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Morphological Evolution of a Pyramidal Sandpile Through *Bifurcation Theory:* a Qualitative Model

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Abstract—A simple first order nonlinear difference equation that has physical relevance to model the morphological evolution of a pyramidal sandpile is used to simulate distinct possible behaviours. As an attempt to furnish the interplay between numerical experiments and theory of morphological evolution, numerical simulations are performed by iterating this difference equation iterating 3×10^4 time steps to illustrate several possible morphological dynamical behaviours of a sandpile by changing the regulatory parameter, λ, that explains the detailed form of exodynamic process. Bifurcation diagram is described as a model to illustrate how the sandpile under dynamics behaves concerning change of regulatory parameter. Computed attractor inter-slip face angles (θ^*) at respective threshold regulatory parameters are depicted on the bifurcation diagram. By considering these θ^*s, an equation is also proposed to compute metric universality. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Sand falls on the *supply area* in the form of particles of various shapes. The accumulation of thick strata of sand and its transformation into the sandpile is a lengthy and complicated process proceeding under the effect of various *exodynamic* processes. These processes are the direct causes for sanddrift, sand withdrawal, sand assemblage and eventually the effect of these processes is the oscillations in the morphology of a sandpile profile. Due to these effects, sand dunes undergo flattening and protrusion. The spatiotemporal organisation of such a sandpile can show many different morphological dynamics because of different morphological constitutions that the sandpile traverses and also due to the type of wind actions. These morphological changes may be according to a rule through which one can explore the morphological dynamics. Moreover, these oscillations are dependent on the regulatory parameter that plays vital role in the present investigation. This parameter can be derived by studying the morphology of a sandpile at specific time intervals. Several papers have appeared during the last decade that address the application of fractal concepts on the studies of geoscientific interest. Behaviour of various systems of geoscientific interest such as electrical conductivity and fractures of rocks to the microcrack population [1], coalescence of fractures [2, 3], and stick-slip behaviour [4] through renormalisation group approach. In particular, several models have been proposed to comprehend the dynamical processes of sandpiles in two dimensions [5, 6]. However, the present paper aims to provide a qualitative model for morphological dynamics of a pyramidal sandpile through bifurcation theory.

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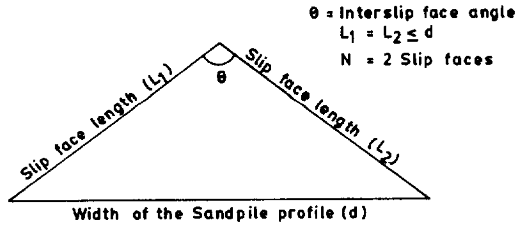

Fig. 1. A pyramidal sandpile profile.

1.1. *A pyramidal sandpile*

The description of the morphology of a sandpile is mainly concerned with its profile that may be described as angular. Such a sandpile is assumed as pyramidal if its slip faces are of equal lengths. Pyramidal sandpiles form due to convection and interferential types of wind. Such sandpiles are common in the Central Asia and Africa. There is a less scope for the movement of sand dune due to convection and interferential type of wind conditions. Such sandpiles cover limited areal extent and owe their composition of the interference of air waves caused by wind reflecting from mountain barriers [7]. The transitions in the sandpile profile may be observed at different type of conditions. It is heuristically justifiable that the degree of unsteady state to fall over is more in the steep sandpiles. In this paper, the profile of a pyramidal sandpile (Fig. 1) with the following specifications is considered.

- Sandpile profile should have a heap with two slipfaces of equal lengths ($L_1 = L_2$). It is symmetric with respect to origin at the center of the base of the heap.
- The d, width of the base of the heap should be greater than the length of a slip face ($d > L_1 = L_2$). This assumption is cogent due to the fact that the length of the slip face is not greater than the width (d), e.g., Bharkan sands and pyramidal sand forms of certain areas of Central Asia and Africa [7].

The width of the sandpile profile (d) is considered as rigid during the progressive sandpile evolution. But, the length of the slip face (L) varies with the continuous accumulation of sand.

