \to qXX + qYXE - EC_0,$$

$$(EX)Y \to qX(EY) \to qXX + (XY)E \to qXX + qYXE - C_0E,$$

$$X(KL) \to X,$$

$$(XK)L \to q^{-1}K(XL) \to (KL)X \to X,$$

$$E(XL) \to q(EL)X \to q^3L(EX) \to q^4LXE,$$

$$(EX)L \to qX(EL) \to q^3(XL)E \to q^4LXE,$$

$$E(XK) \to q^{-1}(EK)X \to q^{-3}K(EX) \to q^{-2}KXE,$$

$$(EX)K \to qX(EK) \to q^{-1}(XK)E \to q^{-2}KXE.$$

This completes the proof of Theorem 2.1. □

This method can also be applied to H_f, and provides a simpler proof than in [12].

We may define a $\mathbb{Z}_{\geqslant 0}$-filtration on A by assigning $\deg E = \deg F = 1$, $\deg K = \deg K^{-1} = 0$, and $\deg X = \deg Y$ to be some sufficiently big number so that, by Theorem 2.1, the associated graded algebra $\operatorname{gr} A$ is a skew-Laurent extension of a quantum affine space, cf. e.g. [2]. Hence, we obtain the following corollary.

Corollary 2.3. *The algebra A is a Noetherian domain.*

3. Standard cyclic modules

Given a $\Bbbk[K, K^{-1}]$-module M and $a \in \Bbbk^\times$, we define $M_a = \{m \in M\colon\ K \cdot m = am\}$ and denote by $\Pi(M)$ the set of *weights*: $\{a \in \Bbbk^\times\colon\ M_a \neq 0\}$. We consider A as a $\Bbbk[K, K^{-1}]$-module on which K^c ($c \in \mathbb{Z}$) acts by conjugation.

Lemma 3.1.

(1) *If M is a $\Bbbk[K, K^{-1}]$-module, then the sum $\sum_{a \in \Bbbk^\times} M_a$ is direct, and K-stable.*
(2) *If M is any A-module, then $A_a M_b \subset M_{ab}$.*
(3) *We have: $A = \bigoplus_a A_a$, and $\Bbbk[K, K^{-1}] \subset A_1$.*

Note that A contains subalgebras $B_+ = \langle E, X, K, K^{-1}\rangle$ and $B_- = \langle F, Y, K, K^{-1}\rangle$. We define N_+ (respectively N_-) to be the nonunital subalgebra of A generated by E, X (respectively F, Y). These are analogs of the enveloping algebras of Borel or nilpotent subalgebras of a semisimple Lie algebra.

Later on, we will use often the "purely CSA" (CSA stands for Cartan subalgebra) map $\xi : A \to \Bbbk[K, K^{-1}]$ defined as follows: write each element $U \in A$ in the PBW basis given in Theorem 2.1, then $\xi(U)$ is the sum of all vectors in $\Bbbk[K, K^{-1}]$, i.e. $U - \xi(U) \in N_- A + A N_+$.

We need some terminology that is standard in representation theory. If M is an A-module, a *maximal vector* is any $m \in M$ that is killed by E, X and is an eigenvector for K, K^{-1}. A *standard cyclic module* is one that is generated by exactly one maximal vector. For each $r \in \Bbbk^\times$, define the *Verma module* $Z(r) := A/(AN_+ + A(K - r \cdot 1))$, cf. [9,12]. It is a free B_--module of rank one, by the PBW Theorem for A, hence isomorphic to $\Bbbk[Y, F]$ and has a basis $\{F^i Y^j\colon\ i, j \geqslant 0\}$. Furthermore, $\Pi(Z(r)) = \{q^{-n} r,\ n \geqslant 0\}$.

The proof of the following proposition is standard—see e.g. [9] or [12].

Proposition 3.2.

(1) *$Z(r)$ has a unique maximal submodule $W(r)$, and the quotient $Z(r)/W(r)$ is a simple module $V(r)$.*
(2) *Any standard cyclic module is a quotient of some Verma module.*

We may identify $U_q(\mathfrak{sl}_2)$ with the subalgebra of A generated by E, F, K and K^{-1}. Let $\mathfrak{Z}(U_q(\mathfrak{sl}_2))$ denote the center of $U_q(\mathfrak{sl}_2)$, and denote by $Z_C(r)$ and $V_C(r)$ the Verma and simple $U_q(\mathfrak{sl}_2)$-module, respectively, of highest weight $r \in \Bbbk^\times$. We note that any $z \in \mathfrak{Z}(U_q(\mathfrak{sl}_2))$ acts on any standard cyclic $U_q(\mathfrak{sl}_2)$-module with highest weight r by the scalar $\xi(z)(r)$, where we evaluate the (finite) Laurent polynomial $\xi(z) \in \Bbbk[K, K^{-1}]$ at $r \in \Bbbk^\times$. Define $c_{0r} = \xi(C_0)(r)$ to be the scalar by which C_0 acts on a $U_q(\mathfrak{sl}_2)$-Verma module $Z_C(r)$.

Now we introduce some more notation. We know that units in the $\Bbbk$-algebra $\Bbbk[K, K^{-1}]$ are all of the form bK^m, where $b \in \Bbbk^\times$ and $m \in \mathbb{Z}$. We denote this set (of all units) by $\Bbbk^\times K^{\mathbb{Z}}$. Moreover, for any $a \in \Bbbk^\times K^{\mathbb{Z}}$, define $\{a\} := \frac{a-a^{-1}}{q-q^{-1}}$. The following identity is now easy to check:

$$a\{b\} - b\{a\} = \{a^{-1}b\} \quad \text{for all } a, b \in \Bbbk^\times K^{\mathbb{Z}}. \tag{3.3}$$

We use the identity in proving the next result, as well as Theorem 5.1 below.

Lemma 3.4. *Suppose $a, b \in \Bbbk^\times K^{\mathbb{Z}}$. Then*

(a) $\{a^{-1}\} = -\{a\}$, *and*
(b) $q^{-1}\{b\} + b = \{qb\}$.

Proof. To prove (a), we set $b = 1$ in (3.3), and to prove (b), we set $a = q^{-1}$ in (3.3). □

We shall write $Z(r) \to V \to 0$ to mean that V is a standard cyclic A-module with highest weight r.

As we shall see, many standard cyclic (respectively Verma, simple) A-modules $Z(r) \to V \to 0$ are a direct sum of a progression of standard cyclic (respectively Verma, simple) $U_q(\mathfrak{sl}_2)$-modules of highest weight $t = r, q^{-1}r, \ldots$, each such module having multiplicity one as well. The specific equations governing such a direct sum $V = \bigoplus_i V_{C,q^{-i}r}$ are the subject of this section.

For $m \geqslant 2$, we define

$$\alpha_{r,m} = \sum_{j=0}^{m-2} \{q^{1-j}r\} c_{0,q^{-j}r}. \tag{3.5}$$

This constant will play a fundamental role in the rest of this paper. (We remark that this constant α_{rm} is different from the constant that was also denoted by α_{rm} in [12].)

Let $\epsilon = \pm 1$ henceforth. We will also need the constant

$$d_{r,m} := \frac{\alpha_{r,m}}{\{q^{2-m}r\}\{q^{3-m}r\}},$$

which is defined for all m, if r is not of the form ϵq^l, or for $2 \leqslant m \leqslant l+1$, if $r = \epsilon q^l$ (where $l \in \mathbb{Z}_{\geqslant 0}$).

Lemma 3.6. *Given $r \in \Bbbk^\times$ and $n \in \mathbb{Z}_{\geqslant 0}$, whenever all terms below are defined, we have*

$$\{q^{1-n}r\} d_{r,n+1} = \{q^{3-n}r\} d_{r,n} + c_{0,q^{1-n}r}.$$

Proof. We have

$$\begin{aligned}
\{q^{2-n}r\}\bigl(\{q^{1-n}r\} d_{r,n+1}\bigr) &= \alpha_{r,n+1} \\
&= \alpha_{r,n} + \{q^{2-n}r\} c_{0,q^{1-n}r} \\
&= \{q^{2-n}r\}\bigl(\{q^{3-n}r\} d_{r,n} + c_{0,q^{1-n}r}\bigr).
\end{aligned}$$

Since all terms in the claim are defined, $\{q^{2-n}r\} \neq 0$ and can be canceled from both sides. □

We now imitate the structure theory in [12, Section 9].

Theorem 3.7. *Let $V = Av_r$ be a standard cyclic module, where v_r is a highest weight vector of weight $r \in \Bbbk^\times$. Suppose that $r \neq q^j$ for $1 \leqslant j \leqslant m-1$, and where $m \in \mathbb{Z}_{\geqslant 0}$. Then we have the following*:

(1) *v_r and $v_{q^{-1}r} := Yv_r$ are $U_q(\mathfrak{sl}_2)$-maximal vectors.*
(2) *Suppose $1 \leqslant n \leqslant m$. Set $t_n = q^{-n}r$. Define inductively*:

$$v_{t_n} := Yv_{t_{n-1}} + d_{r,n}Fv_{t_{n-2}}. \tag{3.8}$$

If $n \geqslant 2$, the following two equalities hold:

$$Xv_{t_{n-1}} = EYv_{t_{n-1}} = -\frac{\alpha_{r,n}}{\{t_{n-3}\}}v_{t_{n-2}}. \tag{3.9}$$

Moreover, v_{t_n} is $U_q(\mathfrak{sl}_2)$-maximal, i.e. $Ev_{t_n} = 0$. It is a maximal vector for the algebra A if and only if $\alpha_{r,n+1} = 0$.
(3) *There exist* monic *polynomials*

$$p_{r,n}(Y,F) = Y^n + c_1FY^{n-2} + c_2F^2Y^{n-4} + \cdots \quad (\textit{where } c_i \in \Bbbk)$$

that satisfy $p_{r,n}(Y,F)v_r = v_{t_n}$.

Proof. The last part is obvious from the defining equations, so we show the rest now.

(1) v_r is A-maximal and hence $Ev_r = 0$. Similarly, $EYv_r = Xv_r = 0$.

(2) We proceed by induction, so we assume that all the statements are true when $n = k$ and we want to show that they are true when $n = k+1$.

(a) By induction, v_{t_k} is $U_q(\mathfrak{sl}_2)$-maximal, so $Xv_{t_k} = (EY - q^{-1}YE)v_{t_k} = EYv_{t_k}$.

(b) If n is 0 or 1, then we are done from the first part (since we may choose to set $v_{t_{-1}} = 0$ if we wish). If $n = k+1$ and $k > 1$, we have

$$\begin{aligned} Xv_{t_k} &= X(Yv_{t_{k-1}} + d_{r,k}Fv_{t_{k-2}}) \\ &= (qYX - C_0)v_{t_{k-1}} + d_{r,k}\bigl(FX - YK^{-1}\bigr)v_{t_{k-2}}. \end{aligned}$$

Using the induction hypothesis, we get

$$\begin{aligned} Xv_{t_k} = {} & qY\bigl(-d_{r,k}\{t_{k-2}\}\bigr)v_{t_{k-2}} - c_{0,t_{k-1}}v_{t_{k-1}} \\ & + d_{r,k}\bigl(-Fd_{r,k-1}\{t_{k-3}\}v_{t_{k-3}} - Y(t_{k-2})^{-1}v_{t_{k-2}}\bigr). \end{aligned}$$

Regrouping terms, we then have

$$\begin{aligned} Xv_{t_k} = {} & -d_{r,k}Yv_{t_{k-2}}\bigl(q\{t_{k-2}\} + (t_{k-2})^{-1}\bigr) \\ & - d_{r,k}\{t_{k-3}\}(d_{r,k-1}Fv_{t_{k-3}}) - c_{0,t_{k-1}}v_{t_{k-1}}. \end{aligned}$$

Now use Lemma 3.4 and regroup terms to get

$$Xv_{t_k} = -d_{r,k}\{t_{k-3}\}(Yv_{t_{k-2}} + d_{r,k-1}Fv_{t_{k-3}}) - c_{0,t_{k-1}}v_{t_{k-1}}.$$

Applying the induction hypothesis again, we get

$$Xv_{t_k} = -d_{r,k}\{t_{k-3}\}v_{t_{k-1}} - c_{0,t_{k-1}}v_{t_{k-1}} = -\frac{\alpha_{r,k+1}}{\{t_{k-2}\}}v_{t_{k-1}}.$$

The last equality here uses Eq. (3.5) and Lemma 3.6. This completes the induction.

(c) By induction, v_{t_k} is killed by E, so

$$EYv_{t_k} = q^{-1}Y(Ev_{t_k}) + Xv_{t_k} = Xv_{t_k} = -d_{r,k+1}\{t_{k-1}\}v_{t_{k-1}}$$

and

$$EFv_{t_{k-1}} = \big(FE + \{K\}\big)v_{t_{k-1}} = \{t_{k-1}\}v_{t_{k-1}}.$$

Hence, the vector $Yv_{t_k} + d_{r,k+1}Fv_{t_{k-1}}$ is indeed killed by E. In other words, $v_{t_{k+1}}$ is a maximal $U_q(\mathfrak{sl}_2)$-vector.

Finally, v_{t_k} is A-maximal if and only if $Xv_{t_k} = 0$, which holds if and only if $\alpha_{r,k+1} = 0$ (use (3.9) for $n = k+1$ and note that $\{t_{k-2}\} \neq 0$). □

Example. Let us take a look at the undeformed case $C_0 = 0$. The following proposition holds under this assumption.

