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Estimation of fractal dimension through morphological decomposition

P. Radhakrishnan [a], Teo Lay Lian [b], B.S. Daya Sagar [b,*]

[a] *Faculty of Information Science and Technology, Melaka Campus, Multimedia University, Jalan Ayer Keroh Lama, 75450 Melaka, Malaysia*

[b] *Faculty of Engineering and Technology, Melaka Campus, Multimedia University, Jalan Ayer Keroh Lama, 75450 Melaka, Malaysia*

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Abstract

Set theory based morphological transformations have been employed to decompose a binary fractal by means of discrete structuring elements such as square, rhombus and octagon. This decomposition provides an alternative approach to estimate fractal dimensions. The fractal dimensions estimated through this morphological decomposition procedure by employing different structuring elements are considerably similar. A color-coding scheme is adapted to identify the several sizes of decomposed non-overlapping disks (DNDs) that could be fit into a fractal. This exercise facilitates to test the number–radius relationship from which the fractal dimension has been estimated for a Koch Quadric, which yield the significantly similar values of 1.67 ± 0.05 by three structuring elements. In addition to this dimension, by considering the number of DNDs of various orders (radii) and the mean diameter of disks (MDDs) of corresponding order, two topological quantities namely number ratio (R_B) and mean diameter ratio (R_L) are computed, employing which another type of fractal dimension is estimated as $\frac{\log R_B}{\log R_L}$. These results are in accord with the fractal dimensions computed through number–radius relationship, and connectivity network of the Koch Quadric that is reported elsewhere.

1. Introduction

Planar shapes can be decomposed into regular shapes of several sizes. For instance, a square shape can be decomposed into shapes such as circle, rhombus, and octagon of various sizes. This decomposition facilitates a procedure to estimate the dimensions, akin to fractal dimension, through a power-law relationship between size or radius, and number of decomposed and disconnected shapes at a given threshold value. The power law relationship is represented as Eq. (1).

$$N(\leqslant S_n) \sim (r)^D \tag{1}$$

where, N, r, and D respectively represent the number of decomposed non-overlapping disks (DNDs) that are smaller than the structuring element of size S_n, radius and the fractal dimension. In addition to this power-law based fractal dimension, topological quantities, where the two basic measures such as number of DNDs and mean diameter of disks (MDDs) of corresponding orders provide another method to estimate fractal dimension. The fractal dimensions that could be derived from these two methods are treated as unique, since only the internal region of the shape is considered.

* Corresponding author.
E-mail address: bsdaya.sagar@mmu.edu.my (B.S. Daya Sagar).

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In other words, if the shape of which the dimension is to be estimated is in binary format, for instance shape with 1s, and its background with 0s, only the region with 1s will be considered by leaving the region with 0s. Mathematical morphological transformations have been earlier applied to estimate fractal dimensions [1–4]. The application of mathematical morphological transformations has been shown to deal with the fractal related studies [5–12]. In what follows in this paper includes basic introduction on mathematical morphology that is required to implement the decomposition procedures, morphological decomposition of fractal, and an alternative procedure to estimate fractal dimension, and the obtained results.

2. Morphological transformations

The discrete binary fractal, M, is defined as a finite subset of Euclidean two-dimensional space, IR^2 that can admit values of 1 and 0. Basic morphological transformations such as dilations and erosions are defined as set transformations that expand and contract a set. These transformations [13] can be visualized as working with two images. The image being processed is referred to as M and other image being a structuring template (S). Each S possesses a designed shape that can be thought of as a probe of M. To estimate the dimension through morphological decomposition procedure, one can consider a probing rule with which the M under study can be decomposed. The probing rules include various structuring templates (Fig. 1). The four basic morphological transformations include *dilation*, *erosion*, *opening* and *closing* are mathematically represented as Eqs. (2)–(5). Dilation and erosions of M by a structuring element S are the union and intersection of the translates M_{-s} of M where s sweeps S [13]. S is assessed to be a compact set having a finite number of points. Thus, only finite number of translations of M are required. A brief description of dilation/erosion procedure follows: For the fractal M, the structuring element S, and the result, three image bit planes are required respectively. The image plane M is shifted in parallel to the result plane according to the s to S. The result plane holds the parallel OR or AND of the shifted version of the image bit plane. After all the points of S have been covered, it will contain the dilation or erosion respectively of the original image. The mathematical representations of these two basic transformations are shown in Eqs. (2) and (3).

