

SHARP NONZERO LOWER BOUNDS FOR THE SCHUR PRODUCT THEOREM

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ABSTRACT. By a result of Schur [J. Reine Angew. Math. 140 (1911), pp. 1–28], the entrywise product $M \circ N$ of two positive semidefinite matrices M, N is again positive. Vybíral [Adv. Math. 368 (2020), p. 9] improved on this by showing the uniform lower bound $M \circ \overline{M} \geq E_n/n$ for all $n \times n$ real or complex correlation matrices M , where E_n is the all-ones matrix. This was applied to settle a conjecture of Novak [J. Complexity 15 (1999), pp. 299–316] and to positive definite functions on groups. Vybíral (in his original preprint) asked if one can obtain similar uniform lower bounds for higher entrywise powers of M , or for $M \circ N$ when $N \neq M, \overline{M}$. A natural third question is to ask for a tighter lower bound that does not vanish as $n \rightarrow \infty$, i.e., over infinite-dimensional Hilbert spaces.

In this note, we affirmatively answer all three questions by extending and refining Vybíral’s result to lower-bound $M \circ N$, for arbitrary complex positive semidefinite matrices M, N . Specifically: we provide tight lower bounds, improving on Vybíral’s bounds. Second, our proof is ‘conceptual’ (and self-contained), providing a natural interpretation of these improved bounds via tracial Cauchy–Schwarz inequalities. Third, we extend our tight lower bounds to Hilbert–Schmidt operators. As an application, we settle Open Problem 1 of Hinrichs–Krieg–Novak–Vybíral [J. Complexity 65 (2021), Paper No. 101544, 20 pp.], which yields improvements in the error bounds in certain tensor product (integration) problems.

1. INTRODUCTION AND MAIN RESULT

1.1. The Schur product theorem and nonzero lower bounds. We begin with a few definitions. A *positive semidefinite matrix* is a complex Hermitian matrix with non-negative eigenvalues. Denote the space of such $n \times n$ matrices by $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$. Given integers $m, n \geq 1$, the *Schur product*, or *entrywise product* of two (possibly rectangular) $m \times n$ complex matrices $A = (a_{jk}), B = (b_{jk})$ equals the $m \times n$ matrix $A \circ B$ with (j, k) entry $a_{jk}b_{jk}$.

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A seminal result by Schur [17] asserts that if M, N are positive semidefinite matrices of the same size, then so is their entrywise product $M \circ N$. This fundamental observation has had numerous follow-ups and applications; perhaps the most relevant to the present short note is the development of the entrywise calculus in matrix analysis, with connections to numerous classical and modern works, both theoretical and applied. (See e.g. the two-part survey [2, 3].) It also extends to positive self-adjoint operators on Hilbert spaces.

The Schur product theorem is often phrased using the *Loewner ordering* on $\mathbb{C}^{n \times n}$ – in which $M \geq N$ if $M - N \in \mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$ – in the following form:

$$(1.1) \quad M \geq \mathbf{0}_{n \times n}, N \geq \mathbf{0}_{n \times n} \implies M \circ N \geq \mathbf{0}_{n \times n}.$$

This is a ‘qualitative’ result, in that it provides a lower bound of $\mathbf{0}_{n \times n}$ for $M \circ N$ for all $M, N \in \mathbb{P}_n$. It is natural to seek ‘quantitative’ results, i.e., nonzero lower bounds. Here are some known bounds: Fiedler’s inequality [5] says $A \circ A^{-1} \geq \text{Id}_n$ if $A \in \mathbb{P}_n$ is invertible. Two more examples, see e.g. [6, 12], are:

$$(1.2) \quad \begin{aligned} M \circ N &\geq \lambda_{\min}(N)(M \circ \text{Id}_n), \text{ if } M \in \mathbb{P}_n \text{ is real and } N = N^T \in \mathbb{R}^{n \times n}, \\ M \circ N &\geq \frac{1}{\mathbf{e}^T N^{-1} \mathbf{e}} M, \text{ if } M, N \in \mathbb{P}_n \text{ and } \det(N) > 0. \end{aligned}$$

Here and below, we use the following **notation** without further reference.

- Given a fixed integer $n \geq 1$, let $\mathbf{e} = \mathbf{e}(n) := (1, \dots, 1)^T \in \mathbb{C}^n$, and $E_n := \mathbf{e}\mathbf{e}^T \in \mathbb{P}_n$.
- We say that a matrix in \mathbb{P}_n is a *real/complex correlation matrix* if it has all diagonal entries 1, and all entries real/complex respectively.
- Given a matrix $M_{n \times n}$ and a subset $J \subset \{1, \dots, n\}$, let $M_{J \times J}$ denote the principal submatrix of M corresponding to the rows and columns indexed by J ; and let $d_M := (m_{11}, \dots, m_{nn})^T$.

This note concerns the recent paper [20], in which Vybíral showed a new lower bound for all $M \circ \overline{M}$, where M is a correlation matrix:

Theorem 1.3 ([20]). *If $n \geq 1$ and $M_{n \times n}$ is a real or complex correlation matrix (so $\overline{M} = M^T$), then $M \circ \overline{M} \geq \frac{1}{n} E_n$.*

Theorem 1.3 is striking in its simplicity (and in that it seems to have been undiscovered for more than a century after the Schur product theorem [17]). There are no obvious upper bounds for the left-hand side, while it is *a priori* intriguing that there is a nonzero lower bound.

Vybíral provided a direct proof, in fact of a more general fact:

Theorem 1.4 ([20]). *Given a matrix $M \in \mathbb{C}^{n \times n}$, let $d_M := (m_{11}, \dots, m_{nn})^T$ be the vector consisting of its diagonal entries. Now if $M \in \mathbb{P}_n$, then $M \circ \overline{M} \geq \frac{1}{n} d_M d_M^T$.*

Vybíral used these results to prove a conjecture of Novak [10] in numerical integration (see Theorem A.7), with applications to positive definite functions and in other areas. See [20] for details.

1.2. The main result. Following the above results, Vybíral asked – at the end of his original 2019 preprint [18] – if Theorem 1.3 admits variants (1) for $M \circ N$ for $N \neq M, \overline{M}$; and (2) for higher powers of M . He answered (1) in his updated paper, as follows.

Theorem 1.5 ([20]).