1.2. *Physical problem and a rule for sandpile morphological dynamics*

The intensity of the cause can be derived from the effect. Such a derived cause might be in terms of various physical forces. The effect of this cause is in terms of deformation that can be quantified by means of an analytical value (e.g. fractal dimension). The fractal dimensions at discrete time intervals enable that whether the morphological dynamics of sandpile is of nonlinear type. However, based on the instinctive argument, it is apprehended that the sandpile morphological dynamics follow nonlinear rules. This intuitive argument may be endured by the fact that the sandpiles protrude and flatten over a time interval due to distinct nature of pyramidal sandpile structure. This argument is also supported by a postulate that the successive phases of profiles of a sandpile undergoing dynamics may be nonoverlapping, moreover the output in terms of fractal dimension of the sandpile undergoing dynamics may not be directly proportional to its input. This phenomenon is due to the relative divergent balance of the sand that is accumulated, and also due to changes in morphological constitution at discrete time intervals. The dynamics of morphological behaviour of a sandpile under different regulatory parameters depend on the initial configuration of the sandpile. By considering the quantified parameter at discrete time intervals, the term called strength of regulatory parameter (SRP) can be derived.

Several assumptions of morphological dynamics seem to be cogent by the fact that the exodynamic processes are always non-systematic (random) that alter the morphological behaviour of a sandpile. To carry out computer simulations to visualise distinct possible behaviours concerning a change in *SRP*, a first order nonlinear difference eqn (1) proposed elsewhere [8] that has physical relevance as the simplest possible model of the formation of a pyramidal sandpile is considered as the basis.

$$\alpha_{t+1} = \lambda\alpha_t(1-\alpha_t), \tag{1}$$

where:

α: normalised fractal dimension of a sandpile profile, $0 \leqslant \alpha \leqslant 1$

λ: strength of regulatory parameter, $0 \leqslant \lambda \leqslant 4$

Equation (1) has been studied extensively [8] and is considered to be a simple model to explain the dynamics in one-dimensional maps, where increasing λ induces period doubling bifurcation leading to chaos. This equation possesses one equilibrium point and the stability of the fixed point and the consequent dynamics exhibited by the systems are dependent on λ alone.

The normalised fractal dimension (α) of the pyramidal sandpile can be obtained by subtracting the topological dimension (D_T) from the fractal dimension as shown in eqn (2).

$$\alpha = [\mathrm{Log}(N)/\mathrm{Log}(d/L)] - D_T, \tag{2}$$

where:

N: 2 (slip faces)