Proposition 3.10. *Assume $C_0 = 0$.*

(1) *Every Verma module $Z(r)$ is a direct sum $Z(r) = \bigoplus_{n\geqslant 0} Z_C(q^{-n}r)$ of $U_q(\mathfrak{sl}_2)$-Verma modules. It has a submodule $Z(q^{-1}r)$, and the quotient $Z_C(r)$ is annihilated by X, Y.*
(2) *The simple module $V(r)$ equals $V_C(r)$ and is annihilated by X, Y.*

Proof. (1) We claim that the structure equations, analogous to those in Theorem 3.7, now become

$$v_{q^{-n}r} = Y^n v_r; \qquad Xv_{q^{-n}r} = Ev_{q^{-n}r} = 0.$$

Firstly, X commutes with Y since $C_0 = 0$, so we have

$$X\big(Y^n v_r\big) = Y^n(Xv_r) = 0.$$

Next,

$$EY^n v_r = q^{-1}Y\big(EY^{n-1}v_r\big) + XY^{n-1}v_r = 0$$

by induction on n. Hence, each $Y^n v_r$ is maximal. We have

$$Z\big(q^{-1}r\big) \xrightarrow{\sim} A\cdot Yv_r \hookrightarrow Z(r)$$

because $B_- = \Bbbk[Y, F]$ is an integral domain. We also have isomorphisms

$$Z(r)/Z\big(q^{-1}r\big) \cong \sum_{n\geqslant 0} \Bbbk F^n v_r \cong Z_C(r).$$

Now $YF^n v_r = F^n Y v_r \in Z(q^{-1}r)$, hence $YF^n \bar{v}_r = 0$, where $\bar{v}_r$ is the image of v_r in the quotient $Z(r)/Z(q^{-1}r)$. Moreover,

$$XF^n\bar{v}_r = FXF^{n-1}\bar{v}_r - YK^{-1}F^{n-1}\bar{v}_r = 0$$

by induction. This proves the last claim of part (1).

(2) Since $Z(r)/Z(q^{-1}r)$ is annihilated by X and Y, the maximal submodule of $Z(r)$ corresponds, in this quotient, to the maximal $U_q(\mathfrak{sl}_2)$-submodule of $Z_C(r)$. □

Standing Assumption. From now on, unless otherwise stated, we assume that q is *not* a root of unity.

In this case, the center $\mathfrak{Z}(U_q(\mathfrak{sl}_2))$ is generated by the Casimir element

$$C := FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

and we will write $C_0 = p(C)$ for some polynomial $p \in \Bbbk[t]$. We also let $c_r = \xi(C)(r)$ to be the scalar by which C acts on the $U_q(\mathfrak{sl}_2)$-Verma module $Z_C(r)$. Thus,

$$c_r = \big(qr + q^{-1}r^{-1}\big)/\big(q-q^{-1}\big)^2 \quad \text{and} \quad c_{0r} = p(c_r). \tag{3.11}$$

The following proposition will play an important role in obtaining the decomposition of category $\mathcal{O}$ (which will be defined later) and in proving that Verma modules have finite length.

Proposition 3.12. *If q is not a root of unity, then $\alpha_{r,m}$ is of the form $(q^m r)^{-N} b(r, q^m)$ for some polynomial $b \in \Bbbk[S, T]$, and some $N \in \mathbb{Z}_{>0}$.*

Proof. It is clear from (3.11) that $c_{0,r} = h(qr)$ for some $h \in \Bbbk[T, T^{-1}]$. Now $g(T) := h(T)\cdot\{T\}$ is also in $\Bbbk[T, T^{-1}]$. We observe, from Eq. (3.5), that

$$\alpha_{r,m} = \sum_{j=0}^{m-2} g\big(q^{1-j}r\big).$$

We write $h(T) = \sum_{i=-M}^{M} b_i T^i$. By [11, Lemma 2.17], we have $b_i = b_{-i}$ for each i. Hence by definition of g, if $g(T) = \sum_{i=-N}^{N} a_i T^i$ (where $N = M+1$), then $a_{-i} = -a_i$. In particular, $a_0 = 0$.

Recall that we are assuming that q is not a root of unity. Interchanging the finite sums, we get

$$\alpha_{r,m} = \sum_{j=0}^{m-2} \sum_{i=-N}^{N} a_i r^i q^{i(1-j)} = \sum_{i=-N}^{N} a_i r^i \sum_{j=0}^{m-2} \left(q^{-i}\right)^{j-1}$$
$$= \sum_{i=-N}^{N} \frac{a_i}{q^{-i}-1} r^i \left(q^{i-mi} - 1\right) q^i.$$

Henceforth, denote by $\sum'$ the summation above with the $i = 0$ term omitted. If we set $b(S, T) = \sum_{i=-N}^{\prime N} \frac{a_i}{q^{-i}-1} q^i S^{N+i}(q^i T^{N-i} - T^N)$, then $\alpha_{r,m} = (q^m r)^{-N} b(r, q^m)$. Here, b is a polynomial in S, T, and we are done. □

4. Finite-dimensional modules

We will first give an example for which the category of finite-dimensional modules over A is not semisimple. Afterwards, assuming that q is not a root of unity, we will give a (rough) classification of all simple finite-dimensional (K-semisimple) modules.

Counterexample to complete reducibility

Consider the module V of dimension 3 spanned over $\Bbbk$ by v_{-1}, v_0, v_1 and with defining relations: $Kv_i = q^i v_i$; v_0 is annihilated by E, X, Y, F; v_1 is killed by E, X; F, Y kill v_{-1}; and finally

$$Fv_1 = v_{-1}, \qquad Ev_{-1} = v_1, \qquad Yv_1 = v_0, \qquad Xv_{-1} = -q^{-1}v_0.$$

In order to satisfy relation (1.6), we set $C_0 = 0$. The space V and $V_0 = \Bbbk v_0 = V(1)$ are easily seen to be A-modules. However, any complement of V_0 in V must contain a vector of the form $v = v_1 + cv_{-1}$, and then $Yv = v_0 \in V_0$. Thus V does not contain a submodule complementary to V_0.

We also remark that the trivial module $V_0 = V(1)$ has no resolution by Verma modules. (Such resolutions have been useful in the theory of semisimple Lie algebras.) For, if we had $Z(r_2) \to Z(r_1) \to V(1)$, then $r_1 = 1$, and then $W(1) = \Bbbk[Y, F](Y, F)v_1$ would be the radical of $Z(r_1) = Z(1)$. But then we must have $Z(r_2) \to W(1) \to 0$, whence $r_2 = q^{-1}$ (by looking at the highest weight in both modules) and $v_{r_2} \mapsto Yv_1$. But then the image of the map is $\Bbbk[Y, F]Yv_1$, and Fv_1 is not in the image of this map.

Recall that we write ϵ for ± 1. Every K-semisimple finite-dimensional simple module is of the form $V(r)$ for some $r = \epsilon q^n$, since $V_C(r)$, and hence $V(r)$, is infinite-dimensional, if r is not of this form. Since char $\Bbbk \neq 2$, every finite-dimensional module is K-semisimple (cf. [11, Section 2.3]).

The main theorem of this section is the following.

Theorem 4.1. *The simple module $V(r)$ is finite-dimensional if and only if $r = \pm q^n$ and there is a* (*least*) *integer $m > 1$ so that $\alpha_{r,n-m+2} = 0$. Furthermore, in this case,*

$$V(r) = \bigoplus_{i=0}^{n-m} V_C\big(q^{-i}r\big).$$

Proof. Suppose $V = V(r)$ is finite-dimensional simple, so $r = \epsilon q^n$, $n \in \mathbb{Z}_{\geqslant 0}$, as was observed above. It must be standard cyclic, so we can apply Theorem 3.7 above to V. By [11, Theorem 2.9], V is a direct sum of simple $V_C(t)$'s, each of which is finite-dimensional, and completely known, by [11, Theorem 2.6].

Clearly, $v_{\epsilon q^{-1}}$ must be zero in V, else V would contain a copy of the infinite-dimensional $U_q(\mathfrak{sl}_2)$-Verma module $Z(\epsilon q^{-1}) = V(\epsilon q^{-1})$ (which is also simple by [11, Proposition 2.5]). So let $v_{\epsilon q^m}$ be the "least" nonzero $U_q(\mathfrak{sl}_2)$-maximal vector in V. Then $v_{\epsilon q^{m-1}} = 0$ in V. Using Theorem 3.7 again, we can consider $v_{\epsilon q^{m-1}}$ to be the image, under the quotient $\pi : Z(r) \twoheadrightarrow V$, of a vector $\tilde{v}_{\epsilon q^{m-1}} \in Z(r)$, defined as in Theorem 3.7. If $\alpha_{r,n-m+2} \neq 0$, then Eq. (3.9) shows that $X\tilde{v}_{\epsilon q^{m-1}}$ is a nonzero multiple of $\tilde{v}_{\epsilon q^m}$, so $\pi(X\tilde{v}_{\epsilon q^{m-1}})$ is nonzero in V, which is a contradiction. Therefore, $\alpha_{r,n-m+2} = 0$.

Furthermore, $v_{\epsilon q^l} \neq 0$ for $l = m, \ldots, n$. Indeed, if $v_{\epsilon q^l} = 0$ for some $m+1 \leqslant l \leqslant n$, then $Xv_{\epsilon q^{l-1}} = 0$ according to Eq. (3.9), but this is a contradiction because $v_{\epsilon q^n}$ is (up to a scalar) the only highest weight vector in V since V is simple. For the same reason, Theorem 3.7 implies that $n - m + 2$ is the smallest positive integer $d > 1$ so that $\alpha_{r,d} = 0$.

Conversely, if there exists a $m \in \mathbb{Z}_{\geqslant 0}$ so that $\alpha_{r,n-m+2} = 0$, then assuming that m is the least such integer, we can give the $U_q(\mathfrak{sl}_2)$ module $V = \sum_{i=m}^{n} U_q(\mathfrak{sl}_2)v_{\epsilon q^i}$ the structure of a (simple) finite-dimensional A-module, using the equations worked out by Theorem 3.7 and [11, Theorem 2.6]. □

We remark that one can write down the Weyl Character Formula for a simple finite-dimensional A-module V, because this formula is known for $V_C(q^{-i}r)$.

5. Verma modules I: Maximal vectors

One of the basic questions about the induced modules $Z(r)$ is: what are their maximal vectors? The main result of this section is a step towards a full answer to this question.

Theorem 5.1. *We consider $Z(r)$ for any $r \in \Bbbk^\times$.*

(1) *If $Z(r)$ has a maximal vector of weight $t = q^{-n}r$, then it is unique up to scalars and $\alpha_{r,n+1} = 0$.*
(2) *We have*: $\dim_{\Bbbk} \operatorname{Hom}_A(Z(r'), Z(r)) = 0$ *or* 1 *for all r, r', and all nonzero homomorphisms between two Verma modules are injective.*

Part (2) follows from the first part and from the fact that $B_- = \Bbbk[Y, F]$ is an integral domain.

Thus, a necessary condition for $Z(r)$ not to be simple (for general $r \in \Bbbk^\times$) is that $\alpha_{r,m} = 0$ for some $m \geqslant 0$. Moreover, if $r \neq \pm q^n$ $(n \in \mathbb{Z}_{\geqslant 0})$, then, from the previous section, this condition is also sufficient, i.e. the converse to part (1) holds as well.

To prove the first part of the theorem, we imitate [12, Lemma 4], and then [12, Section 14]. First, we show the following lemma.

Lemma 5.2. *Let $r \in \Bbbk^\times$. The following equalities hold for $v_r \in Z(r)$:*

$$\big[X, F^n Y^m\big] v_r = -F^n \sum_{j=0}^{m-1} q^j Y^j C_0 Y^{m-1-j} v_r - q^{m+n-1} r^{-1} \big\{q^n\big\} F^{n-1} Y^{m+1} v_r,$$

$$\big[E, F^n Y^m\big] v_r = -F^n \sum_{j=0}^{m-2} \big\{q^{j+1}\big\} Y^j C_0 Y^{m-2-j} v_r + \big\{q^n\big\}\big\{q^{1-m-n} r\big\} F^{n-1} Y^m v_r.$$

Proof.

$$\begin{aligned} X\big(F^n Y^m v_r\big) &= \big[X, F^n Y^m\big] v_r = \big[X, F^n\big] Y^m v_r + F^n X Y^m v_r \\ &= \big[X, F^n\big] Y^m v_r + F^n \sum_{j=0}^{m-1} q^j Y^j (XY - qYX) Y^{m-1-j} v_r \\ &= \big[X, F^n\big] Y^m v_r - F^n \sum_{j=0}^{m-1} q^j Y^j C_0 Y^{m-1-j} v_r. \end{aligned}$$

We have to expand the first term:

$$\begin{aligned} \big[X, F^n\big] Y^m v_r &= \sum_{j=0}^{n-1} F^j [X, F] F^{n-1-j} Y^m v_r \\ &= -\sum_{j=0}^{n-1} F^j Y K^{-1} F^{n-1-j} Y^m v_r \\ &= -\sum_{j=0}^{n-1} q^m r^{-1} q^{2n-2-2j} F^{n-1} Y^{m+1} v_r \\ &= -q^m r^{-1} \left(\sum_{j=0}^{n-1} q^{2(n-1-j)}\right) F^{n-1} Y^{m+1} v_r \\ &= -q^m r^{-1} \frac{q^{2n} - 1}{q^2 - 1} F^{n-1} Y^{m+1} v_r \\ &= -q^{m+n-1} r^{-1} \big\{q^n\big\} F^{n-1} Y^{m+1} v_r. \end{aligned}$$

This proves the first equality of the lemma. We now turn to the second one. We have $E(F^n Y^m v_r) = [E, F^n Y^m] v_r = [E, F^n] Y^m v_r + F^n [E, Y^m] v_r$, so let us compute these two terms separately:

$$\begin{aligned} \big[E, F^n\big] Y^m v_r &= \sum_{i=0}^{n-1} F^i [E, F] F^{n-1-i} Y^m v_r \\ &= \sum_{i=0}^{n-1} F^i \frac{K - K^{-1}}{q - q^{-1}} F^{n-1-i} Y^m v_r \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} F^i \frac{q^{-2(n-1-i)-m} r - q^{2(n-1-i)+m} r^{-1}}{q-q^{-1}} F^{n-1-i} Y^m v_r \\
&= \sum_{i=0}^{n-1} \frac{q^{-2(n-1-i)-m} r - q^{2(n-1-i)+m} r^{-1}}{q-q^{-1}} F^{n-1} Y^m v_r \\
&= \sum_{i=0}^{n-1} \frac{q^{-2i-m} r - q^{2i+m} r^{-1}}{q-q^{-1}} F^{n-1} Y^m v_r \\
&= \frac{\frac{q^{-2n}-1}{q^{-2}-1} q^{-m} r - \frac{q^{2n}-1}{q^{2}-1} q^{m} r^{-1}}{q-q^{-1}} F^{n-1} Y^m v_r \\
&= \{q^n\}\{q^{1-n-m} r\} F^{n-1} Y^m v_r,
\end{aligned}
$$

$$
\begin{aligned}
F^n[E, Y^m] v_r &= F^n \sum_{j=0}^{m-1} q^{-j} Y^j (EY - q^{-1} Y E) Y^{m-1-j} v_r \\
&= F^n \sum_{j=0}^{m-1} q^{-j} Y^j X Y^{m-1-j} v_r \\
&= -F^n \sum_{i=0}^{m-1} q^{-i} Y^i \sum_{j=0}^{m-2-i} q^j Y^j C_0 Y^{m-2-j-i} v_r \\
&= -F^n \sum_{j=0}^{m-2} q^j Y^j \left(\sum_{i=0}^{m-j} q^{-i} Y^i C_0 Y^{-i} \right) Y^{m-2-j} v_r \\
&= -F^n \sum_{k=0}^{m-2} \left(\sum_{i=0}^{k} q^{k-2i} \right) Y^k C_0 Y^{m-2-k} v_r \\
&= -F^n \sum_{k=0}^{m-2} \{q^{k+1}\} Y^k C_0 Y^{m-2-k} v_r \\
&= -F^n \sum_{k=0}^{m-2} \{q^{k+1}\} Y^k C_0 Y^{m-2-k} v_r.
\end{aligned}
$$

This completes the proof of the lemma. □

Convention. An element $v \in Z(r) = \Bbbk[Y, F] v_r \cong \Bbbk[Y, F]$ can be viewed as a polynomial in Y, with coefficients in $\Bbbk[F]$. We now define the *leading term* and *lower order terms* of v to be these terms with respect to the Y-degree.

