Fractal dimension of non-network space of a catchment basin

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[1] Topographically convex regions within a catchment basin represent varied degrees of hill-slopes. The non-network space (M), the characterization of which we address in this letter, is akin to the space that is achieved by subtracting channelized portions contributed due to concave regions from the watershed space (X). This non-network space is similar to non-channelized convex region within a catchment basin. We propose an alternative shape-dependent quantity like fractal dimension to characterize this non-network space. Towards this goal, we decompose the non-network space in two-dimensional discrete space into simple non-overlapping disks (NODs) of various sizes by employing mathematical morphological transformations and certain logical operations. Furthermore, we plot the number of NODs of less than threshold radius against the radius, and compute the shape-dependent fractal dimension of non-network space. *INDEX TERMS:* 1824 Hydrology: Geomorphology (1625); 1848 Hydrology: Networks; 3200 Mathematical Geophysics; 3210 Mathematical Geophysics: Modeling; 3250 Mathematical Geophysics: Fractals and multifractals. **Citation:** Sagar, B. S. D., and L. Chockalingam (2004), Fractal dimension of non-network space of a catchment basin, *Geophys. Res. Lett.*, *31*, L12502, doi:10.1029/2004GL019749.

1. Introduction

[2] Channel network and non-network spaces are two important features within a catchment basin. Channel and ridge connectivity networks possess scale invariant properties. Change in lengths of these networks scales as power of resolution that indicates fractality. Despite the fact that there is no significant change in the areal extent of non-network space under the succession of scale changes, its topological organization varies due to change in the network length. The abstract structure, akin to the network connecting the regional *maxima* of topographically convex regions, explains this phenomenon. The topographically convex and concave regions respectively contain hierarchical concavities and convexities with increasing resolution. With increase in magnification, the increase in the observed length is true with both channel network and abstract network of the non-network space (Figures 1a and 1b). This implies that the scale dependent non-network space that can be represented as abstract structure, from which non-network space can be retrieved, also possesses fractal properties. Verifying Hortonian laws mostly involves the validation of several network models [e.g., *Scheidegger*, 1967; *Shreve*, 1967; *Tokunaga*, 1984; *Howard*, 1990; *Rodriguez-Iturbe and Rinaldo*, 1997; *Gupta and Veitzer*, 2000; *Sagar et al.*, 1998, 2001]. In addition to these laws that iron out much of the details of branched networks, in recent past allometric studies have resulted in several universal power-law relationships [*Maritan et al.*, 1996a, 1996b, 2002; *Sagar and Tien*, 2004]. Network characterization through Hortonian laws, and of late through fractal and multifractal properties receives notable attention, and various significant results have been accomplished. Several researchers relate fractal dimension of a network within a catchment basin to the bifurcation ratio (R_B) and length ratio (R_L) of idealized Hortonian-network trees as $D = \frac{\log R_B}{\log R_L}$ [*Mandelbrot*, 1982; *LaBarbera and Rosso*, 1987; *Tarboton et al.*, 1988; *Takayasu*, 1990; *Rigon et al.*, 1993; *Beer and Borgas*, 1993; *Nikora and Sapozhnikov*, 1993; *Marani et al.*, 1991; *Sagar*, 1996; *Turcotte*, 1997; *Rodriguez-Iturbe and Rinaldo*, 1997; *Sagar et al.*, 1998, 2001]. This non-shape-dependent dimension based on two morphometric quantities explains a space-filling characteristic of the network. Intuitively, it is clear that networks respectively from elongated and radial basins may yield the same fractal dimension D, so it seems that this non-shape-dependent D may be of limited use to relate process with shape of a watershed. Heuristic argument is that the D may be the same for certain elongated and radial basins, as the computed topological quantities do not consider any other properties than network length and number. However, *Tarboton et al.* [1988] provide a parameter $D = d\frac{\log R_B}{\log R_L}$. *Karlinger et al.* [1994] describe the fractal scaling of river networks in the context of both thin and fat fractals. They characterize the fat-fractal dimension as a scaling exponent derived from the behavior of the river-channel area.

[3] In the present letter, we propose a technique to characterize non-network space via a morphological decomposition procedure, which is popular in shape description studies [*Serra*, 1982]. This technique provides a shape-dependent power-law. We consider this geometric approach to characterize non-network space within catchment basins. In what follows, we introduce basic morphological transformations needed to develop a framework (Section 2), isolation of non-network space from the reconstructed basin (Section 3), decomposition of the non-network space into non-overlapping disks (NODs) and computation of a power-law based fractal dimension (Section 4), and concluding remarks (Section 5).