$$\text{Dilation}: \quad M \oplus S = \{m : S_m \cap M \neq \emptyset) = \cup_{s \in S} M_s \tag{2}$$

$$\text{Erosion}: \quad M \ominus S = \{m : S_m \subseteq M) = \cap_{s \in S} M_s \tag{3}$$

In other words, dilation and erosion transformations respectively combines and subtracts M and S using vector addition and subtraction of set elements m and s, respectively, $m = (m_1, \ldots, m_N)$ and $s = (s_1, \ldots, s_N)$ being N-tuples of element co-ordinates. Then, the dilation and erosion of M by S are respectively the set of all possible vector sums and subtractions of pairs of elements, one coming from M and the other from S. This dilation of M by S is defined as the set of all points m such that all translates of m by s (S_m) intersects M as shown in Eq. (2). The erosion transformation of M by S is defined as the set of points m such that the translated S_m is contained in M, and is expressed as in Eq. (3). By employing these two transformations, opening and closing are defined respectively as Eqs. (4) and (5). The dilation followed by erosion is called closing transformation. Cascade of erosion–dilation is called opening transformation.

$$\text{Opening}: \quad M \circ S = (M \ominus S) \oplus S \tag{4}$$

By duality, the closing of M by S comes from dilation first and then erosion.

$$\text{Closing}: \quad M \bullet S = (M \oplus S) \ominus S \tag{5}$$

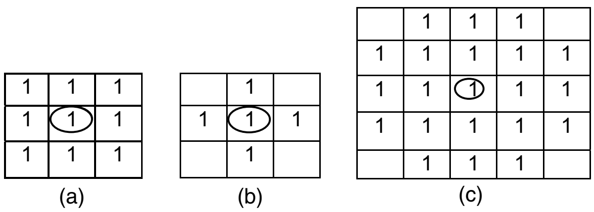

Fig. 1. Discrete structuring elements (a) square, (b) rhombus, and (c) octagon in IR^2.