- (1) If $M = AA^*, N = BB^* \in \mathbb{P}_n$ for $A, B \in \mathbb{C}^{n \times n}$, then $M \circ N \geq \frac{1}{n} w w^*$, with $w := (A \circ B)\mathbf{e}$.
- (2) In particular, setting $B = \mathbf{e}\mathbf{e}^T = E_n$, we have $AA^* \geq \frac{1}{n}(A\mathbf{e})(A\mathbf{e})^*$.

Note, the first part implies Theorem 1.4 (hence Theorem 1.3) by setting $B = \overline{A}$. Thus, Theorem 1.5(1) is currently state-of-the-art.

The lower bound of $1/n$ poses a technical challenge to the functional analyst: Theorem 1.5 cannot be extended to infinite-dimensional Hilbert spaces to yield a nontrivial lower bound. It is thus natural to ask (3) whether there exists a function of M, N (or of A, B) that can improve the constant $1/n$ to a bound that remains nonzero in Hilbert spaces.

The contributions of this short note are as follows:

- Our main result indeed provides an improved bound sought-for above, so that it also extends to a nonzero lower bound in the Hilbert space setting (see Section 2).
- We show this improved bound is tight, and strictly improves on the state-of-the-art Theorem 1.5. We also do not require A, B to be square matrices – or even equi-dimensional.
- The proof we provide is conceptual and ‘coordinate-free’, in contrast to previous direct and ‘computational’ proofs of special cases. (At the same time, our proof uses elementary arguments, hence is self-contained.) In particular, we show that the results here and by Vybíral are all tracial Cauchy–Schwarz inequalities – our proof also explains the *meaning* of our tight bound.

Here is the main result of this note.

Theorem A. Given integers $n, a \geq 1$ and nonzero matrices $A, B \in \mathbb{C}^{n \times a}$, we have the (rank ≤ 1) lower bound:

$$(1.6) \quad AA^* \circ BB^* \geq \frac{1}{\min(\text{rk}(AA^*), \text{rk}(BB^*))} \cdot d_{AB^T} d_{AB^T}^*,$$

and the choice of constant is best possible.

(In fact we do not require A, B to have the same number of columns; see Corollary 1.16 below.) Before proving this theorem, we discuss some special cases, beginning with the solution to an open problem.

Theorem A finds an application in numerical integration (in the spirit of Vybíral’s original result [20] being recently applied to resolve Novak’s conjecture [10]). Specifically, in the recent work by Hinrichs–Krieg–Novak–Vybíral [8], the authors prove two results (Theorems 15 and 16 in *loc. cit.*); the latter states that given integers $n, D \geq 1$, and matrices $A, B \in \mathbb{R}^{n \times D}$ with $AA^T = BB^T$ of rank $r > 0$, we have:

$$(1.7) \quad AA^T \circ BB^T \geq \frac{1}{2r} d_{AB^T} d_{AB^T}^T.$$

The authors then ask (see Open Problem 1 in [8]) if the constant $1/(2r)$ can be improved to $1/r$; this would lead to improved error bounds in certain tensor product integration problems. This Open Problem – as well as both of their aforementioned theorems – are immediate consequences of Theorem A. For instance, in the special

case $AA^T = BB^T$, Theorem A above answers the Open Problem (in particular, improving on [8, Theorem 16]):

Corollary 1.8. *Given arbitrary integers $n, D \geq 1$ and nonzero matrices $A, B \in \mathbb{R}^{n \times D}$, if $AA^T = BB^T = M$ then*

$$AA^T \circ BB^T \geq \frac{1}{\text{rk}(M)} d_{AB^T} d_{AB^T}^T.$$

We next discuss additional special cases of our main result, which were previously proved in the literature.

Remark 1.9 (Specializing to earlier results). Theorem A extends and unifies the preceding results above. It suffices to deduce the ‘state-of-the-art’ Theorem 1.5(1). Letting v_j, w_k denote the columns of $A_{n \times n}, B_{n \times n}$ respectively, we have $AA^* = \sum_j v_j v_j^*$ and $BB^* = \sum_k w_k w_k^*$. Hence

$$AA^* \circ BB^* = \sum_{j,k=1}^n (v_j v_j^*) \circ (w_k w_k^*) \geq \sum_{j=1}^n (v_j \circ w_j)(v_j \circ w_j)^* = (A \circ B)(A \circ B)^*.$$

This shows that the two assertions in Theorem 1.5 are equivalent; and setting $a = n, B = E_n/\sqrt{n}$ in Theorem A yields Theorem 1.5(2).

Remark 1.10. Another special case that Vybíral has separately communicated to us [19], again holds for square matrix decompositions:

$$(1.11) \quad AA^* \circ BB^* \geq \frac{1}{\max(\text{rk}(AA^*), \text{rk}(BB^*))} d_{AB^T} d_{AB^T}^*, \quad \forall A, B \in \mathbb{C}^{n \times n}.$$

More precisely, Vybíral mentioned that given any two positive matrices $M, N \in \mathbb{P}_n$, one has the lower bound (1.11) for every pair of decompositions $M = AA^*, N = BB^*$ for square matrices $A, B \in \mathbb{C}^{n \times n}$. Notice that: (a) this holds only in the special case $a = n$ of Theorem A; (b) the bound in (1.11) is also not tight, as the coefficient of $1/\max$ can be improved to $1/\min$ in Theorem A; and (c) it is not clear if this statement implies Theorem 1.5, or conversely. Our main result, Theorem A, clearly unifies and strengthens all of these variants.

Having discussed the myriad special cases of the theorem, here is a proof (that is self-contained on the one hand, and on the other, explains the tight lower bound):

Proof of Theorem A. The key identity needed to prove (1.6) is algebraic: given any square $n \times n$ matrices M, N and vectors u, v with n coordinates (over a unital commutative ring),

$$(1.12) \quad u^T (M \circ N) v = \text{tr}(N^T D_u M D_v),$$

where D_u for a vector $u \in \mathbb{C}^n$ is the diagonal matrix with (j, j) entry u_j . Thus, pre- and post-multiplying the left-hand side of (1.6) by u^*, u respectively, we compute:

$$u^*(AA^* \circ BB^*)u = \text{tr}(\overline{B} B^T D_{\overline{u}} A A^* D_u) = \text{tr}(N^* N), \quad \text{where } N := A^* D_u \overline{B}.$$