d: width of the stationery base of the sandpile

L: length of the slip face, $L \leqslant d$

D_T: topological dimension

The notion of subtracting the topological dimension (D_T) from the fractal dimension in eqn (2) is to keep the values in normalised scale, $\alpha[0,1]$. A sandpile with high degree of steepness will have a value of $\alpha = 1$, and with less steepness (straight line) will have a value $\alpha = 0$. To study morphological dynamical behaviour of a sandpile, it is necessary to know how much of the total morphological change is accommodated across time intervals. The rate of changes in the fractal dimension of dynamically changing sandpile at discrete time intervals depends on the exodynamic processes. The collective impact of exodynamic processes (cause) which alter sandpile morphology can be defined as a strength of regulatory parameter (*SRP*) by studying the (degree of deformation) effect due to the cause at discrete time intervals. As the fractal dimension enables the characteristic of the sandpile profile that is projected as well as flattened, the parameter, *SRP* can be defined as a numerical value. From the theoretical standpoint, the *SRP* (λ) may be computed by considering the α_t and α_{t+1} to fit the curve $\lambda\alpha_t(1-\alpha_t)$. This λ gives total description of the dynamics of sandpile. The impact of non-systematic exodynamic processes on a sandpile in terms of its dynamical behaviour is investigated through the first order difference eqn (1) of the form, $\alpha_{t+1} = f(\alpha_t)$; the normalised fractal dimension in $t+1$, α_{t+1} is given as some function, f, of the α_t in time t. If this equation were linear ($f = \lambda\alpha$), the α would just increase or decrease exponentially if $\lambda < 1$. Moreover, the fractal dimension tends to increase when at low α and to crash at high α value, corresponding to some nonlinear function, with a hump, of which the quadratic $f = \alpha_{t+1} = \lambda\alpha(1-\alpha)$. It does mean, there is a tendency for the variable α to increase from time 't' to the next when it is small, and for it to decrease when it is large. This tendency is preserved due to $(1-\alpha_t)$ in eqn (1). It is intuitively apparent that the degree of unsteady state to fall over is more in the steep sandpile that possesses high fractal dimension. Hence, as the steepness of sandpile increases the degree of fall over of sand becomes more when compared to the sandpile of lesser steepness. This phenomenon can be compared with *overcrowding* parameter in the context of population dynamics described in the logistic equations. This statement supports the argument that α tends to increase when it is small, and to decrease when it is large. In other words, as the sandpile crest reaches to critical angle, i.e. 90° (steepest of the sandpile), at which the normalised fractal dimension $\alpha = 1$, there is a tendency for α to decrease due to the fact that the unsteady state of sand to fall over is more in steeper sandpiles. On the contrary, when the sandpile profile possesses less fractal dimension, there may be a possibility for it to get protruded due to sand assemblage and due to more *sand holding* capacity in the supply area. When it possesses high fractal dimension, due to unsteady state to fall over is more and this may lead to a decline of the fractal dimension. However, it may also lead to oscillations, or even chaotic fluctuations depending on the nature of exodynamic processes, and the sandpile characteristics.

Equation (1) to compute α_{t+1}, $\lambda\alpha_t(1-\alpha_t)$explains that the normalised status of a sandpile dynamics if α starting at larger than 1, it immediately goes negative at one time step. Moreover, if $\lambda > 4$, the hump of the parabola exceeds 1, thus enabling the initial α value near 0.5 to exceed criticality in two time steps. Therefore, there is a need to restrict the analysis to value of λ between 1 and 4, and values of α between 0 and 1. In the qualitative understanding of dynamical behaviour, value α_{t+1} is obtained from the previous value α_t by multiplying it by $\lambda(1-\alpha_t)$. It is clear that for $\lambda(1-\alpha_t)$ greater than 1, the successive values, viz., α_{t+2}, α_{t+3}, α_{t+4}, ., α_{t+N}, will grow bigger—that is, a change in α_t will get amplified. This is the sandpile protrusion due to sand assemblage. If $\lambda(1-\alpha_t)$ becomes smaller than 1, then the subsequent values must diminish. This is sandpile flattening due to fall over of sand. In what follows, the observations and results of the study of morphological behaviour of sandpile, evolving according to eqn (1) for different *SRP*s, is given for a better understanding.

1.3. α and θ relationship

Equation (2), which is due to Mandelbrot (1982) [9] to compute fractal dimension of Koch generator (Mandelbrot, 1982, p. 64 [9]) that is similar to pyramidal sandpile profile, can be rewritten as eqn (3). This equation computes the fractal dimension of pyramidal sandpile by considering the interslip face angle (*ISFA*). From the θ, the corresponding normalised fractal dimension (*NFD*) can be calculated for the pyramidal sandpile using eqn (3)

$$\alpha = \{\mathrm{Log}N/[\mathrm{Log}[2\mathrm{Sin}(\theta/2)]] - D_T\}, \tag{3}$$

where $90° \leqslant \theta \leqslant 180°$.

The profiles of the sandpiles with 90° (steepest) and 180° (zero–steepness) of inter-slip face angles possess normalised fractal dimensions 1 and 0 respectively. At the critical angle, (θ_{crit}) i.e., 90°, the parameter α attains its peak value, $\alpha = 1$. The corresponding fractal dimension (D) is at its criticality, i.e., $\alpha + D_T = 2$. The range of inter-slip face angle of a sandpile under investigation is between 180° and 90°. A sandpile under dynamics will reach to criticality where the ratio between Log (N) and Log (d/L) becomes 2. At this critical state, the inter-slip face angle becomes 90° (θ_{crit}). For better understanding typical profiles of pyramidal sandpiles are illustrated with normalised fractal dimensions and their corresponding interslip face angles (Figure 2).