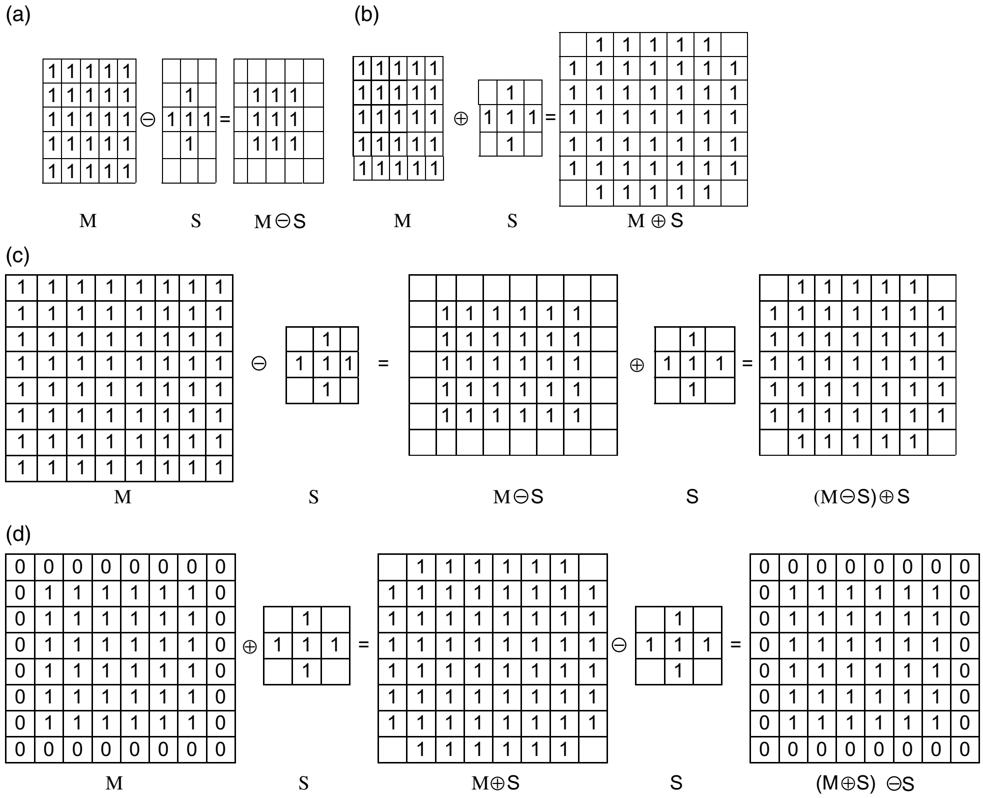

Fig. 2. Diagrammatic representation of basic morphological transformations (a) erosion, (b) dilation, (c) opening, and (d) closing.

In Eqs. (2)–(5), M is a fractal shape, m and s are set elements respectively, $m_1, \ldots, m_n = m$ and $s_1, \ldots, s_n = s$; S_m is translates of all m points by S, and $\oplus$, $\ominus$, $\circ$, $\bullet$ respectively denote Minkowski's addition, subtraction, and symbols for opening and closing. Definition of the morphological transformations, which appear in (2)–(5) can be found in [13]. The Minkowski's addition and subtraction are similar to morphological dilation and erosion respectively under the condition that $S = \widehat{S}$. These transformations can also be carried out according to the multiscale approach [14] in which the cascades of erosion–dilation and dilation–erosion are defined with respect to the structuring element S with scaling factor n. In this approach, the size of the structuring template will be increased from iteration to iteration as shown in Eq. (6).

$$S_n = \underbrace{S \oplus S \oplus S \oplus \cdots \oplus S}_{n\text{times}} \tag{6}$$

Fig. 2 depicts the diagrammatic representation of these morphological transformations that are used in the discrete morphological decomposition procedures described in Section 3.

3. Morphological decomposition: fractal dimension

To estimate the fractal dimension [15] of a fractal, a procedure, based on morphological decomposition that includes systematic use of multiscale *opening* and simple logical operators is adapted. Morphological decomposition of a fractal in to a union of disks is performed in the following way. The set of maximum inscribable disks in the fractal (M), that have the maximum radius, is found. This identified set, the first level, of decomposed fractal (or) conveniently we termed it as nth order decomposed DNDs. The $(n-1)$th order DNDs is obtained in the following way. The nth order DND is subtracted from the M. Then the set of the maximum inscribable DNDs in the remaining of the fractal, that have the maximum radius, is found. This set is the second level of decomposed fractal, (or) $(n-1)$th order DNDs. The first and second level sets of the DNDs are subtracted from M, and the procedure is repeated until the reminder of M becomes empty set. This entire decomposition procedure is described in a mathematical way by following Eq. (7).

$$\left.\begin{array}{l} M_2 = M_1 \setminus M_1 \circ S_n \\ M_3 = M_2 \setminus M_2 \circ S_n \\ M_n = M_{n-1} \setminus M_{n-1} \circ S_n \quad \text{and} \quad M_{n+1} = M_n \setminus M_n \circ S_n \\ \because \bigcup_{n=0}^{N} M_n = M \quad \text{and} \quad \because M_{n+1} = \phi \\ M_{\text{Decomp}} = (M \circ S_n) \cup (M_1 \circ S_n) \cup (M_2 \circ S_n) \cup \cdots \cup (M_{n-1} \circ S_n) \\ \because M_n \subset M_{n-1} \subset \cdots \subset M_3 \subset M_2 \subset M_1 \subset M \end{array}\right\} \tag{7}$$