Lemma 5.3. *Let $r \in \Bbbk^\times$. The following relations are valid in $Z(r)$.*

(1) *Any $z \in \mathfrak{Z}(U_q(\mathfrak{sl}_2))$ acts on $F^n Y^m v_r$ by*

$$
z F^n Y^m v_r = F^n \big(\xi(z)(q^{-m} r) Y^m v_r + l.o.t. \big) \in Z(r)_{q^{-m-2n} r}
$$

(2) *In particular,* $C_0 F^n Y^m v_r = F^n(c_{0,q^{-m}r} Y^m + l.o.t.)v_r$.
(3) *If* $v \in Z(r)_{q^{-m}r}$ *satisfies* $Xv = 0$*, then, up to scalars, we have*

$$v = \left(q^{m-2}r^{-1}\right)Y^m v_r - \left(\sum_{j=0}^{m-1} q^{m-1-j} c_{0,q^{-j}r}\right) F Y^{m-2} v_r + l.o.t.$$

(4) *Similarly, if* $v \in Z(r)_{q^{-m}r}$ *satisfies* $Ev = 0$*, then, up to scalars,*
  (a) $v = F^n v_{\epsilon q^{n-1}}$*, where* $r = \epsilon q^{m-n-1}$ *(for some* $n > 0$*), or*
  (b) $v = \{q^{m-2}r^{-1}\}Y^m v_r - (\sum_{j=0}^{m-1}\{q^{m-1-j}\}c_{0,q^{-j}r})FY^{m-2}v_r + l.o.t.$

Proof. (1) We only need to show this for the case $n = 0$ because $z \in \mathfrak{Z}(U_q(\mathfrak{sl}_2))$. Firstly, by weight considerations, $z \in \mathrm{End}_{\Bbbk}(Z(r)_t)$ for every t. Now recall that $z - \xi(z) = FUE$ for some $U \in U_q(\mathfrak{sl}_2)$. From above, E takes Y^m into lower order terms, whence so does FUE. Therefore $zY^m v_r = \xi(z)Y^m v_r + l.o.t.$, and $\xi(z)$ acts on the vector $Y^m v_r$ by $\xi(z)(q^{-m}r)$, as claimed.

(2) This is now obvious.

(3) We first claim that any vector killed by X must be of the form $Y^m + l.o.t.$ (up to scalars). For, if

$$v = F^n\left(Y^{m-2n} + a_1 F Y^{m-2n-2} + \cdots\right)v_r = F^n Y^{m-2n} + l.o.t.,$$

then, by the above lemma, we have

$$Xv = -q^{m-n-1}r^{-1}\left\{q^n\right\}F^{n-1}Y^{m-2n+1} + l.o.t.$$

and this is nonzero if $n > 0$, because q is not a root of unity. Hence such a v cannot be a solution.

Next, any solution is unique up to scalars, because given two such vectors $v_i = Y^m + l.o.t._i$ (for $i = 1, 2$), we have $X(v_1 - v_2) = 0$. However, $v_1 - v_2 = l.o.t._1 - l.o.t._2$, and hence must be zero from above.

Finally, from Lemma 5.2,

$$\begin{aligned} XY^m v_r &= -\sum_{j=0}^{m-1} q^j Y^j C_0 Y^{m-1-j} v_r \\ &= -Y^{m-1}\sum_{j=0}^{m-1} q^{m-1-j} c_{0,q^{-j}r} + l.o.t. \quad \big(\text{by part (2)}\big). \end{aligned}$$

Similarly, $XFY^{m-2}v_r = -q^{m-2}r^{-1}Y^{m-1} + l.o.t.$ Hence, if $Xv = 0$, then in order that the two highest degree (in Y) terms cancel, v must be of the given form.

(4) Now suppose $Ev = 0$ for some $v \in Z(r)_{q^{-m}r}$. Once again, if

$$v = F^n\left(Y^{m-2n} + a_1 F Y^{m-2n-2} + \cdots\right)v_r,$$

then

$$Ev = \left\{q^n\right\}\left\{q^{1-m+2n-n}r\right\}F^{n-1}Y^{m-2n}v_r + l.o.t.$$

and if this is zero, then $n = 0$, or $r = \pm q^{m-n-1}$.

If $n=0$, then a similar analysis as above reveals that

$$EY^m = -Y^{m-2}\sum_{j=0}^{m-2}\{q^{1+(m-2-j)}\}c_{0,q^{-j}r} + l.o.t.,$$

and

$$EFY^{m-2} = \{q^{2-m}r\}Y^{m-2} + l.o.t.$$

Therefore in this case (by the same argument as above), we must have $v = \{q^{m-2}r^{-1}\}Y^m - bFY^{m-2} + l.o.t.$, where

$$b = \sum_{j=0}^{m-2}\{q^{m-1-j}\}c_{0,q^{-j}r} = \sum_{j=0}^{m-1}\{q^{m-1-j}\}c_{0,q^{-j}r}$$

since $\{q^0\}=0$.

On the other hand, if $n>0$, then $v = F^n v'$, where $v' = Y^{m-2n} + l.o.t. \in Z(r)_{\pm q^{n-1}}$. But $n>0$, so the equations of Theorem 3.7 apply, and we can write v as a sum of vectors in various $U_q(\mathfrak{sl}_2)$-Verma modules. But now, $U_q(\mathfrak{sl}_2)$-theory gives us that $v = F^n v_{\pm q^{n-1}}$, where n satisfies the given conditions. □

Proof of Theorem 5.1. The vector v is maximal if and only if $Ev = Xv = 0$. Hence, v is monic and unique up to scalars according to the previous lemma. Using the last two parts, we can write v in two different ways.

Therefore,

$$\{q^{m-2}r^{-1}\}\sum_{j=0}^{m-1} q^{m-1-j}c_{0,q^{-j}r} = (q^{m-2}r^{-1})\sum_{j=0}^{m-1}\{q^{m-1-j}\}c_{0,q^{-j}r}.$$

Subtracting, we get

$$\sum_{j=0}^{m-1}\big[(q^{m-2}r^{-1})\{q^{m-1-j}\} - q^{m-1-j}\{q^{m-2}r^{-1}\}\big]c_{0,q^{-j}r} = 0.$$

Finally, using Eq. (3.3), we get

$$\sum_{j=0}^{m-1}\{q^{1-j}r\}c_{0,q^{-j}r} = 0, \quad \text{that is, } \alpha_{r,m+1} = 0. \qquad \square$$

6. Verma modules II: Noninteger case

Standing Assumption. From now on, unless otherwise stated, we assume that $C_0 = p(C) \neq 0$, or $p \neq 0$.

Suppose $r \neq \pm q^n$ for any $n \in \mathbb{Z}_{\geqslant 0}$. Then the Verma module $Z(r)$ becomes very easy to describe. We observe that Eqs. (3.9), (3.8) are valid for all n, so the set $\{F^j v_{q^{-i}r}\colon i, j \geqslant 0\}$ is a basis for $Z(r)$.

Theorem 6.1 *(Noninteger power case). Suppose $r \neq \pm q^n$ for any $n \in \mathbb{Z}_{\geqslant 0}$. Then*

(1) *$Z(r)$ is a direct sum of $U_q(\mathfrak{sl}_2)$-Verma modules $Z_C(q^{-i}r)$, one copy for each i.*
(2) *The submodules of $Z(r)$ are precisely of the form $Z(t) = \Bbbk[Y, F]v_t$, where $t = q^{-n}r$ for every root n of $\alpha_{r,n+1}$. In particular, all these submodules lie in a chain, and $Z(r)$ has finite length.*

Proof. The first part is a consequence of Theorem 3.7 and of the observation above. Next, if M is a submodule of $Z(r)$ containing a vector of highest possible weight $t = q^{-n}r$, then we claim that $M = Z(t) = \Bbbk[Y, F]v_t$. To start with, $v_t \in Z(r)$ is the unique maximal vector in $Z(r)$ of weight t up to scalars, by Theorem 5.1 above. Hence $v_t \in M$. We now show that $M \subset k[Y, F]v_t$.

Suppose, to the contrary, that $v \in M$ is of the form

$$v = p(Y, F)v_t + a_1 F^{i_1} v_{qt} + \cdots + a_n F^{i_n} v_r.$$

We may assume that $p(Y, F) = 0$ because $v_t \in M$. We know (by [11, Theorem 2.5]), that the $U_q(\mathfrak{sl}_2)$-Verma modules $Z_C(q^i t)$ are simple, so $E^l v \in \operatorname{span}\{v_{q^i t} \mid 1 \leqslant i \leqslant n\}$ for some $l \gg 0$ and $E^l v \neq 0$. Therefore, since all these vectors are in different K-eigenspaces, we conclude that $v_{q^i t} \in M$ for some $i \geqslant 1$. This is a contradiction since, by assumption, $q^i t$ is not a weight of M if $i \geqslant 1$. □

Remark 6.2. In the above theorem, the successive subquotients are the simple modules $V(t)$, and all the modules described in this section are infinite-dimensional.

7. Verma modules III: Integer case

In Section 6, we assumed that $r \neq \pm q^n$. In this section, we treat the remaining case, namely $r = \pm q^n$. In this case, it may happen that the simple module $V(r)$ is finite-dimensional—see Section 4.

The main result is the following.

Theorem 7.1 *(Integer power case). Suppose $r = \pm q^n$ for $n \in \mathbb{Z}_{\geqslant 0}$. Suppose $0 = n_0 < n_1 < \cdots < n_k \leqslant n+1$ are the roots to $\alpha_{r,n+1}$. Denote $t_i = q^{-n_i}r$. Then*

(1) *$Z(r)$ is a direct sum of $U_q(\mathfrak{sl}_2)$-Verma modules $Z_C(q^{-i}r)$ for $0 \leqslant i < n_k$,* and *the A-Verma module $Z(t_k)$.*
(2) *$Z(r)$ has the following filtration*

$$Z(r) = Z(t_0) \supset W(t_0) \supset Z(t_1) \supset W(t_1) \supset \cdots \supset W(t_{k-1}) \supset Z(t_k) \supset W(t_k)$$

where the successive subquotients are, respectively,

$$V(t_0), V\big(\big(q^3 t_1\big)^{-1}\big), V(t_1), V\big(\big(q^3 t_2\big)^{-1}\big), \ldots, V(t_{k-1}), V\big(\big(q^3 t_k\big)^{-1}\big), V(t_k).$$

(3) *If $Z(t_k)$ is not simple, then it has a unique maximal submodule of the form $Z(t)$ for some $t = \pm q^{-N}$. We then know the composition series of $Z(t_k)$ by Theorem* 6.1 *in this case, or if $t_k = -1$.*

(4) *The $V(t_i)$'s are finite-dimensional (for $0 \leqslant i < k$).*

This theorem is similar to a corresponding one in [12]. We will need the following lemma.

Lemma 7.2. *Suppose that $V(r)$ is finite-dimensional. If $Z(t) \hookrightarrow Z(r)$ is the largest Verma submodule in $Z(r)$ (with $t = q^m$ for some $-1 \leqslant m < n$), and $W(r)$ denotes the unique maximal submodule of $Z(r)$ (so that $Z(r)/W(r) \cong V(r)$), then $W(r)/Z(t) \cong V((q^3t)^{-1})$.*

Proof. From the definition of the Casimir operator C, it follows immediately that $c_r = c_{(q^2r)^{-1}}$, whence $c_{0,r} = c_{0,(q^2r)^{-1}}$. We claim that $\alpha_{r,2n+4} = 0$. Indeed,

$$
\begin{aligned}
\alpha_{r,2n+4} &= \sum_{j=0}^{2n+2} \{q^{1-j+n}\} c_{0,q^{-j+n}} \\
&= \sum_{j=-n-1}^{n+1} \{q^j\} c_{0,q^{j-1}} \\
&= \sum_{j=1}^{n+1} \left(\{q^j\} c_{0,q^{j-1}} + \{q^{-j}\} c_{0,q^{-j-1}}\right), \quad \text{since } \{q^0\} = 0 \\
&= \sum_{j=1}^{n+1} \{q^j\} (c_{0,q^{j-1}} - c_{0,q^{-j-1}}), \quad \text{since } \{a\} + \{a^{-1}\} = 0\ \forall a \\
&= \sum_{l=0}^{n} \{q^{l+1}\} (c_{0,q^l} - c_{0,q^{-l-2}}) \\
&= 0 \quad \text{from above.}
\end{aligned}
$$

Thus, t is the first root after r for $\alpha_{r,n}$, if and only if $(q^3r)^{-1}$ is the first root after $(q^3t)^{-1}$ for $\alpha_{(q^3t)^{-1},n}$.