Consider the inner product on $\mathbb{C}^{a \times a}$, given by $\langle X, Y \rangle := \text{tr}(X^* Y)$, and define the projection

$$(1.13) \quad P := \text{proj}_{(\ker A)^\perp} \upharpoonright_{\text{im}(B^T)};$$

thus $P \in \mathbb{C}^{a \times a}$. We compute:

$$\begin{aligned} \langle P, P \rangle &\leq \min(\dim(\ker A)^\perp, \dim \operatorname{im}(B^T)) = \min(\operatorname{rk}(A^*), \operatorname{rk}(B^*)) \\ &= \min(\operatorname{rk}(AA^*), \operatorname{rk}(BB^*)). \end{aligned}$$

Hence by the Cauchy–Schwarz inequality (for this tracial inner product),

$$\begin{aligned} u^*(AA^* \circ BB^*)u = \langle N, N \rangle &\geq \frac{|\langle N, P \rangle|^2}{\langle P, P \rangle} = \frac{|\operatorname{tr}(APB^T D_{\bar{u}})|^2}{\langle P, P \rangle} = \frac{|u^* d_{APB^T}|^2}{\langle P, P \rangle} \\ &\geq \frac{1}{\min(\operatorname{rk}(AA^*), \operatorname{rk}(BB^*))} u^* d_{APB^T} d_{APB^T}^* u. \end{aligned}$$

But this holds for all vectors u . This shows (1.6) where d_{AB^T} is replaced by d_{APB^T} ; but in fact $APB^T = AB^T$ by choice of P .

Finally, we show the tightness of the bound $1/\min(\operatorname{rk}(AA^*), \operatorname{rk}(BB^*))$ (e.g. over $1/\max$). Choose integers $1 \leq r$ with $r, s \leq n$, and complex block diagonal matrices

$$A_{n \times a} := \begin{pmatrix} D_{r \times r} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{n \times a} := \begin{pmatrix} D'_{s \times s} & 0 \\ 0 & 0 \end{pmatrix},$$

with both D, D' nonsingular. Then $P := \begin{pmatrix} \operatorname{Id}_{\min(r,s)} & 0 \\ 0 & 0 \end{pmatrix}$, and the bound of $1/\min(r, s)$ is indeed tight, as can be verified using the Cauchy–Schwarz identity. □

We end this part with additional remarks, beginning by attaining equality in (1.6).

Example 1.14. Suppose $A = u, B = v$ are nonzero vectors in \mathbb{C}^n . Then (1.6) says:

$$uu^* \circ vv^* \geq d_{uv^T} d_{uv^T}^* = (u \circ v)(u \circ v)^*.$$

Thus, the inequality (1.6) reduces to an equality for rank-one matrices AA^*, BB^* .

Remark 1.15. Another way to consider Theorem A is to start with matrices $M, N \in \mathbb{P}_n(\mathbb{C})$ and then obtain the bound (1.6) for every decomposition $M = AA^*, N = BB^*$. In this case, it is clear that the constant

$$\gamma = \min(\operatorname{rk} M_{J \times J}, \operatorname{rk} N_{J \times J})^{-1}$$

does not change; but the rank-one lower bound can indeed change. Even if one runs over decompositions in terms of *square* matrices A, B (to dispense with the role of P), and assumes $C = \operatorname{Id}_J$, it would be interesting to obtain some understanding of the possible rank-one matrices obtained as lower bounds.

This is also linked to the possibility of obtaining higher-rank lower bounds for $AA^* \circ BB^*$. One way to do so is to realize that the left-hand side of (1.6) is bi-additive in (AA^*, BB^*) , so one can decompose both AA^* and BB^* as sums of lower-rank matrices and obtain rank-one lower bounds for each pair of lower-rank matrices. Example 1.14 is relevant here: it shows that if one writes AA^*, BB^* as sums of rank-one matrices, then each corresponding inequality is an equality, and adding these yields the unique best lower bound of $AA^* \circ BB^*$.

1.3. Refinements using coordinates. We now present several refinements of Theorem A. The first is *a priori* more general, but in fact equivalent:

Corollary 1.16. *Given integers $n, a, b \geq 1$ and nonzero matrices $A \in \mathbb{C}^{n \times a}$, $B \in \mathbb{C}^{n \times b}$, we have the (rank ≤ 1) lower bound:*

$$(1.17) \quad AA^* \circ BB^* \geq \frac{1}{\min(\text{rk}(AA^*), \text{rk}(BB^*))} \cdot d_{A_0 B_0^T} d_{A_0 B_0^T}^*$$

where A_0 appends $p + \max(a, b) - a$ zero-columns to the right of A , and B_0 appends $p + \max(a, b) - b$ zero-columns to the right of B , for some integer $p \geq 0$. Moreover, the choice of constant is best possible.

Proof. Clearly this result implies Theorem A by setting $b = a$ and $p = 0$. Conversely, apply Theorem A to A_0, B_0 , and use $p + \max(a, b)$ in place of a , to obtain:

$$A_0 A_0^* \circ B_0 B_0^* \geq \frac{1}{\min(\text{rk}(A_0 A_0^*), \text{rk}(B_0 B_0^*))} \cdot d_{A_0 B_0^T} d_{A_0 B_0^T}^*$$

Now notice that $A_0 A_0^* = AA^*$ and $B_0 B_0^* = BB^*$. □

The next result refines Theorem A in the following sense: suppose the matrix M has nonzero entries only in the $J \times J$ coordinates (for a nonempty subset $J \subset \{1, \dots, n\}$). Then the bound can in fact be improved:

Theorem 1.18. *Given integers $n, k \geq 1$ and a complex matrix $C_{k \times n}$, let $J \subset \{1, \dots, n\}$ index the nonzero columns of C . Then for all integers $a \geq 1$ and matrices $A \in \mathbb{C}^{n \times a}$, $B \in \mathbb{C}^{n \times a}$ such that $(AA^*)_{J \times J}, (BB^*)_{J \times J}$ are nonzero, we have the (rank ≤ 1) lower bound:*

$$(1.19) \quad C(AA^* \circ BB^*)C^* \geq \gamma(A, B, J) \cdot C d_{AB^T} d_{AB^T}^* C^*$$

where the following choice of scalar $\gamma(A, B, J)$ is best possible:

$$(1.20) \quad \gamma(A, B, J) := \frac{1}{\min(\text{rk}(AA^*)_{J \times J}, \text{rk}(BB^*)_{J \times J})}$$

Clearly, this implies Theorem A by setting $k = n$ and $C = \text{Id}_n$, so that $J = \{1, \dots, n\}$. However, it is essentially also implied by it, as the following proof reveals.