After performing n times of multiscale opening on M, which is subjected to estimation of fractal dimension, the *opened* M needs to be subtracted from the original M. This can be achieved by simple logical operation, which is represented as the symbol ($\setminus$). If $n+1$ times are required to vanish a set M, n times of multiscale openings need to be performed to decompose M and successively achieved subtracted portions of the shape. On each subtracted portion, $n+1$ times of multiscale *opening* should vanish the respective M. Number of subtracted portions that may appear while decomposing M depends on its size and shape, and of the structuring template with its characteristic information. Morphological decomposition gives the possibility to represent a fractal as union of simple objects. Each of these object is completely described by the locus and the radius of the corresponding disks. To have a better understanding of the superficially simple morphological transformations, various steps involved in morphological decomposition, and the subsequent procedures to estimate fractal dimensions through a number–radius relationship (1) and topological quantities (8)–(11), have been shown in flowchart (Fig. 3). The implementation of sequential steps involved in decomposing the shape is also diagrammatically depicted (Fig. 4).

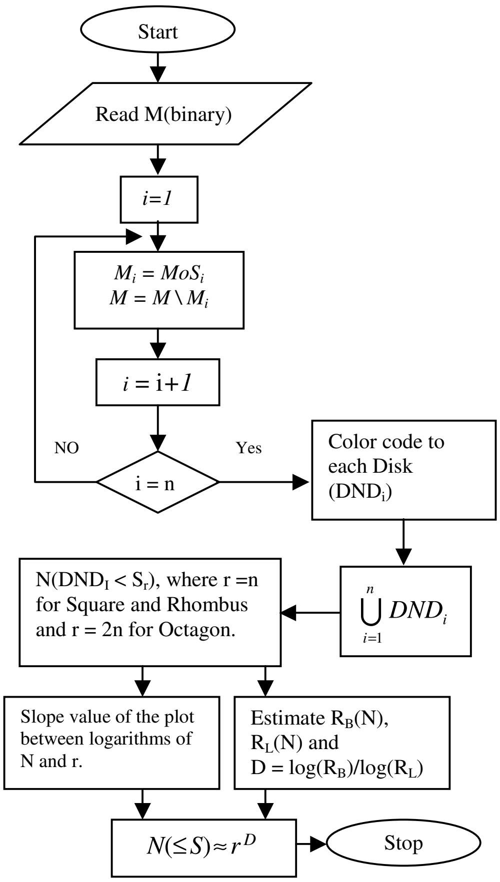

Fig. 3. Flowchart showing the sequential steps.

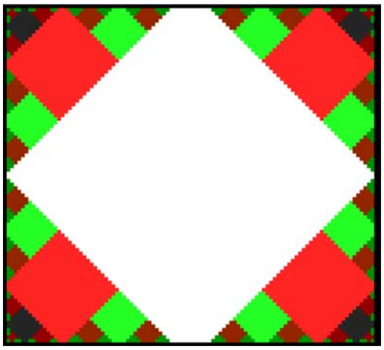

Fig. 4. A square shape that is decomposed by means of a rhombus structuring element.