But now, the quotient $W(r)/Z(t)$ has a vector of highest weight $(q^3t)^{-1}$: if v_{qt} is the lowest $U_q(\mathfrak{sl}_2)$-maximal vector in $V(r)$, and $t = \epsilon q^m$, then $F^{m+2} v_{qt}$ is $U_q(\mathfrak{sl}_2)$-maximal and of highest weight in the quotient. But it has weight $q^{-2m-4}\epsilon q^{m+1} = (q^3t)^{-1}$ as claimed.

Thus, $W(r)/Z(t)$ has a subquotient of the form $V((q^3t)^{-1})$. But one can check that they have the same characters. Hence they are equal. □

Proof of Theorem 7.1. To simplify the notation, let us assume that $r = q^n$; the case $r = -q^n$ is similar. Suppose that the simple module $V(r)$ is finite-dimensional. Then, by Theorem 4.1, $\alpha_{r,n-m+2} = 0$ for some $m > 1$ and $n - (m-1) > 0$. We assume that m is the smallest such integer. Then, from the proof of that same theorem, we know that $v_{q^{m-1}}$ is maximal in $Z(r)$. Therefore, setting $n_1 = n - (m-1)$ and $t_1 = q^{-n_1} r = q^{m-1}$, we get that $Z(t_1) \hookrightarrow Z(r)$.

We can repeat the same procedure with $Z(t_1)$. If its simple top quotient $V(t_1)$ is finite-dimensional, then there exists a (smallest) integer m_1 such that $\alpha_{(m-1)-m_1+2} = 0$ for some $m_1 > 1$ and $(m-1)-(m_1-1) > 0$. Again, $v_{q^{m_1-1}}$ is maximal in $Z(t_1)$, so $Z(t_2) \hookrightarrow Z(t_1)$ where $t_2 = q^{-n_2}t_1 = q^{m_1-1}$ and $n_2 = m-1-(m_1-1)$. Note that $n-(m_1-1) = n-(m-1)+(m-1)-(m_1-1) > 0$.

We can continue repeating this procedure and get a chain of Verma submodules $Z(r) \supset Z(t_1) \supset Z(t_2) \supset \cdots$. Set $n_0 = n$, $m_0 = m$, $d_0 = n-(m-1)+1$ and $d_i = (m_{i-1}-1)-m_i+2$ for $i \geqslant 1$. Since $n-(m_i-1) > 0$ for all $i \geqslant 1$ (as noted in the previous paragraph for $i=1$), this procedure must stop for some positive integer k. This means that $Z(t_k) \subset Z(r)$, but the top quotient of $Z(t_k)$ is not finite-dimensional.

Using Theorem 3.7, this proves part (1). We can now apply Lemma 7.2 to each successive inclusion $Z(t_i) \subset Z(t_{i-1})$, and part (2) is proved. Part (4) follows from the first two parts.

It remains to show part (3); namely, that $W(t_k) = 0$ or $Z(t)$ for some t. So suppose $Z(t_k)$ is not simple. Let v_t be the highest possible maximal vector in $Z(t_k)$, that is not of weight t_k (i.e. it has "smaller" weight). Thus $t = q^{-n}t_k$ for some n, and $v_t = Y^n v_{t_k} + l.o.t.$, from Lemma 5.3 above.

Now, any weight vector $v_x \in W(t_k)$ is (upto scalars) of the form $g(Y,F)v_t + F^l h(Y,F)v_{t_k}$, where h is *monic* in Y. (This follows from the Euclidean algorithm for polynomials $(\Bbbk[F])[Y]$, because v_t is monic.) Further, $l > 0$, since we are not considering the case $t_k = \pm q^{-1}$, which we know by Section 6.

To show $W(t_k) = Z(t)$, we must prove that $h = 0$ for each such v_x. Suppose not. Let $v_x \in W(t_k)$ be a weight vector of highest possible weight x, such that $h \neq 0$. Now, $Ev_x \in W(t_k)$, so by maximality of x, $Ev_x \in Z(t) = \Bbbk[Y,F]v_t$, hence $E(v_x - g(Y,F)v_t) \in Z(t)$. Hence, we get that $E(F^l h(Y,F)v_{t_k}) \in Z(t)$.

This is in the $U_q(\mathfrak{sl}_2)$-span of $v_{t_k}, Yv_{t_k}, Y^2v_{t_k}, \ldots, Y^{n-1}v_{t_k}$, so if it is in $Z(t)$, then it must be zero, by the PBW Theorem. Hence $E(F^l h(Y,F)v_{t_k}) = 0$. But now, part (4) of Lemma 5.3 above, gives us that $F^l h(Y,F)v_{t_k} = F^l v_{\pm q^{l-1}}$.

Hence we finally get that $v' = F^l v_{\pm q^{l-1}} \in W(t_k)$. Hence $X^l v' \in W(t_k)$. From the following lemma, this means that (up to a nonzero scalar), $v_{\pm q^{-1}} \in W(t_k)$. But t was "lower" than $\pm q^{-1}$ from above, hence this is a contradiction, and no such v_x exists. □

Lemma 7.3.

(1) $[F^{j+1}, X] = q^j\{q^{j+1}\}F^jYK^{-1}$.
(2) *If* $r = \epsilon q^n$, *then* $F^{j+1}v_{\epsilon q^j}$ *is* $U_q(\mathfrak{sl}_2)$*-maximal* (*for each* $-1 \leqslant j \leqslant n$), *and* $X(F^{j+1}v_{\epsilon q^j}) = -\{\epsilon q^{j+1}\}F^j v_{\epsilon q^{j-1}}$.

Proof. For the first part, we compute, using the defining relations:

$$\begin{aligned}\left[F^{j+1}, X\right] &= \sum_{i=0}^{j} F^{j-i}[F,X]F^i = \sum_{i=0}^{j} F^{j-i}YK^{-1}F^i = \sum_{i=0}^{j} F^{j-i}Yq^{2i}F^iK^{-1} \\ &= \sum_{i=0}^{j} q^{2i}\cdot F^jYK^{-1} = \frac{q^{2j+2}-1}{q^2-1}F^jYK^{-1} = q^j\left\{q^{j+1}\right\}F^jYK^{-1}\end{aligned}$$

as claimed. Next, suppose $r = \epsilon q^n$ for some n. We then compute:

$$\begin{aligned}
E \cdot F^{j+1} v_{\epsilon q^j} &= F^{j+1} \cdot E v_{\epsilon q^j} + \sum_{i=0}^{j} F^{j-i}[E, F] F^i v_{\epsilon q^j} \\
&= 0 + \sum_{i=0}^{j} F^{j-i} \frac{K - K^{-1}}{q - q^{-1}} F^i v_{\epsilon q^j} \\
&= \sum_{i=0}^{j} F^{j-i} \frac{\epsilon}{q - q^{-1}} \big(q^{j-2i} - q^{2i-j}\big) F^i v_{\epsilon q^j} \\
&= \frac{\alpha \epsilon}{q - q^{-1}} F^j v_{\epsilon q^j},
\end{aligned}$$

where $\alpha = \sum_{i=0}^{j} q^{j-2i} - q^{2i-j} = (q^j + q^{j-2} + \cdots + q^{-j}) - (q^{-j} + q^{-j+2} + \cdots + q^j) = 0$. Thus $F^{j+1} v_{\epsilon q^j}$ is $U_q(\mathfrak{sl}_2)$-maximal as claimed.

Finally, we show the last assertion, for which we need the first part of this lemma, as well as Eqs. (3.9), (3.8). We compute:

$$\begin{aligned}
X F^{j+1} v_{\epsilon q^j} &= F^{j+1} X v_{\epsilon q^j} - q^j \big\{q^{j+1}\big\} F^j Y K^{-1} v_{\epsilon q^j} \\
&= F^{j+1} \cdot \left(-\frac{\alpha_{r,n-j+1}}{\{\epsilon q^{j+2}\}} v_{\epsilon q^{j+1}}\right) - q^j \big\{q^{j+1}\big\} \epsilon q^{-j} F^j Y v_{\epsilon q^j} \\
&= -\frac{\alpha_{r,n-j+1}}{\{\epsilon q^{j+2}\}} F^{j+1} v_{\epsilon q^{j+1}} - \big\{\epsilon q^{j+1}\big\} F^j Y v_{\epsilon q^j} \\
&= -F^j \big\{\epsilon q^{j+1}\big\} \left(Y v_{\epsilon q^j} + \frac{\alpha_{r,n-j+1}}{\{\epsilon q^{j+1}\}\{\epsilon q^{j+2}\}} F v_{\epsilon q^{j+1}}\right) \\
&= -\big\{\epsilon q^{j+1}\big\} F^j \big(Y v_{\epsilon q^j} + d_{r,n-j+1} F v_{\epsilon q^{j+1}}\big) \\
&= -\big\{\epsilon q^{j+1}\big\} F^j v_{\epsilon q^{j-1}}
\end{aligned}$$

and we are done. □

8. Category $\mathcal{O}$

Our goal in this section is to show that the category $\mathcal{O}$ (defined below) is a highest weight category in the sense of [5] and that it can be decomposed into a direct sum of subcategories ("blocks"), each of which contains only finitely many simple modules.[1] We retain our assumption that $C_0 \neq 0$.

Definition 8.1. The category $\mathcal{O}$ consists of all finitely generated A-modules with the following properties:

(1) The K-action is diagonalizable with finite-dimensional weight spaces.
(2) The B_+-action is locally finite.

[1] The original paper [3] achieves this using the eigenvalues of the Casimir operator. However, unlike their case, we will see later that our algebra A has trivial center (if $C_0 \neq 0$). Therefore, such an approach fails and we have to do more work (similarly to [12]).

Given $r \in \Bbbk^\times$, we claim there exist only finitely many $t = q^{-n}r$ such that $Z(t) \hookrightarrow Z(r)$. If we have such an embedding, then $\alpha_{r,n+1} = 0$ by Theorem 5.1, so we have to see that this is true for only finitely many n if r is fixed. Proposition 3.12 says that $\alpha_{r,n+1}$ is a nonvanishing function, multiplied by a polynomial in q^{n+1}, if r is fixed. This polynomial can be factored as $\prod_{i=1}^{L}(q^{n+1} - z_i)$ where $z_1, \dots, z_L$ are the roots of the polynomial. This will be zero only for values of n such that $q^{n+1} = z_i$ for some i; since q is assumed not to be a root-of-unity, there are only finitely many such n.

We claim also that, fixing r, there are only finitely many s of the form $s = q^n r$ such that $Z(r) \hookrightarrow Z(s)$. This is because, if we have such an embedding, then $\alpha_{s,n+1} = 0$ by Theorem 5.1 and since for fixed r, $\alpha_{s,n+1} = \alpha_{q^n r,n+1}$ is (as above) essentially a polynomial in q^{n+1} (by looking at the expansion of $b(S,T)$ in Proposition 3.12), it vanishes for only finitely many values of n.

Let us fix r and consider the *maximal* $n \geqslant 0$ so that $\alpha_{q^n r,n+1} = 0$. That such an N exists follows from the observation (in the previous paragraph) that the set of such n is finite. Set $r_0 = q^N r$, so $\alpha_{r_0,N+1} = 0$.

Define $S(r)$ to be the set of all $t = q^{-m} r_0$, so that $\alpha_{r_0,m+1} = 0$. This is a finite set.

We now introduce a graph structure on $\Bbbk^\times$ by connecting t and r by an edge if and only if $Z(r)$ has a simple subquotient $V(t)$ or $Z(t)$ has a simple subquotient $V(r)$. The component of this graph containing r is denoted $T(r)$.

Proposition 8.2.

(1) *If $t \in S(r)$, then $S(t) = S(r)$.*
(2) *For each $r \in \Bbbk^\times$, $T(r) \subset S(r)$. In particular, $T(r)$ is finite for each r.*
(3) *Every Verma module has finite length.*

Proof. The proof of part (1) is in two parts. The first one is the following equality:

$$\alpha_{q^n r,n+m+1} = \alpha_{q^n r,n+1} + \alpha_{r,m+1}. \tag{8.3}$$

We provide a proof of this equality using the definition of α:

$$\begin{aligned}
\alpha_{q^n r,n+m+1} &= \sum_{j=0}^{n+m-1} \{q^{1-j}(q^n r)\} c_{0,q^{-j}(q^n r)} \\
&= \sum_{j=0}^{n-1} \{q^{1-j}(q^n r)\} c_{0,q^{-j}(q^n r)} + \sum_{j=n}^{n+m-1} \{q^{1-j}(q^n r)\} c_{0,q^{-j}(q^n r)} \\
&= \alpha_{q^n r,n+1} + \sum_{i=0}^{m-1} \{q^{1-i-n}(q^n r)\} c_{0,q^{-i-n}(q^n r)} \\
&= \alpha_{q^n r,n+1} + \sum_{i=0}^{m-1} \{q^{1-i} r\} c_{0,q^{-i} r} \\
&= \alpha_{q^n r,n+1} + \alpha_{r,m+1}.
\end{aligned}$$

Now suppose $t \in S(r)$, so $t = q^{-l} r_0$ and $\alpha_{r_0,l+1} = 0$. We define t_0 similarly to r_0, so, in particular, $t_0 = q^T t$ and $\alpha_{t_0,T+1} = 0$. We claim that $t_0 = r_0$, which implies that $S(t) = S(r)$.

Note that, by the maximality of t_0, $l \leqslant T$. We have to show that $l = T$, so assume that, on the contrary, $l < T$. Equation (8.3) along with $\alpha_{t_0,T+1} = \alpha_{r_0,l+1} = 0$ implies that $\alpha_{t_0,T-l+1} = 0$. This last equality, now in conjunction with $\alpha_{r_0,N+1} = 0$ and Eq. (8.3), yields $\alpha_{t_0,T-l+N+1} = 0$. Since $t_0 = q^{N+T-l} r$ and $N + T - l > N$, this contradicts the maximality of N. Therefore, $T = l$.