Proof. A preliminary observation is that if $(AA^*)_{J \times J} = 0$ then $\text{Id}_J (AA^*) \text{Id}_J = 0$, where $\text{Id}_J \in \mathbb{P}_n$ has diagonal entries $\mathbf{1}_{i \in J}$. But then the submatrix $A_{J \times \{1, \dots, a\}} = 0$, hence

$$C d_{AB^T} = C \text{Id}_J d_{AB^T} = 0.$$

Thus the matrices on both sides of (1.19) are zero, and so the coefficient is irrelevant. The same conclusion is obtained by a similar argument if $(BB^*)_{J \times J} = 0$.

We now prove (1.19). First observe that $C = C \text{Id}_J$, so that (1.19) for (C, A, B) follows from (1.19) for $(C = \text{Id}_J, A, B)$. But this is precisely (1.19) for the matrices $(\text{Id}_J, \text{Id}_J A, \text{Id}_J B)$. In other words, by restricting to the $J \times J$ principal submatrices on both sides, we may assume without loss of generality that $J = \{1, \dots, n\}$ and $C = \text{Id}_n$; the hypotheses imply A, B are nonzero. This is precisely Theorem A. □

We conclude this section by observing that (1.19) can be extended to Schur products of any number of positive matrices. Here are two sample results:

Corollary 1.21. *Let $m, n, l \geq 1$ and matrices $M_1, \dots, M_m \in \mathbb{P}_n$. Given a partition of $\{1, \dots, m\}$ into subsets $J_1 \sqcup \dots \sqcup J_{2l}$, let*

$$M'_j := \circ_{i \in J_j} M_i, \quad 1 \leq j \leq 2l.$$

Now if $M'_j = A_j A_j^*$ for all $j \geq 1$, with each A_j square and nonzero, then we have the (rank ≤ 1) lower bound:

$$M'_1 \circ \dots \circ M'_{2l} \geq \prod_{j=1}^k \frac{1}{\min(\text{rk}(M'_j), \text{rk}(M'_{j+l}))} \mathbf{w} \mathbf{w}^*, \quad \text{where } \mathbf{w} := \circ_{j=1}^l d_{A_j A_{j+l}^T}.$$

While this result implies (1.19) for $l = 1, n = k$, and $C = \text{Id}_n$, it is also implied by it, via the ‘monotonicity’ of the Schur product: if $A \geq B$ and $A' \geq B'$, then $A \circ A' \geq B \circ A' \geq B \circ B'$.

Theorem 1.22. *Given a vector $u = (u_1, \dots, u_n)^T \in \mathbb{C}^n$, let D_u denote the diagonal matrix whose diagonal entries are the coordinates u_1, \dots, u_n of u ; and let $J(u) \subset \{1, \dots, n\}$ denote the nonzero coordinates of u , i.e., $\{j : 1 \leq j \leq n, u_j \neq 0\}$.*

Now let $k \geq 1$, and fix vectors $u_1, y_1, \dots, u_k, y_k \in \mathbb{C}^n$ such that $\mathbf{w} := (u_1 \circ y_1) \circ \dots \circ (u_k \circ y_k)$ is nonzero. Then we have the (rank ≤ 1) lower bound: for all matrices $M_1, \dots, M_k \in \mathbb{P}_n$,

$$(1.23) \quad \begin{aligned} & (D_{u_1} M_1 D_{u_1}^* \circ D_{y_1} \overline{M}_1 D_{y_1}^*) \circ \dots \circ (D_{u_k} M_k D_{u_k}^* \circ D_{y_k} \overline{M}_k D_{y_k}^*) \\ & \geq \frac{1}{\text{rk}(M_{J(\mathbf{w}) \times J(\mathbf{w})})} (\mathbf{w} \circ d_{M_1} \circ \dots \circ d_{M_k}) (\mathbf{w} \circ d_{M_1} \circ \dots \circ d_{M_k})^*, \end{aligned}$$

where $M := M_1 \circ \dots \circ M_k$. Note, if the principal submatrix $M_{J(\mathbf{w}) \times J(\mathbf{w})} = \mathbf{0}$ then $\mathbf{w} \circ d_{M_1} \circ \dots \circ d_{M_k}$ is also zero, so the coefficient is irrelevant.

Moreover, the coefficient $\frac{1}{\text{rk}(M_{J(\mathbf{w}) \times J(\mathbf{w})})}$ is best possible for all $u_j, y_j \in \mathbb{C}^n$ for which $\mathbf{w} \neq 0$, and all M_1, \dots, M_k for which $M_{J(\mathbf{w}) \times J(\mathbf{w})} \neq \mathbf{0}$.

Theorem 1.22 is a tighter refinement of the Schur product theorem than Theorem 1.4, which is the special case with $k = 1$ and $u_1 = y_1 = \mathbf{e}$. Moreover, Theorem 1.22 can (and does) extend to provide nonzero lower bounds in infinite-dimensional Hilbert spaces, unlike Theorems 1.3 and 1.4. We leave the proof to the interested reader, as it is similar to (and follows from) theorems above.

Remark 1.24. Define for a nonzero vector $d \in \mathbb{C}^n$, the ‘level set’

$$\mathcal{S}_d := \{(M_1, \dots, M_k) : M_j \in \mathbb{P}_n \ \forall j, \ d_{M_1} \circ \dots \circ d_{M_n} = d\}.$$

Then a consequence of Theorem 1.22 for $u_j = y_j = \mathbf{e} \ \forall j$, is that (1.23) provides a uniform lower bound on each set \mathcal{S}_d (i.e., which depends only on d). In fact the case of M a correlation matrix in [20], is a special case of this consequence for $d = \mathbf{e}$ (and $k = 1$).

We conclude with a ‘negative’ remark, which shows that one cannot deviate very far from the above hypotheses on the matrices in question.