1.4. Computation of inter-slip face angle (θ) of a sandpile profile under dynamics

By incorporating the dynamical rule, i.e., eqn (1), which is intuitively justifiable to model sandpile morphological evolution, in eqn (3), following eqn (4) is proposed. By considering the parameters such as normalised fractal dimension to describe the change in morphology of the sandpile, α at time 't', and the *SRP* (λ) in eqn (4), the behaviour of morphological dynamics of a sandpile that may range from stable to chaotic can be quantified. Equation (4) includes these

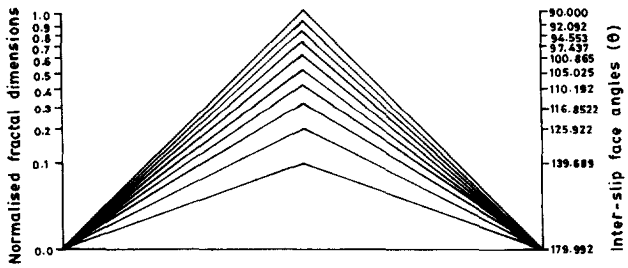

Fig. 2. Pyramidal sandpile profiles with different normalised fractal dimensions and their corresponding interslip face angles.

specifications to record the changing interslip face angle (θ) of a sandpile under evolution according to eqn (1).

$$\theta_{t+1} = 2\mathrm{Sin}^{-1}\frac{10^{\{\mathrm{Log}N/[\lambda\alpha_t(1-\alpha_t)+D_\mathrm{T}]\}}}{2}, \tag{4}$$

where:
$0 \leqslant \lambda \leqslant 4$,
$0 \leqslant \alpha \leqslant 1$,
θ = Inter-slip face angle.

With α_t and λ as 0.5 and 4, respectively, the α and θ values of sandpile under evolution at time $t+1$ become 1 and 90° at one single time step respectively. Once the inter-slip face angle reaches its criticality, the sandpile may become either stable, or toppled as time progresses.

Instead of α one can consider θ values to carry out simulations for modelling. Using eqn (3) in eqn (4), eqn (5) is proposed in which the *ISFA* at time t, is considered instead of the *NFD* to compute the *ISFA*s at time $t+1$, ., $t+n$ of the sandpile undergoing dynamics according to first order difference equation as a dynamical rule. The *ISFA*s at time $t+1$ can be computed by considering θ at time t as some function defined as follows

$$\theta_{t+1} = 2\mathrm{Sin}^{-1}\frac{10\exp\mathrm{Log}N/\{\lambda\{\mathrm{Log}N/[\mathrm{Log}[2\mathrm{Sin}(\theta_t/2)]]-D_\mathrm{T}\}\{1-\{\mathrm{Log}N/[\mathrm{Log}[2\mathrm{Sin}(\theta_t/2)]]-D_\mathrm{T}\}\}+D_\mathrm{T}\}}{2}, \tag{5}$$

where:
$0 \leqslant \lambda \leqslant 4$,
$90^\circ \leqslant \theta \leqslant 180^\circ$.

In eqn (5), the strength of regulatory parameter is a constant parameter. However, the emphasis can also be given to carry out the simulations to understand the possible dynamics by understanding the dynamics of the time dependent control parameter.

1.5. Bifurcation phenomenon

Changes in the regulatory parameter result variations in the behaviour of the morphological dynamics of a sandpile ranging from steady state to periodicity and chaotic. The random behaviour of sandpile, from its inception of the formation is not only due to randomness of the exodynamic processes that influence the sandpile incessantly but also due to the morphological constitution of the sandpile. This study opens a way to reconstruct the initial sandpile by