The proof of part (2) is also in two steps. First, we need the following observation: Theorems 6.1 and 7.1 state that if $V(t)$ is a subquotient of $Z(r)$, then $t = q^{-m} r$ for some root m of $\alpha_{r,m+1}$. The second step consists in showing that $S(t) = S(r)$ if $V(t)$ is a subquotient of $Z(r)$. From part (1), it is enough to show that $t \in S(r)$. If $V(t)$ is a simple subquotient of $Z(r)$ with $t = q^{-m} r$, then $\alpha_{r,m+1} = 0$, and combining this with $\alpha_{r_0,N+1} = 0$ and Eq. (8.3), we get $\alpha_{r_0,m+N+1} = 0$. Since $t = q^{-m-N} r_0$, this means exactly that $t \in S(r)$.

The general case of an arbitrary $t \in T(r)$ follows from the specific case that we just considered.

Part (3) is a consequence of part (2) and of the fact that the simple quotients of a Verma module occur with finite multiplicities, which, in turn, is a consequence of the fact that the weight spaces of every Verma module are finite-dimensional. □

Definition 8.4. A finite filtration $M = F^0 \supset F^1 \supset \cdots \supset F^r = \{0\}$ of a module $M \in \mathcal{O}$ is said to be *standard* if F^i / F^{i+1} is a Verma module for all i.

We construct some useful modules which admit such a filtration. Let $a \in \Bbbk^\times$, $l \in \mathbb{Z}_{\geqslant 0}$; define $Q(l)$ to be the A-module induced from the B_+-module B_+/N_+^l, and define $Q(a,l)$ to be the A-module induced from the B_+-module $\mathcal{B}_{a,l} := B_+/((K-a), N_+^l)$, so $Q(a,l) = A \otimes_{B_+} \mathcal{B}_{a,l}$ is a quotient of $Q(l) = A \otimes_{B_+} B_+/N_+^l$. Notice that the modules $Q(a,l)$ all have standard filtrations, because $N_+^j \mathcal{B}_{a,l} / N_+^{j+1} \mathcal{B}_{a,l}$ is a B_+-module on which N_+ acts trivially, and $\Bbbk[K, K^{-1}]$ semisimple.

Moreover, given any module $M \in \mathcal{O}$ and a weight vector $m \in M$ of weight a, there exists a nonzero homomorphism $f : Q(a,l) \to M$ for some l, taking $\bar{1}$ to m, where $\bar{1}$ is the generator $1 \otimes 1$ in $Q(a,m)$. This is because N_+ acts nilpotently on m.

Proposition 8.5. *Every module in $\mathcal{O}$ is a quotient of a module which admits a standard filtration.*

Proof. Let M be an arbitrary module in $\mathcal{O}$. Since M is finitely generated over A, M is Noetherian according to Corollary 2.3. Choose a nonzero weight vector m_1 in the weight space M_{a_1} (for some $a_1 \in \Bbbk^\times$), and an arbitrary nonzero homomorphism $f_1 : Q(a_1, l_1) \to M$ for some l_1, and set $N_1 = \operatorname{im}(f_1)$. If $N_1 \neq M$, choose another homomorphism $f_2 : Q(a_2, l_2) \to M$ such that $N_1 \subsetneq N_1 + N_2$, where $N_2 = \operatorname{im}(f_2)$ (this is possible by the remark above).

Repeating this procedure, we get an increasing chain of submodules $N_1 \subsetneq N_1 + N_2 \subsetneq N_1 + N_2 + N_3 \subsetneq \cdots$ which must stabilize since M is Noetherian. This implies that $M = N_1 + N_2 + \cdots + N_k$ for some k. It was observed above that $Q(a_1, l_1) \oplus \cdots \oplus Q(a_k, l_k)$ has a standard filtration. □

Proposition 8.6. *Every module in $\mathcal{O}$ has finite length.*

Proof. This is an immediate consequence of Proposition 8.5 and of the fact that Verma modules have finite length—see Proposition 8.2. □

We introduce the following partial order on $\Bbbk$: $t \leqslant s$ if and only if $t = q^l s$ for some $l \in \mathbb{Z}_{\leqslant 0}$.

Proposition 8.7.

(1) *If* $s \notin T(r)$*, then* $\operatorname{Ext}^1_{\mathcal{O}}(V(r), V(s)) = 0$ *and* $\operatorname{Ext}^1_{\mathcal{O}}(Z(r), Z(s)) = 0$.
(2) *Simple modules have no self-extensions.*

We omit the proof of the preceding proposition, which is same as the corresponding statements for H_f; see [12, Theorem 4]. We only need to show the existence of a "good" duality functor $\mathscr{F}$ as in [12, Section 2]. We do this now.

Remark 8.8. It is easy to check that the following define an anti-involution i of A:

$$\mathsf{i}(E) = -FK, \qquad \mathsf{i}(F) = -K^{-1}E, \qquad \mathsf{i}(K) = K, \qquad \mathsf{i}\bigl(K^{-1}\bigr) = K^{-1},$$
$$\mathsf{i}(X) = Y, \qquad \mathsf{i}(Y) = X.$$

Note, in particular, that $\mathsf{i}(C) = C$.

Definition 8.9. Define the *duality functor* $\mathscr{F}$ on $\mathcal{O}$ as follows: if $M \in \mathcal{O}$, then let $\mathscr{F}(M)$ be the linear span of all K-semisimple vectors in M^*. The A-module structure on $\mathscr{F}(M)$ is given, using the anti-involution i, by

$$(am^*)(m) := m^*\bigl(\mathsf{i}(a)m\bigr) \quad \text{for all } a \in A,\ m \in M,\ m^* \in \mathscr{F}(M).$$

As in [12, Proposition 2], the duality functor $\mathscr{F}$ is exact, contravariant, takes simple objects in $\mathcal{O}$ to themselves, and preserves the formal characters and the set of composition factors, of any (finite length) object in $\mathcal{O}$.

Definition 8.10. Define $\mathcal{O}(r)$ to be the subcategory of all the modules whose simple subquotients $V(t)$ satisfy $t \in T(r)$.

Corollary 8.11. *We have a decomposition* $\mathcal{O} = \bigoplus_r \mathcal{O}(r)$.

Proof. This is an immediate consequence of the vanishing of the $\operatorname{Ext}^1_{\mathcal{O}}$ in the previous proposition. □

Proposition 8.12. *The category* $\mathcal{O}$ *has enough projective objects.*

Proof. Consider a component $T(r)$: we know it is finite. Pick $s \in \Bbbk^\times$. Since $T(r)$ is finite, there exists an integer n_s such that $N_+^{n_s} v = 0$ for any $v \in M_s$ and any $M \in \mathcal{O}(r)$, where M_s is the weight space of M of weight s. For each such s, choose such an n_s.

Since $\operatorname{Ext}^1_{\mathcal{O}}(Z(r_1), Z(r_2)) = 0$ if $T(r_1) \neq T(r_2)$ according to Proposition 8.7, it follows that $Q(s, n_s)$ decomposes as a direct sum $Q(s, n_s) = \bigoplus_r Q(s, n_s, r)$, where $Q(s, n_s, r)$ is a submodule, all of whose successive subquotients are in $T(r)$. It should be noted that $Q(s, n_s, r) = 0$ if s is not of the form $s = q^l r$. Set $P(s, r) = Q(s, n_s, r)$.

We claim that $P(s, r)$ is projective in $\mathcal{O}$. Indeed, $\operatorname{Hom}_{\mathcal{O}}(P(s, r), V(t)) = 0$ if $t \notin T(r)$, and if $M \in \mathcal{O}(r)$, then $\operatorname{Hom}_{\mathcal{O}}(P(s, r), M) = \operatorname{Hom}_{\mathcal{O}}(Q(s, n_s), M) = M_s$. Since the K-action on M is diagonalizable, $M \to M_s$ is an exact functor from $\mathcal{O}$ to the category of vector spaces. Therefore, $P(s, r)$ is projective. □

Let $V(s)$ be a simple A-module. Then $P(s,s)$ admits an epimorphism onto $V(s)$. Since $P(s,s)$ has finite length, we can express it as the direct sum of finitely many indecomposable projective modules. This implies that there exists an indecomposable direct summand $P(s)$ with a nonzero homomorphism $P(s) \to V(s)$. This module $P(s)$ also admits a standard filtration since it is a direct summand of a module with such a filtration [3].

Proposition 8.13. *The category $\mathcal{O}$ is a highest weight category.*

Proof. The only two points that we have to prove are the following:

(1) If $V(t)$ is a subquotient of $Z(r)$, then $t \leqslant r$ and the multiplicity $[Z(r) : V(r)]$ of $V(r)$ as a subquotient of $Z(r)$ is one.
(2) If $Z(r)$ appears as a subquotient in a standard filtration of $P(s)$, then $s \leqslant r$. Moreover, $Z(s)$ appears exactly once in any such filtration.

The statement (1) is a consequence of the observation that if t is a weight of $Z(r)$, then $t \leqslant r$. Moreover, the weight space of r in $Z(r)$ has dimension one.

The second part follows from the following vanishing result: if $r \nleqslant s$, then we have $\operatorname{Ext}^1_{\mathcal{O}}(Z(r), Z(s)) = 0$, which can be proved exactly as the analogous result (Proposition 4) in [12]. Another approach (see [12, Proposition 11]) is to use the construction of $P(s)$ as a direct summand of $P(s,s)$. $\square$

9. Block decompositions in highest weight categories

In a highest weight category like $\mathcal{O}$, it is possible to define a block decomposition in several different ways. We now show why all these ways yield the same decomposition. To begin with, we can define a decomposition using Verma modules; this is exactly the one given by the sets $T(r)$, and we rephrase this as the condition

$$G_{VZ}: \quad \text{“}V(t) \text{ is a subquotient of } Z(r).\text{”}$$

Thus, $S_{VZ}(r)$ is the graph component of $\Bbbk^\times$ containing r, where we join r and t by an edge if $V(t)$ is a subquotient of $Z(r)$, or $V(r)$ is a subquotient of $Z(t)$.

Recall that there exists an exact contravariant duality functor $\mathscr{F}$ (as mentioned above) that takes simple objects to themselves, and preserves the set of composition factors of any object of finite length in $\mathcal{O}$.

Using this functor $\mathscr{F}$, we can now define $A(r) = \mathscr{F}(Z(r))$ and $I(r) = \mathscr{F}(P(r))$ to be the co-standard and (indecomposable) injective modules, respectively. In a highest weight category like $\mathcal{O}$, every projective module has a standard filtration as above, and BGG Reciprocity also holds (cf. [6]). In other words, $[P(r) : Z(t)] = [Z(t) : V(r)]$ for all t, r.

Definitions.

(1) We define the property G_{ab} by

$$G_{ab}: \quad \text{“}a(t) \text{ is a subquotient of } b(r).\text{”}$$

(2) Given a, b as above, we introduce a graph structure on $\Bbbk^\times$ as follows: connect r and t by an edge r–t if G_{ab} holds for the pair (r,t) or (t,r). Under this structure, we define the connected component of $\Bbbk^\times$ containing r, to be the block $S_{ab}(r)$.
(3) We also have the *categorical definition of linking*: We say r and t are *linked* if there is a chain of *indecomposable* objects $V_i \in \mathcal{O}$ and nonzero maps $f_i \in \mathrm{Hom}_{\mathcal{O}}(V_{i-1}, V_i)$ such that

$$V_0 = V(t) \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} V_n = V(r).$$

(4) We now define the final graph structure on $\Bbbk^\times$ as follows: $B(r)$ is the connected component of $\Bbbk^\times$ containing r, where edges denote linked objects.

We remark that the V_i's need to be indecomposable, otherwise any two objects of $\mathcal{O}$ are linked by $0 \to M \to M \oplus N \to N \to 0$. Also note that the definition of linking is clearly symmetric, using the duality functor $\mathscr{F}$.

We now explain why certain block decompositions of $\Bbbk^\times$ are the same. Using the duality functor, it is easy to see that the conditions G_{VZ} and G_{VA} are the same; hence we have $S_{VZ}(r) = S_{VA}(r)$ for all r. Similarly, G_{VP} and G_{VI} are equivalent, as are G_{ZP} and G_{AI}. We now have the following result.

Theorem 9.1.

(1) *$T(r) = S_{VZ}(r) \subset S_{ZP}(r) \subset S_{VP}(r) \subset B(r)$ for all r.*
(2) *In fact, $T(r) = B(r)$.*

In particular, all the various block classifications coincide in our case.

Proof. (1) We proceed step by step. The philosophy is to show that two vertices connected by an edge in a graph structure are connected in a bigger one.

Suppose r,t are edge-connected in $T(r) = S_{VZ}(r)$. Then, by BGG Reciprocity, $[P(r) : Z(t)] = [Z(t) : V(r)] > 0$. Hence $S_{VZ} \subset S_{ZP}$. Next, if $P(r)$ has a subquotient $Z(t)$, then it clearly has a subquotient $V(t)$ as well. Thus $S_{ZP} \subset S_{VP}$.

Finally, suppose $P(r)$ has a simple subquotient $V(t)$. We show that r and t are linked. We have a sequence of maps

$$0 \to N \to M \to V(t) \to 0, \qquad N \hookrightarrow M \hookrightarrow P(r) \twoheadrightarrow V(r).$$

Since $V(t)$ is indecomposable, we can choose M to be indecomposable as well. Since $\mathcal{O}$ is finite length, we have $V(s) \hookrightarrow N \hookrightarrow M$ for some s. Hence, using the duality functor $\mathscr{F}$, we now have the following sequence of maps linking $V(t)$ and $V(r)$:

$$V(t) \cong \mathscr{F}\big(V(t)\big) \hookrightarrow \mathscr{F}(M) \twoheadrightarrow \mathscr{F}\big(V(s)\big) \cong V(s) \hookrightarrow M \hookrightarrow P(r) \twoheadrightarrow V(r)$$

and we are done.