Remark 1.25. Given the above results, a natural question is if even the original identity $M \circ \overline{M} \geq \frac{1}{n} d_M d_M^T$ of Vybřal holds more widely. A natural extension to

explore is from matrices $M \circ \overline{M}$ to the larger class of *doubly non-negative matrices*: namely, matrices in \mathbb{P}_n with non-negative entries. In other words, given a doubly non-negative matrix $A \in \mathbb{P}_n$, is it true that

$$A \geq \frac{1}{n} d_A^{\circ 1/2} (d_A^{\circ 1/2})^T, \quad \text{where } d_A^{\circ 1/2} := (a_{11}^{1/2}, \dots, a_{nn}^{1/2})^T?$$

While this question was not addressed in [20], it is easy to verify that it is indeed true for 2×2 matrices. However, here is a family of counterexamples for $n = 3$; we leave the case of higher values of n to the interested reader. Consider the real matrix

$$A = \begin{pmatrix} a & c & d \\ c & b & c \\ d & c & a \end{pmatrix}, \quad \text{where } a, b > 0, c \in [\sqrt{ab/2}, \sqrt{ab}), d = \frac{2c^2}{b} - a < a.$$

These bounds imply A is doubly non-negative. Now we compute:

$$A - d_A^{\circ 1/2} (d_A^{\circ 1/2})^T = \frac{1}{3} \begin{pmatrix} 2a & 3c - \sqrt{ab} & 3d - a \\ 3c - \sqrt{ab} & 2b & 3c - \sqrt{ab} \\ 3d - a & 3c - \sqrt{ab} & 2a \end{pmatrix}.$$

Straightforward computations show that all entries and 2×2 principal minors of this matrix are non-negative; but its determinant equals

$$\frac{2}{3}(a - d)(2\sqrt{abc} + bd - 3c^2) = \frac{-2}{3}(a - d)(\sqrt{ab} - c)^2 < 0.$$

This shows that one cannot hope to go much beyond the above test-set of matrices $M \circ \overline{M}$, along the lines of the lower bound in (1.23).

1.4. An upper bound. While an upper bound on $M \circ N$ is not the focus of the present paper, we provide one for completeness. The following statement depends separately on M, N , not using $M \circ N$:

Proposition 1.26. *Given matrices $M, N \in \mathbb{P}_n(\mathbb{C})$, let $J \subset \{1, \dots, n\}$ comprise the indices j such that $m_{jj}, n_{jj} > 0$, and let D_M, D_N denote the diagonal matrices $M \circ \text{Id}_n, N \circ \text{Id}_n$ respectively. Also suppose $C_J(M)$ denotes the $J \times J$ ‘correlation’ matrix with (j, k) entry $m_{jk} / \sqrt{m_{jj}m_{kk}}$, and similarly for $C_J(N)$. Then,*

$$M \circ N \leq \max_{j \in J} (\|C_J(M)_{*j}\| \cdot \|C_J(N)_{*j}\|) \cdot D_M D_N,$$

where C_{*j} for a matrix $C \in \mathbb{C}^{J \times J}$ denotes its j th column.

Note that this bound is indeed attained. In fact when M, N are diagonal matrices, we obtain an *equality* of matrices.

Proof. First note that the matrices $M \circ N$ and $D_M D_N$ have nonzero entries only in the $J \times J$ locations. Thus we may assume $J = \{1, \dots, n\}$ without loss of generality. Next, $M_{J \times J} = \sqrt{D_M} C_J(M) \sqrt{D_M}$, and similarly for $N = N_{J \times J}$. Thus, if one shows the result with M, N replaced by $C_J(M), C_J(N)$ respectively (in which case D_M, D_N are replaced by Id_n), then the general result follows. Thus, we assume henceforth that $J = \{1, \dots, n\}$ and M, N have all diagonal entries 1. Now $M \circ N \leq \lambda_{\max}(M \circ N) \text{Id}_n$ by the spectral theorem, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue. But this yields

$$\lambda_{\max}(M \circ N) \leq \max_{j \in J} \sum_{k \in J} |m_{jk} n_{jk}| \leq \max_{j \in J} \|M_{*j}^J\| \cdot \|N_{*j}^J\|$$

by Gershgorin’s circle theorem and the Cauchy–Schwarz inequality. □

2. EXTENSION TO HILBERT SPACES

As mentioned in the discussion preceding Theorem A, we now extend that result to Hilbert spaces. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a real or complex Hilbert space with a *fixed* orthonormal basis $\{e_x : x \in X\}$ – so its span is dense in \mathcal{H} . We begin by recalling a few well-known notions, both basic and more advanced. In what follows, $A, B : \mathcal{H} \rightarrow \mathcal{H}'$ are linear maps, with \mathcal{H}' another Hilbert space with orthonormal basis $\{f_y : y \in Y\}$:

- (1) The *adjoint* $A^* : \mathcal{H}' \rightarrow \mathcal{H}$ of A is given by: $\langle A^* f_y, e_x \rangle := \langle f_y, A e_x \rangle$ for all $x \in X, y \in Y$. We will also freely use u^* for a vector $u \in \mathcal{H}$ to denote the linear functional $\langle u, \cdot \rangle$.
- (2) The *transpose* of A is $A^T : \mathcal{H}' \rightarrow \mathcal{H}$, given by: $\langle A^T f_y, e_x \rangle := \langle A e_x, f_y \rangle$ for $x \in X, y \in Y$. The *conjugate* $\bar{A} : \mathcal{H} \rightarrow \mathcal{H}'$ is precisely $(A^*)^T = (A^T)^*$, given by $\langle f_y, \bar{A} e_x \rangle := \langle A e_x, f_y \rangle$.
- (3) The *Schur product* of A, B is the operator $A \circ B$ determined by: $\langle f_y, (A \circ B) e_x \rangle := \langle f_y, A e_x \rangle \langle f_y, B e_x \rangle$ for all $x \in X, y \in Y$.
- (4) We say A is *bounded* if A maps bounded sets into bounded sets. Denote the collection of such bounded linear maps by $\mathcal{B}(\mathcal{H}, \mathcal{H}')$, and by $\mathcal{B}(\mathcal{H})$ if $\mathcal{H}' = \mathcal{H}$. The operator norm of $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is $\|A\| := \sup\{\|Ax\|_{\mathcal{H}'} : \|x\|_{\mathcal{H}} \leq 1\}$.
- (5) We say A is *Hilbert–Schmidt* if its Hilbert–Schmidt/Frobenius norm is finite:

$$\sum_{x \in X} \|A e_x\|^2 < \infty.$$

Denote the set of Hilbert–Schmidt operators by $\mathcal{S}_2(\mathcal{H}, \mathcal{H}')$ (the Schatten 2-class), and by $\mathcal{S}_2(\mathcal{H})$ if $\mathcal{H}' = \mathcal{H}$.