2. Morphological Transformations

[4] The landscape with varied topographic relief consists of catchment basins of several sizes and shapes. We define catchment basin as the topographical space within the boundary. One of the unique connectivity networks in a

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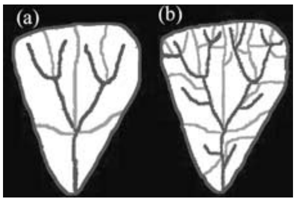

Figure 1. Schematic of a catchment basin with channel (blue) and abstract structure (red) of non-network space at varied resolutions. See color version of this figure in the HTML.

catchment basin is channel network (C). Further, we define channel network and non-network space within a catchment basin as: if C in 2-dimensional Euclidean discrete space Z^2 belongs to the set A, the pixel representing channel network in the basin is white; otherwise it is black. The complement of C (C^c) represents the channel network background. We define a symmetric template as follows: $S^s = [-s{:}s \in S\}$, where S^s is obtained by rotating S by 180° on the plane. Next, we explain simple morphological transformations such as binary erosion, dilation, opening, and closing on the sets, the application of which will be shown to reconstruct a basin from channel network. We perform these transformations [*Serra*, 1982] by means of S that is symmetric with respect to the origin, octagonal in shape and has the size of 5 × 5 (Figure 2b). We define *erosion* of C by S as the set of three points c such that the translated S_c is contained in the original set C, and is equivalent to intersection of all the translates expressed as $C \ominus S = \{c{:}S_c \subseteq C\} = \bigcap_{s \in S} C_{-S}$.

[5] We define *dilation* of C by S as the set of all those points c such that the translated S_c intersects C, and is equivalent to the union of all the translates expressed mathematically as $C \oplus S = \{c{:}S_c \cap C \neq \emptyset\} = \bigcup_{s \in S} C_{-S}$.

[6] To avoid confusion, we refer to $C \ominus S$ and $C \oplus S$ simply as erosion and dilation. The dilation with an elementary structuring template expands the set with a uniform layer of elements while the erosion operator eliminates a layer from the set. We denote recursive erosions and dilations as $(C \ominus S) \ominus S \ominus \ldots \ominus S = (C \ominus S_{ri})$ and $(C \oplus S) \oplus S \oplus \ldots \oplus S = (C \oplus S_{ri})$ respectively. We further represent respectively opening and closing transformations by employing erosion and dilation of C by S as $C \circ S = ((C \ominus S) \oplus S))$ and $C \bullet S = ((C \oplus S) \ominus S))$.

[7] We illustrate these transformations in Figure 2. We apply recursive erosions and dilations to perform mutliscale opening and closing transformations as $(C \circ S_{ri}) = [(C \ominus S_{ri}) \oplus S_{ri})]$ and $(C \bullet S_{ri}) = [(C \oplus S_{ri}) \ominus S_{ri})]$, where ri is the number of times the transformations are repeated. We encourage the Reader to refer to [*Matheron*, 1975; *Serra*, 1982] for morphological transformations and their numerous applications.

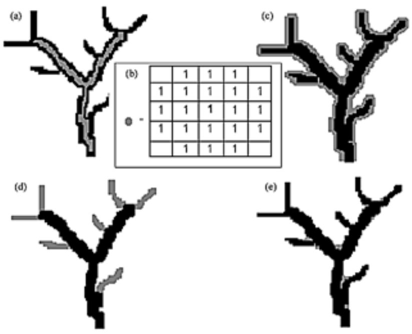

Figure 2. Morphological (a) erosion, (b) structuring template of 5 × 5 size, (c) dilation, (d) opening, and (e) closing of C by S. The red color indicates the change after the respective transformation. See color version of this figure in the HTML.

3. Non-network Space of a Catchment Basin

[8] Let C be the object in Figure 3a and $S \in Z^2$ that in Figure 2b. Channel network and its complementary space are respectively represented with white and black pixels. Finding out the non-network space, within a catchment basin, from channel network consists of

[9] **(a)** application of multiscale closing to reconstruct the basin, and

[10] **(b)** subtraction of channel network from the reconstructed basin achieved in step (a).