(2) Since $T(r)$ is finite for all r, results from the previous section tell us that $\mathcal{O} = \bigoplus_r \mathcal{O}(r)$, where each $\mathcal{O}(r)$ is a highest weight category. Now, suppose $V(r)$ and $V(t)$ are linked via a chain $V_0 = V(r) \to V_1 \to \cdots \to V_n = V(t)$. Since each V_i is indecomposable, it is in a unique block $\mathcal{O}(s)$. However, since there are nonzero homomorphisms in between successive V_i's, all

the V_i's are in the same block. In particular, $V(t) \in \mathcal{O}(r)$, so $B(r) \subset T(r)$ for each r, and hence is finite as well. Combining this with part (1) yields $T(r) = B(r)$. □

10. Semisimplicity

As we saw in Section 4, finite-dimensional K-semisimple A-modules are not A-semisimple for some A (or $C_0 = p(C)$). However, we have the following result, that tells us when the result *does* hold.

Theorem 10.1. *The following are equivalent*:

(1) *For each $n \in \mathbb{Z}_{\geqslant 0}$ (and each $r = \epsilon q^n$), $T(r) \subset \{r, s, (q^3 s)^{-1}, (q^3 r)^{-1}\}$, where $s = \epsilon q^m$ for some $m + 1 \in \mathbb{Z}_{\geqslant 0}$.*
(2) *For each $n \in \mathbb{Z}_{\geqslant 0}$ (and each $r = \epsilon q^n$), the equation $\alpha_{r,m} = 0$ has at most one root m satisfying $2 \leqslant m \leqslant n + 1$.*
(3) *For each $r \in \Bbbk^\times$, there is at most one finite-dimensional simple module in $\mathcal{O}(r)$ (up to isomorphism).*
(4) *Every finite-dimensional A-module is completely reducible.*

Proof. Since char $\Bbbk \neq 2$, every finite-dimensional module is K-semisimple, and hence it is an object of the category $\mathcal{O} = \bigoplus_r \mathcal{O}(r)$.

Note from Lemma 7.2 that $t \in S(r)$ if and only if $(q^3 t)^{-1} \in S(r)$, for every r of the form ϵq^n, $n \in \mathbb{Z}_{\geqslant 0}$. This explains the structure of the set in (1) above (since $T(r) \subset S(r)$).

Next, observe that the set of isomorphism classes of simple modules in a block $T(r)$ is in bijection with the block $T(r)$ itself (under the map $t \mapsto V(t)$ for each $t \in T(r)$), and every simple module $V(r)$ is actually in a block $T(r)$. We first show that the first three assertions are equivalent.

If (1) holds, then any simple module in $\mathcal{O}(r)$ is one of the following:

$$V(r), V(s), V\big((q^3 r)^{-1}\big), V\big((q^3 s)^{-1}\big).$$

Since only $V(r)$ is finite-dimensional in the above list, and that too only when $r = \epsilon q^n$ for $n \in \mathbb{Z}_{\geqslant 0}$, (1) implies (3).

Similarly, if (1) does not hold then $S(r)$ contains $t_i = \epsilon q^{m_i}$ $(i = 0, 1, 2)$, where we assume without loss of generality that $m_0 = n > m_1 > m_2$. Then $\alpha_{r,m}$ has two roots by definition of $S(r)$, so (2) does not hold either. In other words, (2) implies (1).

Now suppose that (2) does not hold. Thus there are at least two roots of α_{rm}. By Theorem 3.7, there are weight vectors v_{t_1}, v_{t_2}, say of weights $t_i = \epsilon q^{m_i}$, in the Verma module $Z(r)$, with $-1 \leqslant m_2 < m_1 \leqslant m_0 = n$. But then by Theorem 4.1, there are (at least) two nonisomorphic finite-dimensional simple modules, namely $V(r) = V(t_0)$ and $V(t_1)$ in $\mathcal{O}(r)$, by Theorem 7.1 and Eq. (8.3). Thus (3) does not hold either, meaning that (3) implies (2), and the first three assumptions are shown to be equivalent.

Now suppose (3) holds. We show complete reducibility. Suppose M is a finite-dimensional A-module. Since the category $\mathcal{O}$ splits up into blocks, we have $M = \bigoplus_r M(r)$, where $M(r) \in \mathcal{O}(r)$. Each $M(r)$ is finite-dimensional, hence so are all its subquotients. Hence by assumption, all subquotients of $M(r)$ are of the form $V(r)$. Since $V(r)$ has no self-extensions in $\mathcal{O}$ by Proposition 8.7, this shows (using [12, Proposition A.1] and induction on length, for instance) that $M(r)$ is actually a direct sum of copies of $V(r)$. Hence $M(r)$ is semisimple.

Finally, suppose (1) does not hold; we show that (4) does not hold either. As in Theorem 7.1, let $r = \epsilon q^n$, and let $0 = n_0 < n_1 < \cdots < n_k \leqslant n+1$ be the various roots of $\alpha_{r,m+1}$. Since (1) fails, we have $k \geqslant 2$.

Given $i \geqslant j$, we now define the module $W(i,j)$ to be the A-submodule generated by $\{F^{b+1} v_{\epsilon q^b}\colon n - n_i \leqslant b \leqslant n - n_j\}$ and $Z(t_i)$. For example, $W(i,i) = Z(t_i)$ is a Verma module, and $W(i+1,i) = W(t_i)$ is its unique maximal submodule.

We now consider the filtration

$$Z(t_0) = W(0,0) \supset W(t_0) = W(1,0) \supset W(2,0).$$

This gives a short exact sequence

$$0 \to W(1,0)/W(2,0) \to Z(t_0)/W(2,0) \to Z(t_0)/W(1,0) \to 0$$

or, in other words,

$$0 \to V(t_1) \to Z(t_0)/W(2,0) \xrightarrow{\varphi} V(t_0) \to 0.$$

The middle term is thus a finite-dimensional module of length 2. We claim that there does not exist a splitting of the map φ. This is easy to show: any complement to $V(t_1)$, if it exists, is also K-semisimple, and hence contains the highest weight vector v_{t_0}. But v_{t_0} generates the entire module $Z(t_0)/W(2,0)$, so there cannot exist a complement, and (4) fails, as claimed. □

Remark 10.2. Note that the condition (2) above depends on the polynomial p, or in other words, on the central element $C_0 = p(C)$, by means of the polynomial $\alpha_{r,m}$. Furthermore, there *are* central elements C_0 in $U_q(\mathfrak{sl}_2)$, that satisfy the condition above. We give such an example now. (Note that in the case $C_0 = 0$, complete reducibility was violated.)

Example of complete reducibility

Standing Assumption. For this example, q is assumed to be transcendental over $\mathbb{Q}$.

Take $p(C) = C_0 = (q - q^{-1})^3 C - (q - q^{-1})(q^{-2} + q^2)$. (Note that if A satisfies complete reducibility (as above) for this p, then it does so for any scalar multiple of p.) We now show that the only finite-dimensional simple module is $V(q) = V_C(q)$, of dimension 2 over $\Bbbk$.

Let us first calculate $\alpha_{r,m}$, using the (computations in the) proof of Proposition 3.12. Clearly, we have $h(qT) = [(qT + q^{-1}T^{-1}) - (q^{-2} + q^2)](q - q^{-1})$, so

$$g(qT) = h(qT)\{qT\} = \left[(qT)^2 - (qT)^{-2}\right] - \left(q^{-2} + q^2\right)\left(qT - q^{-1}T^{-1}\right).$$

Hence $g(T) = (T^2 - T^{-2}) - (q^{-2} + q^2)(T - T^{-1})$. Summing up, as in the proof of Proposition 3.12, we obtain that $\alpha_{r,m}$ equals

$$\left[\frac{r^2(q^{2-2m} - 1)q^2}{q^{-2} - 1} - \frac{r^{-2}(q^{2m-2} - 1)q^{-2}}{q^2 - 1}\right] - \left(q^{-2} + q^2\right)\left[\frac{r(q^{1-m} - 1)q}{q^{-1} - 1} - \frac{r^{-1}(q^{m-1} - 1)q^{-1}}{q - 1}\right].$$

We take the "best possible" common factor. Then we get that this equals

$$\frac{(q^{m-1}-1)}{q^2r^2(q^2-1)}\big[q^6r^4q^{2-2m}\big(q^{m-1}+1\big)-\big(q^{m-1}+1\big) \\ -\big(q^{-2}+q^2\big)\big(q^4r^3q^{1-m}(q+1)-(q+1)rq\big)\big].$$

Put $q^{m-1}=T$. Then we get

$$\begin{aligned}&\frac{T-1}{q^2r^2(q^2-1)}\big[(T+1)\big(\big(q^3r^2T^{-1}\big)^2-1\big)-\big(q^{-2}+q^2\big)qr(q+1)\big(q^3r^2T^{-1}-1\big)\big]\\ &\quad=\frac{(T-1)}{q^2r^2(q^2-1)}\frac{q^3r^2-T}{T^2}\big[(T+1)\big(q^3r^2+T\big)-\big(q^{-2}+q^2\big)qr(q+1)T\big]\\ &\quad=\beta\big[(T+1)\big(q^3r^2+T\big)-\big(q^{-2}+q^2\big)qr(q+1)T\big],\quad \text{say.}\end{aligned}$$

We now show that condition (1) (of Theorem 10.1) is satisfied. If we fix $n\in\mathbb{Z}_{\geqslant 0}$ and $r=\epsilon q^n$, then we want to show that there is at most one root m of the equation $\alpha_{r,m}=0$, for this is equivalent to condition (1), by Theorem 3.7.

We know that $m\geqslant 2$, and q is not a root of unity, hence most of the terms in β above are nonzero. The only term we need to consider is q^3r^2-T. However, if $r=\epsilon q^n$, then this equals $q^{3+2n}-q^{m-1}$, and for this to vanish, we need $m=2n+4$. Clearly, this is impossible, since we desire $\alpha_{\epsilon q^n,m}$ to vanish for some $2\leqslant m\leqslant n+1$. Thus $\beta\neq 0$, so we can cancel it.

We thus need to show that if we fix $r=\epsilon q^n$, then there is at most one solution of the form $T=q^{m-1}$, to the equation

$$(T+1)\big(q^3r^2+T\big)-\big(q^{-2}+q^2\big)qr(q+1)T=0 \tag{10.3}$$

where $2\leqslant m\leqslant n+1$.

Clearly, there are no solutions when $n=0$, since $n+1<2$. The next case is $n=1$. The equation then becomes

$$(T+1)\big(q^5+T\big)=\epsilon T\big(q^4+1\big)(q+1).$$

We need $2\leqslant m\leqslant 2$ to be a solution, i.e. $T=q$.

Taking $\epsilon=1$, we get $T^2+q^5=T(q+q^4)$, which holds for $T=q,q^4$. Hence there is a unique root $T=q$, as desired.

On the other hand, if $\epsilon=-1$, then evaluating at $T=q$, and canceling $(q+1)(q^5+q)$ from both sides (since q is not a root of unity), we get $1=-1$, a contradiction since char $\Bbbk\neq 2$. Hence, there is no root in this case.

Finally, take $n>1$. We claim, in fact, that there is no root of Eq. (10.3), of the form $T=q^{m-1}$. Simply plug in $T=q^{m-1}$ and $r=\epsilon q^n$ above, and multiply both sides by q; we get

$$q\big(q^{m-1}+1\big)\big(q^{2n+3}+q^{m-1}\big)=\epsilon\big(q^4+1\big)(q+1)q^{n+m-1}.$$

By the assumption that q is transcendental, we must have the highest degree terms on both sides to be the same. On the right-hand side, the highest degree is $4+1+(n+m-1)=n+m+4$. On the left-hand side, we have $m-1 \leqslant n < 2n+3$, so the highest degree is $1+(m-1)+(2n+3) = 2n+m+3$. These are equal only when $n=1$, so there is no root for $n>1$.

We conclude that $V(q)$ is the unique finite-dimensional simple A-module (because char $\Bbbk \neq 2$). Since it has no self-extensions (by Proposition 8.7 above), every finite-dimensional module is a direct sum of copies of $V(q)$, and hence, completely reducible.

Finally, we mention that we have similar results and (counter)examples in the case of H_f (cf. [12]). Complete reducibility holds if and only if every block $\mathcal{O}(r)$ contains at most one finite-dimensional simple module, if and only if every $T(r)$ intersects $\mathbb{Z}$ in at most four elements.

11. Center

We will show in this section that the center of A is trivial if $C_0 \neq 0$. Consequently, we cannot use the same approach as in [3] to decompose $\mathcal{O}$ into blocks. This is why we had to follow a different approach in Section 8.

Theorem 11.1. *The center of A is the set of scalars $\Bbbk$ when $C_0 \neq 0$.*

The proof is in two parts. The first part is the following proposition.

Proposition 11.2. *If $a \in \mathfrak{Z}(A)$, then $\xi(a) \in \Bbbk$, i.e. the purely CSA part of a is a scalar.*

For the sake of simplicity, we denote $\mathcal{A} = \Bbbk[K, K^{-1}]$. Following [11, Section 1.6], for $j \in \mathbb{Z}$, we define the operator $\gamma_j : \mathcal{A} \to \mathcal{A}$ by: $\gamma_j(\varphi(K)) = \varphi(q^j K)$. Now define $\eta_j : A \to \mathcal{A}$ by $\eta_j(a) = \gamma_j(\xi(a))$. For example, $\eta_0(C_0) = \gamma_0(\xi(C_0)) = \xi(C_0)$.

Set $a_0 = q\eta_0(C_0) + \eta_{-1}(C_0)$. We claim that $a_0 \neq 0$ if $C_0 \neq 0$. The Casimir is $C = \xi(C) + FE$, and $C_0 = p(C) \in p(\xi(C)) + F \cdot A \cdot E$. Hence $\xi(C_0) = p(\xi(C)) \neq 0$.