- (6) For $\mathcal{H} = \mathcal{H}'$, a Hilbert–Schmidt operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is *trace class* if the sum of the singular values of $\sqrt{A^* A}$ is convergent. For such an operator, its *trace* is defined to be $\text{tr}(A) := \sum_{x \in X} \langle e_x, A e_x \rangle$.
- (7) Given a vector $u \in \mathcal{H}$, the corresponding *multiplier* $M_u : \mathcal{H} \rightarrow \mathcal{H}$ is given by: $\langle e_x, M_u e_y \rangle := \delta_{x,y} \langle e_x, u \rangle$ for all $x, y \in X$. In other words, M_u is a diagonal operator with respect to the given basis $\{e_x\}$, with the corresponding coordinates of the vector u as its diagonal entries.

Next, we collect together some well-known properties of these operators; see e.g. [7].

Lemma 2.1. *Suppose $(\mathcal{H}, \langle \cdot, \cdot \rangle, \{e_x : x \in X\})$ is as above, and $u, v \in \mathcal{H}$. Also fix another Hilbert space \mathcal{H}' with a fixed orthonormal basis $\{f_y : y \in Y\}$.*

- (1) *The space $\mathcal{S}_2(\mathcal{H})$ is a two-sided $*$ -ideal in $\mathcal{B}(\mathcal{H})$, which contains the multipliers M_u .*
- (2) *The subspace $\mathcal{S}_2(\mathcal{H}, \mathcal{H}') \subset \mathcal{B}(\mathcal{H}, \mathcal{H}')$ contains all rank-one operators $\lambda uv^* := \lambda u \langle v, \cdot \rangle$ for $\lambda \in \mathbb{C}, v \in \mathcal{H}, u \in \mathcal{H}'$. Moreover, $*$: $\mathcal{S}_2(\mathcal{H}, \mathcal{H}') \rightarrow \mathcal{S}_2(\mathcal{H}', \mathcal{H})$.*
- (3) *If $A \in \mathcal{S}_2(\mathcal{H}, \mathcal{H}'), B \in \mathcal{S}_2(\mathcal{H}', \mathcal{H})$, then AB, BA are trace class, and their traces coincide.*
- (4) *The assignment $(A, B) \mapsto \text{tr}(A^* B)$ is an inner product on $\mathcal{S}_2(\mathcal{H}, \mathcal{H}')$.*
- (5) *$\mathcal{S}_2(\mathcal{H})$ is closed under taking Schur products (with respect to $\{e_x : x \in X\}$).*

- (6) If $A = \sum_{j=1}^k \lambda_j u_j v_j^*$ is of finite rank for $u_j, v_j \in \mathcal{H}$, then A is trace class and $\text{tr}(A) = \sum_{j=1}^k \lambda_j \langle v_j, u_j \rangle$.
- (7) The multipliers $M_u, u \in \mathcal{H}$ pairwise commute and are Hilbert–Schmidt.

A simple observation is that Hilbert–Schmidt operators are closed under composition:

Corollary 2.2. *Suppose $\mathcal{H}^{(j)}$ is a Hilbert space with a fixed orthonormal basis (indexed by) $X^{(j)}$, for $j = 1, 2, 3$. If $A^{(1)} : \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(2)}$ and $A^{(2)} : \mathcal{H}^{(2)} \rightarrow \mathcal{H}^{(3)}$ are Hilbert–Schmidt, then so is their composition $A^{(2)}A^{(1)} : \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(3)}$.*

Proof. Since $A^{(2)}$ is bounded and $A^{(1)}$ is Hilbert–Schmidt, we compute directly:

$$\sum_{x \in X^{(1)}} \|A^{(2)}A^{(1)}e_x\|_{\mathcal{H}^{(3)}}^2 \leq \|A^{(2)}\|^2 \sum_{x \in X^{(1)}} \|A^{(1)}e_x\|_{\mathcal{H}^{(2)}}^2 < \infty. \quad \square$$

We require a few more notions:

Definition 2.3. Let \mathcal{H}, X be as above.

- (1) Given an operator $A : \mathcal{H} \rightarrow \mathcal{H}$ and a subset $J \subset X$, define its ‘principal submatrix’ $A_{J \times J} : \mathcal{H} \rightarrow \mathcal{H}$ via:

$$\langle e_x, A_{J \times J} e_y \rangle := \mathbf{1}_{x \in J} \mathbf{1}_{y \in J} \langle e_x, A e_y \rangle, \quad \forall x, y \in X.$$

Notice, $A_{J \times J}$ is precisely the compression $P_J A P_J$, where P_J is the orthogonal projection onto the closed subspace $\mathcal{H}_J \subset \mathcal{H}$ spanned by $\{e_j : j \in J\}$.

- (2) For $A \in \mathcal{S}_2(\mathcal{H})$, define its ‘diagonal vector’ $d_A \in \mathcal{H}$ via: $\langle e_x, d_A \rangle := \langle e_x, A e_x \rangle$.
- (3) An operator $A \in \mathcal{B}(\mathcal{H})$ is *positive* if $A = A^*$ (self-adjoint) and $\langle u, Au \rangle \geq 0$ for all $u \in \mathcal{H}$.

With these preparations, we are ready to extend Theorem A to Hilbert–Schmidt operators:

Theorem 2.4. *Fix \mathcal{H}, X as above, and Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Suppose $C_1, C_2 \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H})$ and $C_3 \in \mathcal{S}_2(\mathcal{H}_2, \mathcal{H})$, and define $J := \{x \in X : C_3^* e_x \neq 0\}$. If $(C_1 C_1^*)_{J \times J}, (C_2 C_2^*)_{J \times J}$ are nonzero, then*

$$(2.5) \quad C_3^* (C_1 C_1^* \circ C_2 C_2^*) C_3 \geq \gamma(C_1, C_2, J) \cdot C_3^* d_{C_1 C_2^T} d_{C_1 C_2^T}^* C_3,$$

where $\gamma(C_1, C_2, J)$ is as in (1.20). Moreover, the coefficient $\gamma(C_1, C_2, J)$ is best possible.