3.1. Dimension based on two topological quantities

By employing the two topological quantities such as N(DNDs) and MDDs, number ratio (R_B) and MDD ratio (R_L) can be computed by Eqs. (8) and (9).

$$R_B(n,N) = \frac{\text{Number of DND}(n,N)}{\text{Number of DND}(n+1,N)}, \quad n = 1,2,\ldots,N-1 \tag{8}$$

$$R_L(n,N) = \frac{\text{MDD of } S(n,N)}{\text{MDD of } S(n-1,N)}, \quad n = 2,3,\ldots,N \tag{9}$$

Eqs. (10) and (11) are to compute $R_B(N)$ and $R_L(N)$.

$$R_B(N) = \frac{\sum_{n=1}^{N} R_B(n-N)}{N-1} \tag{10}$$

$$R_L(N) = \frac{\sum_{n=1}^{N} R_L(n,N)}{N-1} \tag{11}$$

By considering $R_B(N)$ and $R_L(N)$ computed by considering DNDs and MDDs for the three structuring elements, fractal dimensions (D) can be computed by following Eq. (12).

$$D = \frac{\log R_B}{\log R_L} \tag{12}$$

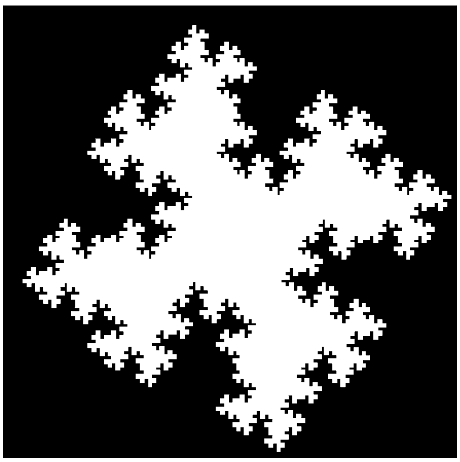

Fig. 5. Binary Koch quadric fractal.