Thus, if $\eta_0(C_0) = \xi(C_0) = \alpha_n K^n + l.o.t.$, then

$$a_0 = q\eta_0(C_0) + \eta_{-1}(C_0) = \alpha_n\big(q + q^{-n}\big)K^n + l.o.t.$$

Clearly, $\alpha_n \neq 0$, so if $a_0 = 0$ we must have $q^{-n} = -q$, or $q^{n+1} = -1$, whence $q^{2n+2} = 1$. Since q is not a root of unity, this means that $n = -1$, hence

$$a_0 = 2\alpha_{-1} q K^{-1} + l.o.t. \neq 0,$$

which is a contradiction, since char $\Bbbk \neq 2$.

Before proving the proposition above, we need the following lemma.

Lemma 11.3. *We have the following commutation relations in A:*

(1) *If $U \in A_{q^j}$ (i.e. $KUK^{-1} = q^j U$) and $\varphi(K) \in \mathcal{A}$, then $\varphi(K)U = U\gamma_j(\varphi(K)) = U\eta_j(\varphi(K))$. Further, when written in the PBW basis,*
(2) *the component in $Y \cdot \mathcal{A}$ of $[X, Y^2]$ is $-Ya_0$,*
(3) *the component in $\mathcal{A}$ of $[E, Y^2]$ is $-\eta_0(C_0)$,*
(4) *the component in $\mathcal{A} \cdot X$ of $[X^2, Y]$ is $-a_0 X$,*
(5) *the component in $\mathcal{A}$ of $[F, X^2]$ is $-\eta_0(C_0)K^{-1}$.*

Proof of the lemma. (1) This is obvious.

(2) We compute: $[X, Y^2] = XY^2 - Y^2X$, so the component in $Y \cdot \mathcal{A}$ is obtained only from XY^2. We have $XY^2 = (qYX - C_0)Y = qY(XY) - C_0Y = qY(qYX - C_0) - C_0Y = q^2Y^2X - qYC_0 - C_0Y$ from the defining relations.

We need to rewrite C_0Y in the PBW basis and find the component in $Y \cdot \mathcal{A}$. Clearly, $(C_0 - \xi(C_0))Y \in A \cdot EY = A \cdot (X + q^{-1}YE)$, and hence this contributes nothing. So the only contribution is from $\xi(C_0)Y$, which from above equals $Y\xi(C_0)(q^{-1}K) = Y\eta_{-1}(C_0)$.

In conclusion, we obtain that the desired component is $-qY\xi(C_0) - Y\eta_{-1}(C_0) = -Ya_0$, as claimed.

(3) Once again, we compute: Y^2E can give no such component, so the only component from $[E, Y^2]$ comes from $EY^2 = (X + q^{-1}YE)Y = XY + q^{-1}Y(X + q^{-1}YE)$. Once again, the only contribution comes from $XY = qYX - C_0$, and hence the component of $[E, Y^2]$ in $\mathcal{A}$ is $-\xi(C_0) = -\eta_0(C_0)$.

(4) This is similar to above: $[X^2, Y]$ and X^2Y have the same component, which comes from $X^2Y = X(qYX - C_0) = q(XY)X - XC_0 = q^2YX^2 - qC_0X - XC_0$. The contribution of XC_0 comes from $X\xi(C_0) = \eta_{-1}(C_0)X$, and the contribution from C_0X is $\xi(C_0)X = \eta_0(C_0)X$. Hence the total contribution is $-q\eta_0(C_0)X - \eta_{-1}(C_0)X = -a_0X$.

(5) Finally, the component in $\mathcal{A}$ comes from $-X^2F = -X(FX - YK^{-1}) = -XFX + (XY)K^{-1} = -XFX + qYXK^{-1} - C_0K^{-1}$. Clearly, only the last term has a nonzero component in $\mathcal{A}$, which is $-\xi(C_0)K^{-1}$, as claimed. □

Proof of the proposition. Given $a \in \mathfrak{Z}(A)$, we write a as a linear combination of PBW basis elements. Note that $KaK^{-1} = a$, whence the only basis elements that can contribute to a are of the form $F^aY^bK^cX^dE^e$ where $2a + b = d + 2e$.

We can write a in the form

$$a = \xi(a) + Yb_1X + Y^2b_2X^2 + Fb_3X^2 + Fb_4E + Y^2b_5E + h.o.t.$$

Here, $h.o.t.$ denotes higher order terms in E, X (i.e. $h.o.t.$ is in the left ideal generated by E^2, EX, X^3) and the b_i's are Laurent polynomials in K.

Step 1. Obtain equations relating the coefficients b_i.

We now use the fact that a commutes with X, Y, E, F to equate various coefficients to zero. We have to consider six different cases.

Case 1. The component in $Y \cdot \mathcal{A} \cdot E$, of $[X, a]$, is zero.

Clearly, if $b \in A$, then $[X, bX] = XbX - bX^2 \in A \cdot X$, by the PBW Theorem. Similarly, $[X, bE^2] \in AE^2$, and $[X, bEX] \in A \cdot EX$. Hence $[X, h.o.t.]$ still gives us only higher order terms. In fact, from this analysis, we see that we only need to consider $[X, Fb_4E + Y^2b_5E]$ for the above coefficient. We have

$$[X, Fb_4E] = [X, F]b_4E + F[X, b_4E] = [X, F]b_4E + F[X, b_4]E + Fb_4[X, E],$$

and the second and third terms are clearly in $A \cdot EX$. Hence we only need to consider the first term. The same is true for $[X, Y^2b_5E]$.

Hence we conclude that, to compute the above coefficient, we only need to look at

$$[X, F]b_4E + \big[X, Y^2\big]b_5E.$$

From the lemma, the contribution is $-YK^{-1}b_4E - Ya_0b_5E$. If this is to be zero, then we obtain

$$b_4 = -Ka_0b_5. \tag{11.4}$$

Case 2. The component in $Y \cdot \mathcal{A} \cdot X^2$ of $[X, a]$ is zero.

Once again, by a similar analysis, we see that we only need to look at

$$Y[X, b_1]X + [X, F]b_3X^2 + \big[X, Y^2\big]b_2X^2$$

and the contribution is $Y[\eta_{-1}(b_1) - \eta_0(b_1)]X^2 - YK^{-1}b_3X^2 - Ya_0b_2X^2$ from the lemma. If this is to be zero, then we obtain

$$b_3 = K\big(\eta_{-1}(b_1) - \eta_0(b_1)\big) - Ka_0b_2. \tag{11.5}$$

Case 3. The component in $\mathcal{A} \cdot X$ of $[X, a]$ is zero.

In this case the contribution comes from $[X, \xi(a)] + [X, Y]b_1X$. Using the lemma, we simplify this to $(\eta_{-1}(a) - \eta_0(a))X - \eta_0(C_0)b_1X = 0$. Hence

$$\eta_{-1}(a) - \eta_0(a) = \eta_0(C_0)b_1. \tag{11.6}$$

Case 4. The component in $\mathcal{A} \cdot E$, of $[E, a]$, is zero.

In this case we look at $([E, \xi(a)] + [E, F]b_4 + [E, Y^2]b_5)E$, which, from the lemma above, contributes $(\eta_{-2}(a) - \eta_0(a) + \{K\}b_4 - \eta_0(C_0)b_5)E$. If this is zero, then we get

$$\eta_{-2}(a) - \eta_0(a) = -\{K\}b_4 + \eta_0(C_0)b_5 = \big(\eta_0(C_0) + K\{K\}a_0\big)b_5 \tag{11.7}$$

where the last equality follows from Eq. (11.4) above.

Case 5. The component in $\mathcal{A} \cdot X^2$ of $[E, a]$ is zero.

In this case we look at $[E, Y]b_1X + [E, Y^2]b_2X^2 + [E, F]b_3X^2$, which, from the lemma above, contributes $Xb_1X - \eta_0(C_0)b_2X^2 + \{K\}b_3$. If the contribution from this is zero, then we get

$$\eta_0(C_0)b_2 = \eta_{-1}(b_1) + \{K\}b_3. \tag{11.8}$$

Case 6. The component in $F \cdot \mathcal{A} \cdot X$ of $[Y, a]$ is zero.

In this case the contribution comes from $-Fb_3[X^2, Y] - Fb_4[E, Y]$. Using the lemma, we simplify this to $Fa_0b_3X - Fb_4X = 0$. Hence

$$b_4 = a_0b_3. \tag{11.9}$$

Step 2. Solve the above system for the b_i's.

We now use these equations. From Eqs. (11.4) and (11.9), we get that $a_0(b_3 + Kb_5) = 0$. We proved at the beginning of this section that $a_0 \neq 0$. Hence $b_3 = -Kb_5$.

Multiplying Eq. (11.5) by $\eta_0(C_0)$, and using Eq. (11.8), we get

$$\eta_0(C_0)b_3 = \eta_0(C_0)K\big(\eta_{-1}(b_1) - \eta_0(b_1)\big) - Ka_0\big(\eta_{-1}(b_1) + \{K\}b_3\big)$$

so that

$$\big(\eta_0(C_0) + K\{K\}a_0\big)b_3 = -K\big[\big(a_0 - \eta_0(C_0)\big)\eta_{-1}(b_1) + \eta_0(C_0)\eta_0(b_1)\big]$$

and this equals $-K(\eta_0(C_0) + K\{K\}a_0)b_5$ because $b_3 = -Kb_5$. Using Eq. (11.7), and dividing by $-K$, we finally get

$$\begin{aligned}\eta_{-2}(a) - \eta_0(a) &= \big(a_0 - \eta_0(C_0)\big)\eta_{-1}(b_1) + \eta_0(C_0)\eta_0(b_1)\\ &= \big[(q-1)\eta_0(C_0) + \eta_{-1}(C_0)\big]\eta_{-1}(b_1) + \eta_0(C_0)\eta_0(b_1)\\ &= (q-1)\eta_0(C_0)\eta_{-1}(b_1) + \big(\eta_{-1}(C_0)\eta_{-1}(b_1) + \eta_0(C_0)\eta_0(b_1)\big).\end{aligned}$$

Thus we finally get, using (11.6),

$$\eta_{-2}(a) - \eta_0(a) = (q-1)\eta_0(C_0)\eta_{-1}(b_1) + \big(\eta_{-2}(a) - \eta_{-1}(a)\big) + \big(\eta_{-1}(a) - \eta_0(a)\big)$$

so that

$$(q-1)\eta_0(C_0)\eta_{-1}(b_1) = 0.$$

The above holds in $\mathcal{A}$. Since $(q-1)\eta_0(C_0) = (q-1)\xi(C_0) \neq 0$ by assumption, $\eta_{-1}(b_1) = 0$. Finally, applying η_{-1} to Eq. (11.6), we get that $\eta_{-2}(a) = \eta_{-1}(a)$. But if $\xi(a) = \sum_i \alpha_i K^i$, then this gives $\alpha_i q^{-i} = \alpha_i q^{-2i}$ for all i. Since q is not a root of unity, the only nonzero coefficient is α_0 and $\xi(a) = \alpha_0$ is indeed a scalar, as claimed. □

To complete the proof that the center is trivial, we use the PBW form of the basis. The lemma below says that for any "purely non-CSA" element $\beta \neq 0$, we can find $w_r \in Z(r)$ (for "most" $r \neq \pm q^n$) so that $\beta w_r \neq 0$ in $Z(r)$. In fact, we explicitly produce such a w_r.

Suppose we are given $\beta \in A$ so that $\xi(\beta) = 0$, and $\beta \neq 0$. We can write β in the PBW form $\beta = \sum_i \beta_i p_i(K) X^{d_i} E^{e_i}$. Here, $\beta_i \in \Bbbk[Y, F]$ and p_i's are Laurent polynomials in one variable. Choose i so that $e = e_i$ is the least among all e's, and among all j's with $e_j = e$, the least value of d_j is $d = d_i$. Without loss of generality, we may assume $i = 0$.

Lemma 11.10. *There exists a finite set $T \subset \Bbbk$ with $0 \in T$ such that if $r \neq \pm q^n$, $r \notin T$ and if $w_r = F^e v_{q^{-d}r}$, then $\beta w_r \in \Bbbk^\times \beta_0 v_r$.*

Proof. We work in the Verma module $Z(r)$, where $r \neq \pm q^n$ for any $n \geqslant 0$. We define $w_r = F^e v_{q^{-d}r}$ and compute $X^{d_i} E^{e_i} w_r$.

Since $v_{q^{-d}r}$ is annihilated by E, it generates a $U_q(\mathfrak{sl}_2)$-Verma module $Z_C(q^{-d}r)$, and by $U_q(\mathfrak{sl}_2)$-theory we observe that $E^eF^ev_{q^{-d}r}$ is a nonzero scalar multiple of $v_{q^{-d}r}$ (by [11, Proposition 2.5] the Verma module is simple, so the only vector killed by E is $v_{q^{-d}r}$).

Next, using Eq. (3.9), an easy induction argument shows that

$$X^d v_{q^{-d}r} = (-1)^d \prod_{i=1}^{d} \frac{\alpha_{r,d+2-i}}{\{q^{1+i-d}r\}} v_r. \tag{11.11}$$

For each fixed i, the expression $\alpha_{r,d+2-i}$ is a nonzero (Laurent) polynomial in r, hence it has a finite set of roots. We now define the finite set T of "bad points." Recall that we wrote $\beta = \sum_i \beta_i p_i(K)X^{d_i}E^{e_i}$. Define T to be the union of the (finite) set of roots of p_0, 0, and the (finite) set of roots r of all the $\alpha_{r,d+2-i}$ for $1 \leqslant i \leqslant d$.

Finally, we compute $X^{d_i}E^{e_i}w_r$. There are two cases:

(a) $e_i > e$, in which case $E^{e_i}w_r = E^{e_i-e-1}(E^{e+1}F^ev_{q^{-d}r}) = 0$ by $U_q(\mathfrak{sl}_2)$-theory; or

(b) $e_i = e$ $(i = 0)$, in which case $X^{d_i}E^ew_r = X^{d_i-d}(X^dE^eF^ev_{q^{-d}r})$. From above, if $r \notin T$, then this is $X^{d_i-d}cv_r$ for some nonzero scalar c. Thus, we get a nonzero vector if and only if $d_i = d$ since v_r is maximal.