[11] Let S be a discrete probing rule, with $S \in Z^2$, bounded, convex, symmetric and containing the origin. We consider a realistic network (Figure 3a) to employ the framework to show the applications of various steps mentioned in subsequent phases. We employ multiscale closing to reconstruct the basin from channel network as $X = \bigcup_{i=0}^{N} C \bullet S_{ri}$, where $C \subset X$. We define non-network space (M) within a basin (X) as a bounded, binary valued discrete space object as $M = [X \backslash C] \subset Z^2$, where C is a channel network. By subtracting the channel network from the bounded reconstructed basin X, we obtain non-network space M

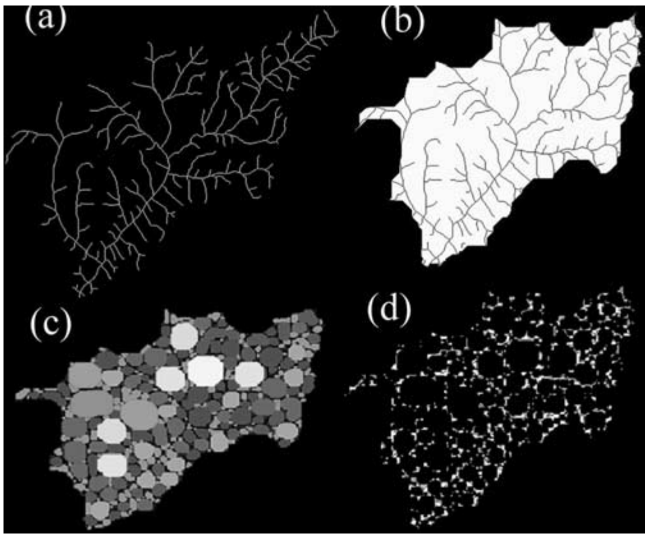

Figure 3. (a) 5th order channel network (C) of Durian Tungal catchment basin, basin X is reconstructed from this channel network via multiscale morphological closing transformation, (b) $M = X \backslash C$ is non-network space within a catchment basin, (c) decomposition of non-network space (M) into non-overlapping disks of octagon shape of several sizes, and (d) transition lines between the packed objects. See color version of this figure in the HTML.

(Figure 3b) of the basin. Decomposition of non-network space provides valuable insight for modeling and understanding catchment basins. The characterization of such scale-dependent topological organization of non-network space has hitherto received little attention.

4. Decomposition of Non-network Space and Dimension Computation

[12] The topographically significant regions in the non-network space include regions with varied degrees of slope, narrow regions with steep gradient, and the corner-portions adjacent to the stream confluence. Characterization of these zones is of importance from the point of understanding the involved geomorphic process within a catchment basin. We estimate the fractal dimension of the non-network space through the following two steps:

[13] **Step 1:** Morphological decomposition of non-network space of catchment basin is someway related to a packing process. In several studies, packing of space (e.g., pore) is done [*Manna and Herrmann*, 1991; *Dodds and Weitz*, 2002, 2003; *Lian et al.*, 2004; *Radhakrishnan et al.*, 2004]. We transform the complex non-network space (M) into "simpler" NODs by defining these simple components as convex polygons with certain characteristics. Here, we use a symmetric octagonal structuring element (Figure 2b) as the simple probing component to convert M into NODs by employing morphological decomposition according to the recursive relation depicted as follows. This recursive relation includes three steps as (a) $M_i = [(M - M'_{i-1})\Theta S_{r_i}] \oplus S_{r_i}$, (b) $M'_i = \bigcup_{0<j\leq i} M_j$, and (c) $M'_o = \phi$, where r_i at step i is the radius of the maximum inscribable disks S_{ri} in any of the connected components of $M - M'_{i-1}$. We give a non-mathematical description of the morphological decomposition of non-network space of the basin into *NOD*s in the following way. We obtain the set of maximum inscribable disk(s) that has (have) the maximum radius in M. We consider this set as the first level decomposed disk(s) in the decomposition. The second set of the maximum inscribable disks in the portion of the basin is that obtained by subtracting the first cluster from M. The procedure is repeated on the portion of the basin that is obtained by subtracting the first and second decomposed disks, until the reminder of the non-network space becomes an empty set. The more regular is the set M, the smaller is the number of categories of regular type *NOD*s of different sizes. Here, we decompose the space in a non-overlapping way with an octagon. The non-overlapping disks $[M_1,\ldots,M_n]$ whose union is M, as $M = \bigcup_{i=1}^{n} M_i$. M_i is a simple set that is equal to discrete rule S of size r_i : $M_i = S_{ri}$, where r_i is the same integer as in the relation $M_i = S \oplus S \oplus \ldots \oplus S$ (ri times, where $\oplus$ denotes Minkowski addition). Figure 3 illustrates the decomposition process explained in its sequential phases, as (a) network of the Durian Tungal catchment, (b) $X \backslash C$ non-network space, (c) decomposed-coded non-network space, and (d) transition lines just before the M becomes empty. One can extend this procedure to 3-dimensional non-network space by seeding 3-D templates, in a non-overlapping way, of varied sizes allowing each template to grow until it collides with the surface of a topographically convex region.