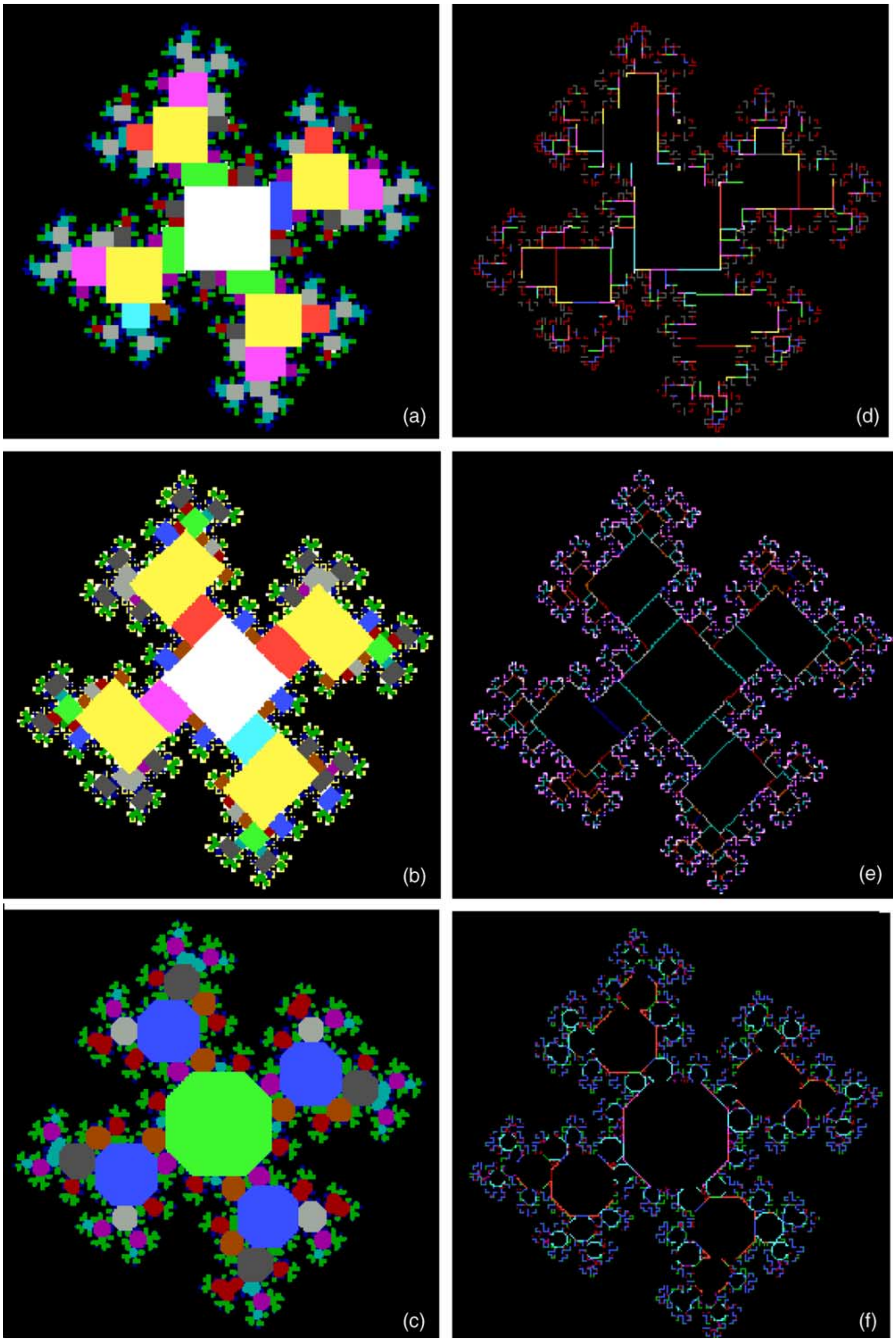

Fig. 6. Fractal decomposition (a, b, c) by means of square, rhombus and octagon respectively, (d, e, f) the transition lines between the color-coded decomposed regions.

Table 1
Fractal dimensions estimated from number–radius power-law relationship

Primitive structuring template	Cycle no./radius	Cumulative number of DNDs ($N(\leqslant S)$)	Log(r)	Log $N(\leqslant S)$	Fractal dimension (D)
Square	42	1	1.623249	2.782473	1.6726
	27	5	1.431364	2.49693	
	15	9	1.176091	2.09691	
	13	12	1.113943	1.929419	
	12	13	1.079181	1.826075	
	11	16	1.041393	1.732394	
	10	17	1	1.716003	
	8	26	0.90309	1.414973	
	7	52	0.845098	1.230449	
	6	54	0.778151	1.20412	
	5	67	0.69897	1.113943	
	4	85	0.60206	1.079181	
	3	125	0.477121	0.954243	
	2	314	0.30103	0.69897	
	1	606	0	0	
Rhombus	63	1	1.799341	1.908485	1.6199
	40	5	1.60206	1.812913	
	22	6	1.342423	1.78533	
	21	7	1.322219	1.681241	
	20	8	1.30103	1.612784	
	15	12	1.176091	1.322219	
	11	21	1.041393	1.079181	
	10	41	1	0.90309	
	9	48	0.954243	0.845098	
	7	61	0.845098	0.778151	
	6	65	0.778151	0.69897	
	5	81	0.69897	0	
Octagon	26	1	1.414973	2.502427	1.6754
	15	5	1.176091	1.892095	
	8	9	0.90309	1.763428	
	6	13	0.778151	1.579784	
	5	21	0.69897	1.322219	
	4	38	0.60206	1.113943	
	3	58	0.477121	0.954243	
	2	78	0.30103	0.69897	
	1	318	0	0	

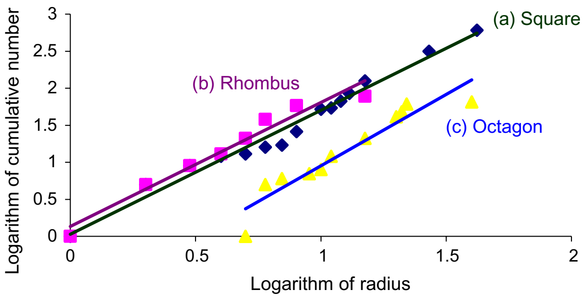

Fig. 7. Fractal plots between number of decomposed portions and radius of the structuring elements (a) square, (b) rhombus, and (c) octagon.