Sketch of proof. If both $(C_1 C_1^*)_{J \times J}$ and $(C_2 C_2^*)_{J \times J}$ have infinite rank, then (the denominator on) the right-hand side vanishes and the inequality reduces to the Schur product theorem. It is when at least one of these ranks is finite that the theorem provides a nonzero lower bound. In this case, one combines the proofs of Theorems A and 1.18; as there are subtleties given the infinite-dimensionality, we provide some details. First note that if $A \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ for a Hilbert space \mathcal{H}' (with \mathcal{H} as in the theorem), then

$$C_3^* A A^* C_3 = C_3^* P_J A A^* P_J C_3 = C_3^* (P_J A) (P_J A)^* C_3,$$

where the orthogonal projection P_J (onto the closed subspace \mathcal{H}_J) is as in Definition 2.3(1). Also note that post-composition by P_J sends the space $\mathcal{S}_2(\mathcal{H}', \mathcal{H})$ to $\mathcal{S}_2(\mathcal{H}', \mathcal{H}_J)$, and also sends finite-rank operators to finite-rank operators. Thus,

it suffices to prove the theorem without the C_3, C_3^* , and with C_1, C_2 replaced by $P_J C_1, P_J C_2 \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_J)$ respectively. This essentially reduces the situation to $J = X$, i.e., to Theorem A over (\mathcal{H}, X) – here we use that $\text{rk}(T_{J \times J}) \leq \text{rk}(T)$ for all $J \subset X$ and operators T of finite rank.

Thus, we assume henceforth that $J = X$, and repeat the proof of Theorem A carefully. First notice by Corollary 2.2 that $M = C_1 C_1^*, N = C_2 C_2^* \in \mathcal{S}_2(\mathcal{H}, \mathcal{H})$, hence so is $C_1 C_1^* \circ C_2 C_2^*$ by Lemma 2.1. We now use the key identity (1.12) applied to these M, N ; firstly, this makes sense as at least one of M, N is now of finite rank, so that the right-hand side has finite rank and hence is trace class by Lemma 2.1. Second, the identity (1.12) specialized to $M = C_1 C_1^*, N = C_2 C_2^*$ holds because both sides are additive and continuous in $u, v \in \mathcal{H}$ and hence can be reduced to (the easily verifiable case of) $u = e_x, v = e_y$. Thus we obtain (with M_u in place of D_u , and $v = u$):

$$\langle u, (C_1 C_1^* \circ C_2 C_2^*)u \rangle = \text{tr}(\overline{C_2} C_2^T M_{\bar{u}} C_1 C_1^* M_u),$$

where $\bar{u} \in \mathcal{H}$ is defined via: $\langle e_x, \bar{u} \rangle := \langle u, e_x \rangle$. But this equals $\text{tr}(N^* N)$ by Lemma 2.1, where $N := C_1^* M_u \overline{C_2}$ now (instead of $A^* D_u \overline{B}$). This is justified because at least one of C_1, C_2 has finite rank, hence so does N ; now N is trace class by Lemma 2.1 (and hence in $\mathcal{S}_2(\mathcal{H}, \mathcal{H})$).

For the same reasons, the same properties are satisfied by the projection P defined as in (1.13) (with C_1, C_2 in place of A, B). Now the remainder of the proof of Theorem A goes through with minimal modifications. □

Remark 2.6. It is natural to ask if Theorem 2.4 follows from Theorem 1.22 by restricting all operators in question to some common finite-dimensional space, e.g. the column space of the matrix on the left side. However, for infinite X this is not clear, because such a subspace need not contain a subset of $\{e_x : x \in X\}$ as a basis, and our Schur product is with respect to this basis $\{e_x : x \in X\}$.

APPENDIX A. FURTHER RAMIFICATIONS

We provide here a few related but somewhat peripheral observations.

A.1. Entrywise polynomial preservers in fixed dimension. The above results reinforce the subtlety of the entrywise calculus. As observed by Pólya–Szegő [11, Problem 37], the Schur product theorem implies that every convergent power series $f(x)$ with real non-negative Maclaurin coefficients, when applied entrywise to positive matrices of all sizes with all entries in the domain of f , preserves matrix positivity. A famous result by Schoenberg [16] and its strengthening by Rudin [13] provide the converse for $I = (-1, 1)$: there are no other such positivity preservers. These works have led to a vast amount of activity on entrywise preservers – see e.g. [2] for more on this.

If one restricts to matrices of a fixed dimension n , the situation is far more challenging and a complete characterization remains open even for $n = 3$. In this setting, partial results are available when one restricts the class of test functions, or the class of test matrices in \mathbb{P}_n – see [3] for details.

We restrict here to a brief comparison of Vybíral’s Theorem 1.3 with basic results in our recent work [9] with Tao and its ‘baby case’ [1] with Belton–Guillot–Putinar. These latter two papers study entrywise polynomial maps that preserve positivity on \mathbb{P}_n for fixed n , and we show in them that for real matrices in \mathbb{P}_n with entries in

$(0, \epsilon)$ (resp. (ϵ, ∞)) for any $\epsilon > 0$, if an entrywise polynomial preserves positivity on such matrices of rank one, then its first (resp. last) n nonzero Maclaurin coefficients must be positive. Contrast this with Theorem 1.22 (or Theorem 1.3 together with the Schur product theorem), which shows that for all real correlation matrices in \mathbb{P}_n , of a fixed dimension n , the polynomials $x^{2k} - 1/n$, $k \geq 1$ preserve matrix positivity when applied entrywise.

One hopes that this contrast, together with Remark 1.24 and the work [20], will lead to further new bounds and refined results for the entrywise calculus on classes of positive matrices.

Remark A.1. On the topic of the entrywise calculus: notice that if one applies (1.6) with A, B to be the positive square roots of M, N respectively, then

$$(A.2) \quad M \circ N \geq \frac{1}{\min(\text{rk}(M), \text{rk}(N))} d_{\sqrt{M}\sqrt{N}} d^*_{\sqrt{M}\sqrt{N}}, \quad \forall M, N \in \mathbb{P}_n(\mathbb{C}), \quad n \geq 1.$$

This provides a connection (and a ‘tight’ one) between the entrywise and functional calculus.

A.2. Positive definite functions and related kernels. As Vybíral remarks in [20], if g is any positive definite function on \mathbb{R}^d , or on a locally compact abelian group G , then Theorem 1.3 immediately implies a sharpening of the ‘easy half of Bochner’s theorem’ for $|g|^2$. We elaborate on this and other applications through the following unifying notion:

Definition A.3. Given a set X and a sequence of positive matrices $\mathcal{M} = \{M_n \in \mathbb{P}_n : n \geq 1\}$, a *complex positive kernel on X with lower bound \mathcal{M}* is any function $K : X \times X \rightarrow \mathbb{C}$ such that for all integers $n \geq 1$ and points $x_1, \dots, x_n \in X$, the matrix $(K(x_i, x_j))_{i,j=1}^n \geq M_n \geq \mathbf{0}_{n \times n}$.