Thus, $\beta w_r = c\beta_0 p_0(K)v_r = c\beta_0 p_0(r)v_r$. Hence $\beta w_r = (cp_0(r))(\beta_0 v_r)$ and $cp_0(r) \neq 0$ for all $r \notin T$, $r \neq \pm q^n$. □

Proof of Theorem 11.1. Suppose $a = \xi(a) + \beta \in \mathfrak{Z}(A)$, $\beta \notin \mathcal{A}$ and $\beta \neq 0$. Let us look at how a acts on $w_r = F^e p_{d,r}(Y,F)v_r$ (as above), with $r \notin T$ and $r \neq \pm q^n$. We know $\beta w_r = f(r)\beta_0 v_r$, $f(r) \in \Bbbk^\times$. Now, $a(F^e p_{d,r}) = (F^e p_{d,r})a$, since a is central. Thus, $aw_r = F^e p_{d,r}(Y,F)av_r$, i.e. $\xi(a)w_r + \beta w_r = F^e p_{d,r}(Y,F)\xi(a)v_r + F^e p_{d,r}(Y,F)\beta v_r = \xi(a)(r)w_r + 0 = \xi(a)(r)w_r$.

Thus, $f(r)\beta_0 v_r = (\xi(a)(r) - \xi(a)(q^{-n}r))w_r$ for some n, i.e.

$$\bigl(\xi(a)(r) - \xi(a)\bigl(q^{-n}r\bigr)\bigr)F^e p_{d,r}(Y,F) = f(r)\beta_0, \quad \text{for all } r \notin T,\ r \neq \pm q^n.$$

But from the above proposition, $\xi(a)$ is a constant, so $\beta_0 = 0$ because $f(r) \neq 0$. This contradicts our assumption that $\beta \neq 0$. Therefore, $\beta = 0$ and we conclude that $a = \xi(a) \in \Bbbk^\times$ so that the center is trivial. □

12. Counterexamples

We provide counterexamples for two questions:

(1) Is every Verma module $Z(r)$ a direct sum of $U_q(\mathfrak{sl}_2)$-Verma modules

$$Z_C(r) \oplus Z_C\bigl(q^{-1}r\bigr) \oplus \cdots?$$

(2) If $\alpha_{r,n+1} = 0$, is it true that $Z(q^{-n}r) \hookrightarrow Z(r)$?

The answers to both questions are: no.

(1) The structure equations guarantee, for $r = \epsilon q^n$, that $v_{\epsilon q^{-1}}$ can be defined, and is $U_q(\mathfrak{sl}_2)$-maximal. However, if $Z(r)$ is to decompose into a direct sum of $Z_C(r')$'s (as above), then we need a monic polynomial $h(Y,F) = Y^{n+2} + l.o.t.$, so that $v_{\epsilon q^{-2}} = h(Y,F)v_r$ is $U_q(\mathfrak{sl}_2)$-maximal.

Now, $EYv_{\epsilon q^{-1}} = Xv_{\epsilon q^{-1}} = -\alpha_{r,n+2}v_\epsilon$, by Eq. (3.9). By $U_q(\mathfrak{sl}_2)$-theory, $EF^{l+1}v_{\epsilon q^{2l}} \in \Bbbk^\times F^l v_{\epsilon q^{2l}}$ for each $l > 0$. Thus, if there exists a $U_q(\mathfrak{sl}_2)$-maximal vector, it has to be a linear combination of $Yv_{\epsilon q^{-1}}$ and Fv_ϵ. However, $EFv_\epsilon = 0$, so the only way $Y^{n+2} + l.o.t.$ is $U_q(\mathfrak{sl}_2)$-maximal, is if $\alpha_{r,n+2} = 0$. By definition of α, this holds if and only if $\alpha_{r,n+3} = 0$.

We conclude that $Z(\epsilon q^n)$ has a $U_q(\mathfrak{sl}_2)$-Verma component $Z_C(\epsilon q^{-2})$ only if $\alpha_{r,n+3} = 0$. Hence (1) fails in general.

(2) This requires some calculations. By definition, we see that $\alpha_{\epsilon,4} = 0$. We now show that $Z(\epsilon)$ does not always have a Verma submodule $Z(\epsilon q^{-3})$.

By Proposition 5.3, if there exists a maximal vector of weight ϵq^{-3}, then (up to scalars) it *must* be $v' = v_{\epsilon q^{-3}} = (Y^3 - bFY)v_\epsilon$, where

$$b = \epsilon\big((q + q^{-1})c_{0,\epsilon} + c_{0,\epsilon q^{-1}}\big).$$

We now calculate what happens when this vector is also killed by X. From the proof of Proposition 5.3, we know that $Xv' = b'Fv_\epsilon$, because the coefficient of Y^2v_ϵ was made to equal zero. We now show that b' is not always zero.

Clearly,

$$XFYv_\epsilon = \big(FX - YK^{-1}\big)Yv_\epsilon = F(XYv_\epsilon) - \epsilon q^{-1}Y^2v_\epsilon = -\big(Fc_{0,\epsilon} + \epsilon q^{-1}Y^2\big)v_\epsilon.$$

But

$$\begin{aligned} XY^3v_\epsilon &= \big(qYXY^2 - C_0Y^2\big)v_\epsilon \\ &= \big(q^2Y^2XY - qYC_0Y - C_0Y^2\big)v_\epsilon \\ &= -q^2c_{0,\epsilon}Y^2v_\epsilon - qYc_{0,\epsilon q^{-1}}Yv_\epsilon - C_0Y^2v_\epsilon. \end{aligned}$$

Hence we only need to look at $C_0Y^2v_\epsilon$, to find the coefficient of Fv_ϵ.

The basic calculation is this: $EY^2v_\epsilon = XYv_\epsilon = -c_{0,\epsilon}v_\epsilon$. Hence,

$$CY^2v_\epsilon = -c_{0,\epsilon}Fv_\epsilon + c_{\epsilon q^{-2}}Y^2v_\epsilon = -c_{0,\epsilon}Fv_\epsilon + c_\epsilon Y^2v_\epsilon,$$

by definition of c_r. An easy induction argument now shows that

$$\begin{aligned} C_0Y^2v_\epsilon &= p(C)Y^2v_\epsilon \\ &= \frac{-c_{0,\epsilon}}{c_\epsilon - c_{0,\epsilon}}\big(p(c_\epsilon) - p(c_{0,\epsilon})\big)Fv_\epsilon + p(c_\epsilon)Y^2v_\epsilon \\ &= \frac{-c_{0,\epsilon}}{c_\epsilon - c_{0,\epsilon}}\big(c_{0,\epsilon} - p(c_{0,\epsilon})\big)Fv_\epsilon + c_{0,\epsilon}Y^2v_\epsilon \\ &= -aFv_\epsilon + c_{0,\epsilon}Y^2v_\epsilon, \quad \text{say.} \end{aligned}$$

Hence, we conclude that the coefficient of Fv_ϵ in $Xv' = X(Y^3 - bFY)v_\epsilon$ is $bc_{0,\epsilon} + a$, and this should be zero if v' is maximal. Simplifying, we get

$$c_{0,\epsilon}(c_\epsilon - c_{0,\epsilon})\epsilon\big((q + q^{-1})c_{0,\epsilon} + c_{0,\epsilon q^{-1}}\big) + c_{0,\epsilon}\big(c_{0,\epsilon} - p(c_{0,\epsilon})\big) = 0. \tag{12.1}$$

But this is not always satisfied: take $p(T)=\beta T$ for some $\beta \in \Bbbk$, $\beta \neq 0, 1$. Then the above condition reduces to

$$\frac{(q+q^{-1})^2+2}{(q-q^{-1})^2}+1=0$$

which simplifies to $2q^6=2$. However, since char $\Bbbk \neq 2$ and q is not a root of unity, this is not true. So at least for some $p(C)$, this condition is false.

13. Classical limit

The algebra A specializes to the symplectic oscillator algebra H_f of [12] as $q \to 1$; this is what we formalize in this section. Let k be a field of characteristic 0. Let $\Bbbk = k(q)$ be the field of rational functions on k and let $R \subset \Bbbk$ be the k-subalgebra of rational functions regular at the point $q=1$. Recall from [12] that

$$\Delta_0 := 1+f(\Delta), \qquad \Delta := \big(FE+H/2+H^2/4\big)/2,$$

where $f \in k[t]$. H_f is the k-algebra with generators X, Y, E, F, H with relations: $\langle E, F, H\rangle$ generate $U(\mathfrak{sl}_2)$, $[E,X]=[F,Y]=0$, $[E,Y]=X$, $[F,X]=Y$, $[H,X]=X$, $[H,Y]=-Y$ and $[Y,X]=1+f(\Delta)$.

We write Δ_0 as

$$\Delta_0 = f_0\big(FE+(H+1)^2/4\big)$$

for some f_0, a polynomial in one variable with coefficients in k. We will explain how H_f is the limit of A as $q \to 1$.

Our algebra A is fixed, and in particular, so is the polynomial p. Write C_0 as

$$C_0 = f_0\left(FE+\frac{Kq+K^{-1}q^{-1}-2}{(q-q^{-1})^2}\right)$$

for some polynomial f_0. The coefficients of f_0 are in k, but the limiting process works so long as they are in R. We will follow the approach in [10] and use the notation on p. 48; in particular,

$$\big(K^m;n\big)_q := \frac{K^m q^n - 1}{q-1}, \quad m,n \in \mathbb{Z}.$$

We define A_R to be the R-subalgebra of A generated by the elements X, Y, E, F, K, K^{-1}, $(K;0)_q$, and set

$$A_1 := \big(R/(q-1)R\big) \otimes_R A_R = A_R/(q-1)A_R. \tag{13.1}$$

The elements $(K^m;n)_q$ are all in A_R. This happens in the case $n=0$, because

$$K^m(K;0)_q = \big(K^{m+1};0\big)_q - \big(K^m;0\big)_q,$$

so by induction it follows that $(K^m;0)_q \in A_R$. For general n, we now conclude that

$$\left(K^m;n\right)_q = K^m(1;n)_q + \left(K^m;0\right)_q \in A_R.$$

Proposition 13.2. *The algebra* A_1 (*defined in* (13.1)) *is isomorphic to* H_f.

Proof. Denote by $\bar{X}$, $\bar{Y}$, $\bar{E}$, $\bar{F}$, $\bar{K}^m$ ($m \in \mathbb{Z}$) the images of X, Y, E, F, K^m under $A_R \to A_1$. Then the image of $K^m - 1$ equals the image of $(q-1)(K^m;0)_q \in A_R$. Thus $K^m - 1 \mapsto 0$ in A_1, so $K^m \mapsto 1$ under $A_R \to A_1$, for all $m \in \mathbb{Z}$.

Define the element $\bar{H}$ in A_1 to be the image of $(K;0)_q$ under the projection $A_R \to A_1$. The element C_0 is in A_R: this follows from the observation that

$$\frac{Kq + K^{-1}q^{-1} - 2}{(q-q^{-1})^2} = \frac{K^{-1}q(K;1)_q^2}{(q+1)^2} \tag{13.3}$$

is in A_R, which in turn is a consequence of [10, Lemma 3.3.2]. The image of $\frac{K^{-1}q(K;1)_q^2}{(q+1)^2}$ under the projection $A_R \to A_1$ is $(\bar{H}+1)^2/4$:

$$\frac{K^{-1}q(K;1)_q^2}{(q+1)^2} = \frac{K^{-1}q}{(q+1)^2}\left(q(K;0)_q + 1\right)^2$$

and we know that $K \to 1$ and $(K;0)_q \to \bar{H}$.

Therefore, because of our choice of f, $\bar{X}$ and $\bar{Y}$ satisfy the relation

$$\bar{Y}\bar{X} - \bar{X}\bar{Y} = f_0\left(\bar{F}\bar{E} + (\bar{H}+1)^2/4\right).$$

It is clear that, in A_1, we have the relations $\bar{E}\bar{X} = \bar{X}\bar{E}$, $\bar{E}\bar{Y} - \bar{Y}\bar{E} = \bar{X}$, $\bar{F}\bar{X} - \bar{X}\bar{F} = \bar{Y}$ and $\bar{F}\bar{Y} = \bar{Y}\bar{F}$. Therefore, we have an epimorphism $H_f \to A_1$. These two rings have a filtration where $\deg(X) = \deg(Y) = 1$, $\deg(E) = \deg(F) = \deg(H) = 0$ and similarly with $\bar{X}$, $\bar{Y}$, $\bar{E}$, $\bar{F}$, $\bar{H}$.

We can identify $\bar{E}$ with E, $\bar{F}$ with F and $\bar{H}$ with H, because we know from [10] that $\bar{E}$, $\bar{F}$, $\bar{H}$ generate a subalgebra isomorphic to $U(\mathfrak{sl}_2)$.

We can view the map $H_f \to A_1$ as a map of $U(\mathfrak{sl}_2)$-modules. Now, $\mathrm{gr}(H_f) = k[X,Y] \rtimes U(\mathfrak{sl}_2)$. Also, $\mathrm{gr}(A_1) = k[X,Y] \rtimes U(\mathfrak{sl}_2)$ since $\mathrm{gr}(A_R) = R[X,Y] \rtimes U_q(\mathfrak{sl}_2)$. The associated graded map $\mathrm{gr}(H_f) \to \mathrm{gr}(A_1)$ is the identity map from $k[X,Y] \rtimes U(\mathfrak{sl}_2)$ to $k[X,Y] \rtimes U(\mathfrak{sl}_2)$. Hence, H_f is isomorphic to A_1. □

Let $r = \pm q^n$ where $n \in \mathbb{Z}$ and let V be a standard cyclic A-module with highest weight r and highest weight vector v_r. We define the R-form of V to be the A_R-module $V_R := A_R \cdot v_r$. Set $V^1 := R/(q-1) \otimes_R V_R$, so V^1 is an A_1-module.

Proposition 13.4. V_1 *is an* H_f *standard cyclic module with highest weight* n *and highest weight vector* v_r. *Furthermore, if* V *is a Verma module, then so is* V_1.

Proof. This is clear from the previous proposition. □

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