[14] The decomposition of non-network space throws some light on classification and characterization of landscape morphology from the point of its surface roughness to further understand about geomorphic activity. The regions of varied degrees of geomorphic activity within a basin can be linked with hill-slope processes. Various categories in the non-network space can be better segmented through the various size-categories of NODs. The non-network space in between the network segments with lesser diverging angle is the region that we achieve with decreasing number of multiscale closings. Smaller category NODs occupy non-network space that is surrounded by dense network segments, the diverging angles between which are relatively less, and the zones adjacent to the channel confluence. The diverging angle of channels determines the topology of confluence. The higher the diverging angle, the larger the disk that can be inscribed, and *vice versa*. This description enables that various categories of NODs can be related to different degrees of topographically convex regions [*Dietrich et al.*, 1992; *Montgomery and Dietrich*, 1992] within a catchment basin.

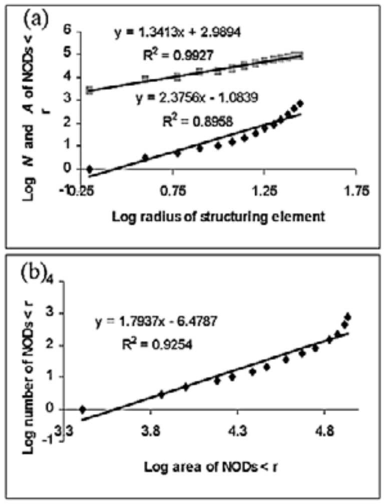

Figure 4. (a) Double-logarithmic plot between the radii of structuring templates and corresponding number and area of NODs, and (b) double logarithmic plot between area and number of NODs with increasing radius of structuring element. See color version of this figure in the HTML.

[15] **Step 2:** We determine power-law exponents for the NODs' number and size distributions by means of a connection to the decay of non-network space of catchment basin. Based on the assumption that the shape of the non-network space alters the number and size distributions of NODs, these exponents are strongly shape dependent. We compute the number of NODs smaller than the specified threshold radius of structuring template and their contributing areas respectively denoted as $N\lfloor NODs(<S_{r_i})\rfloor$ and $A\lfloor NODs(<S_{r_i})\rfloor$. By employing these numbers, their contributing areas, and the corresponding radius of template, we derive simple power-law relationships for a realistic catchment basin. We plot double logarithmic graphs, the slopes of the best-fit lines (α_N and α_A) respectively for number-radius, and area-radius relationships yield 2.37 and 1.34 (Figure 4a) from the relationships as $N\lfloor NODs(<S_{r_i})\rfloor$ (or) $A\lfloor NODs(<S_{r_i})\rfloor \sim r^{\alpha_N (or) \alpha_A}$, where r = radius of template, and α = slope of the best-fit line. These slope values

of the best-fit lines provide shape dependent dimensions as $D_N = \alpha_N - 1$, and $D_A = \alpha_A$.

[16] We compute D_N and D_A for non-network space (Figure 3b), which yield 1.37 and 1.34 (Figure 4a). We also show a power-law relationship, with an exponent value 1.79, between the area and number of NODs observed with increasing radius of structuring template (Figure 4b). However, the ratio of logarithms of bifurcation and mean length ratios of the network yields fractal dimension of 1.77. This shape-dependent dimension provides an insight, if it can be related to other dimensions estimated via linear aspects of the branched networks.

5. Conclusions

[17] This letter addresses two aspects: (a) to generate non-network space (*M*) from the basin (*X*) reconstructed from channel network such that the channel network is contained in *X*, and (b) to decompose *M* into NODs to compute an alternative shape dependent dimension. To achieve these goals, we use set-theory and topology-based mathematical transformations that have hitherto been relatively less employed in geophysics. It would be interesting to compute a spectrum of similar quantities both in 2-, and 3-dimensions, by employing a family of various symmetric and asymmetric probing rules, for the basins possessing varied degrees of self-affinity (decreasing circularity ratio) to establish a relationship between the shape-dependent dimension, and the geometric and morphometric characteristics. This framework allows systematically characterizing and validating the topological properties of the non-network space of various realistic and simulated networks *via* shape-dependent dimension. We also find that the dimensions computed by means of two topological quantities, and area-number relationship are significantly similar. We infer that these dimensions of 1.77 and 1.79 for this case explain space-filling characteristics of networks. These statements infer from a single case need further validation.

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