Note, positive definite functions/kernels are special cases with $M_n = \mathbf{0}_{n \times n}$. By Theorem 1.22:

Proposition A.4. *Suppose $k \geq 1$, and for each $1 \leq j \leq k$, the function K_j is a complex positive kernel on a set X_j , with common lower bound $\{\mathbf{0}_{n \times n} : n \geq 1\}$. Also suppose $K_j(x_j, x_j) = \ell_j > 0 \forall x_j \in X_j, 1 \leq j \leq k$. Then the kernel \mathbf{K} on $X_1 \times \dots \times X_k$ given by*

$$\mathbf{K}((x_1, \dots, x_k), (x'_1, \dots, x'_k)) := \prod_{j=1}^k K_j(x_j, x'_j) K_j(x'_j, x_j), \quad x_j, x'_j \in X_j,$$

is complex positive on $X_1 \times \dots \times X_k$ with lower bound $\{\frac{1}{n} \prod_{j=1}^k \ell_j \cdot E_n : n \geq 1\}$.

This setting and result unify several different notions in the literature, as we now explain:

- (1) *Positive definite functions on groups:* Here X is a group with identity e_X , and K is the composite of the map $(x, x') \mapsto x^{-1}x'$ and a function $g : X \rightarrow \mathbb{C}$ satisfying: $g(x^{-1}) = \overline{g(x)}$. Then the hypotheses of Proposition A.4 apply in this case, with $\ell := g(e_X)$.

For instance, in [20] the author uses the positive definiteness of the cosine function¹ on \mathbb{R} to apply Theorem 1.3 and prove a conjecture of Novak [10]

¹On a related note: Vybíral mentions in [20] that $\cos(\cdot)$ is positive definite on \mathbb{R}^1 using Bochner’s theorem. A simpler way to see this uses trigonometry: given reals x_1, \dots, x_n , the matrix $(\cos(x_i - x_j))_{i,j=1}^n = uu^T + vv^T$, where $u = (\cos x_j)_{j=1}^n$ and $v = (\sin x_j)_{j=1}^n$.

– see Theorem A.7 below. This now follows from Proposition A.4 – we present here a more general version than in [20]:

Proposition A.5. *Let μ_1, \dots, μ_k be finite non-negative Borel measures on X , and g_l the Fourier transform of μ_l for all l . Then,*

$$\left(\prod_{l=1}^k |g_l(x_i^{-1}x_j)|^2\right)_{i,j=1}^n \geq \frac{1}{n} \prod_{l=1}^k g_l(e_X)^2 \cdot E_n.$$

- (2) *Positive semidefinite kernels on Hilbert spaces:* Here $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space over \mathbb{R} or \mathbb{C} , and K is the composite of the map $(x, x') \mapsto \langle x, x' \rangle$ and a function $g : \mathbb{C} \rightarrow \mathbb{C}$ satisfying: $g(\bar{z}) = \overline{g(z)}$. (See e.g. the early work by Rudin [13], which classified the positive semidefinite kernels on \mathbb{R}^d for $d \geq 3$, and related this to harmonic analysis and to the entrywise calculus.) In this case Theorem 1.22 applies; if one restricts to kernels that are positive definite on the unit sphere in X , then Proposition A.4 applies here as well, with $\ell := g(1)$ – and thus applies to *covariance kernels*, widely used in the (statistics) literature.
- (3) *Positive definite functions on metric spaces:* In this case, (X, d) is a metric space, and K is the composite of the map $(x, x') \mapsto d(x, x')$ and a function $g : [0, \infty) \rightarrow \mathbb{R}$. This was studied by several experts including Bochner, Weil, and Schoenberg. For instance, Schoenberg observed in [14] that $\cos(\cdot)$ is positive definite on unit spheres in Euclidean spaces, and went on to classify in [16] the positive definite functions $f \circ \cos$ on spheres of each fixed dimension d . The $d = \infty$ case is the aforementioned ‘converse’ to the Schur product theorem (i.e., it shows that the Pólya–Szegő observation above is ‘sharp’).

We conclude with a specific example, which leads to another result similar to Novak’s conjecture (shown by Vybíral). A well-known result of Schoenberg [15] says that the Gaussian kernel $\exp(-\lambda x^2)$ is positive definite on Euclidean space for all $\lambda > 0$.² (In fact Schoenberg shows this characterizes Hilbert space $\ell^2(\mathbb{N})$, i.e., the completion of $\bigcup_{d \geq 1} (\mathbb{R}^d, \|\cdot\|_2)$.) Thus:

Proposition A.6. *Given $x_{l1}, \dots, x_{ln} \in \ell^2(\mathbb{N})$ for $l = 1, \dots, k$, the $n \times n$ real matrix with (i, j) entry $\prod_{l=1}^k \exp(-\|x_{li} - x_{lj}\|^2) - \frac{1}{n}$ is positive semidefinite.*

This is similar to Novak’s conjecture, now shown by Vybíral:

Theorem A.7 ([10, 20]). *Given $x_{l1}, \dots, x_{ln} \in \mathbb{R}$ for $l = 1, \dots, k$, the $n \times n$ real matrix with (i, j) entry $\prod_{l=1}^k \cos^2(x_{li} - x_{lj}) - \frac{1}{n}$ is positive semidefinite.*

²On a related note: Schoenberg [15] shows the positive definiteness of the Gaussian kernel using Fourier analysis. In the spirit of the preceding footnote, we provide a purely matrix-theoretic proof in three steps – we also include this in the recent survey [2]:

- (1) A result of Gantmacher–Krein says square generalized Vandermonde matrices $(x_j^{\alpha_k})$ have positive determinant if $0 < x_1 < x_2 < \dots$ and $\alpha_1 < \alpha_2 < \dots$ are real.
- (2) This implies an observation of Pólya: the Gaussian kernel is positive definite on \mathbb{R}^1 . Indeed, given $x_1 < x_2 < \dots$, the matrix $(\exp(-(x_j - x_k)^2))$ equals DVD , where D is the diagonal matrix with diagonal entries $\exp(-x_j^2)$, and $V = (\exp(2x_j x_k))$ is a generalized Vandermonde matrix.
- (3) The positivity of the Gaussian kernel on every Euclidean space \mathbb{R}^d , hence on Hilbert space $\ell^2(\mathbb{N})$, now follows from Pólya’s observation via the Schur product theorem.

The two results are similar in that Novak's conjecture uses $\cos(\cdot)$ and \mathbb{R}^1 in place of $\exp(-(\cdot)^2)$ and $\ell^2(\mathbb{N})$ respectively. Both results follow from Proposition A.4.

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