Table 2
Fractal dimensions estimated through two topological quantities

Structuring element	Order-wise no. of DNDs							Order-wise MDDs							R_B	R_L	$\log R_B / \log R_L$
	1	2	3	4	5	6	7	1	2	3	4	5	6	7			
Square	292	189	68	41	8	4	1	2	4	8	15	28	54	84	2.8513	1.870	1.6725
Rhombus	33	27	13	7	1			8	16	32	64	126			3.0390	1.9921	1.6127
Octagon	240	56	12	8	1			4	18	27	61	104			4.6130	2.4910	1.6754

4. Results and conclusion

As a sample study to implement the framework thus described in the previous section, a Koch Quadric [15] binary fractal is considered. The geometrical properties of M and its complement (M^c) are subjected to the morphological functions explained briefly earlier. This fractal of size 512×512 (Fig. 5) is decomposed into simpler non-overlapping shapes, of several sizes, by employing discrete square, rhombus, and octagon structuring elements. The fractal after decomposition by means of these structuring elements has been color-coded for better understanding, and shown in Fig. 6(a), (b), (c) respectively. These color-coded DNDs of all orders, achieved respectively by these structuring elements, and the interfaces between the different orders of these DNDs are shown in Fig. 6(a)–(f). The number of decomposed patterns of square, rhombus, and octagon of respective sizes has been given in the Table 1. The smaller the size of the primary pattern that is used to decompose the fractal, the larger the number of cycles that the fractal is required to be decomposed. Hence, it is apparent that the number of cycles is more while decomposing with rhombus, and followed by square and octagon. This is due to the fact that the size of the primary structuring element of octagon is larger than that of square, and of rhombus. The sequence of cycles can also be visualized as growth stages of fractal. The number–radius power-law relationship is shown for the fractal that is decomposed with these structuring templates. The power-law relationship represents $N(\leqslant S_n) \sim (r)^D$. The power exponent D stands for the fractal dimension. The D is estimated from the graphs plotted between the logarithms of radius and the number of decomposed portions of all sizes as 1.67 with all the three structuring elements. The graphical plots are given in Fig. 7. The fractal dimension estimated from the number–radius power law relationship yields the considerably similar values of 1.67 (Table 1) with all the three different structuring elements. However, the fractal dimension estimated by box dimension method yields the value of 1.72, whereas the box counting dimension for the boundary of the same fractal is estimated as 1.5.

After redistributing the total DNDs of all size categories of M, that are decomposed by means of discrete square, rhombus, and octagon structuring elements, the total number of orders (N) yield respectively 7, 5, and 5. The mean diameters of the corresponding disks are also estimated from the number of cycles performed in each level of decomposition. Number of DNDs existing in between the redistributed successive levels of decomposition with corresponding orders is computed. By employing these two topological quantities, number ratio (R_B), and MDD ratio (R_L) are computed by following Eqs. (8)–(11). By considering $R_B(N)$ and $R_L(N)$ computed by considering DNDs and MDDs for the three structuring elements, fractal dimensions (D) are also computed by following Eq. (12). The results are given in Table 2. These results are in accord with the fractal dimensions computed through number–radius relationship, and connectivity network of the Koch Quadric that is reported elsewhere. Method of estimating fractal dimension through morphological decomposition is most appropriate to characterize pore structures or porous media [16]. For instance, a section containing pore and grain regions, to estimate the fractal dimension of the pore, this method decomposes only pore region without considering the grain part.

A new fractal dimension estimation technique is proposed. It is based on mathematical morphology and it can decompose binary fractals into a union of simple disks. This method, based on morphological decomposition is unique in the sense that it considers the topological region rather than its geometric boundary.

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