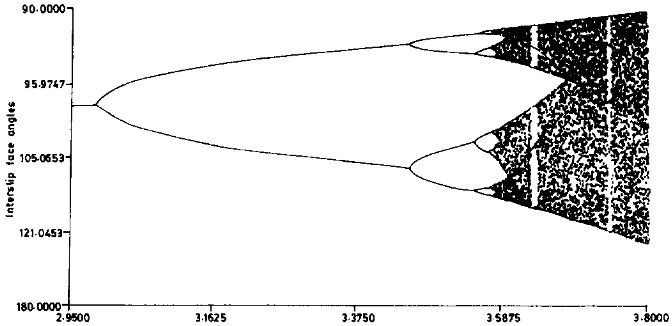

Fig. 3. Bifurcation diagram showing various possibilities of sandpile morphological dynamics. The value λ that measures the regulatory parameter which controls the morphological dynamics of sandpile. The evolution of the sandpile system can be segregated as period one, period two, and chaotic. Period one: When λ is between 1 to 3, the sandpile profile protrudes, and the dynamics attracting to a fixed point. Period two: The sandpile profile oscillates between two points when λ is between 3 and 3.46. The sandpile protrusion and flattening will occur periodically. Chaotic: The behaviour of sandpile is such that the sandpile profiles at different time periods do not overlap exactly. Here, the sandpile protrusion and flattening may occur randomly as time progresses.

considering the equilibrium status, and the critical angle of the sandpile. The historical track along which the morphology of sandpile evolves as the regulatory parameter grows can be characterised by a sequence of stable regions where deterministic laws dominate and of unstable ones near the bifurcation points where the system can choose between or among them one possible future. This mixture composes the morphological account of the sandpile formation.

In Fig. 3, bifurcation diagram is shown for various possible dynamical behaviours, viz., stable, unstable, chaotic, of a sandpile under dynamics. The evolution types of sandpile transformations can be segregated as period one, period two, and chaotic. As λ is varied, changes in the qualitative behaviour of the system can occur. Such qualitative behaviour can be seen in the bifurcation diagram (Fig. 3) in which the attractor interslip face angles (*AISFA*s) are plotted against regulatory parameter λ. In this bifurcation diagram, as $\lambda \in [0,3]$, dynamical behaviour possesses a stable fixed point. As λ is increased past 3, the behaviour becomes unstable and two new stable periodic points appear. The morphological behaviour of sandpile under dynamics follows periodicity where both protrusion and flattening are involved subsequently. The dynamics become unstable each spawning two new stable periodic points of period 4 as λ is further increased from 3.46. Through this bifurcation diagram, the dynamics of morphological behaviour path can be found out with respect to the regulatory parameters. This diagram (Fig. 3) not only portrays the type of dynamical behaviour of the sandpile with respect to the regulatory parameter, but also the critical states in terms of inter-slip face angles of the sandpile under dynamics. The number of critical states that a sandpile reaches under the dynamics during the influence of exodynamic process depends on the starting initial sandpile configuration and the regulatory parameter (λ). For every value of λ, there will be an attracting point. This attracting point in the present context can be represented in two ways, (a) the normalised fractal dimension, (b) inter-slip face angle of a sandpile under dynamics. The values on abscissa represent inter-slip face angles. These angles are computed by eqn (5). This diagram (Fig. 3) contributes the information regarding the morphological history of sandpile formation, provided the initial state of the sandpile and the regulatory parameter that controls the morphological dynamics are precisely computed.

1.6. Attracting interslip face angles (θ^*s)

The threshold regulatory parameter is the value at which the sandpile under dynamics produce critical state (s) or attractor (s). The threshold regulatory parameters are—$\lambda_1 = 3$; $\lambda_2 = 3.46$;

Table 1. The attractor interslip face angles of sandpile at respective threshold regulatory parameters after 3×10^4 iterations

Threshold Regulatory parameters	$\alpha = 0.00001$ $\theta_{init} = 179.57334°$							
3				98.601081°				
3.46		93.588105°			109.48°			
3.569	92.307411°		105.01353°		93.787106°		112.10937°	
3.57	92.3°	105.23263°	93735048°	111.98573°	92.541161°	102.72656°	94.379436°	113.5926°

$\lambda_3 = 3.569$; and $\lambda_4 = 3.57$. The attractor inter-slip face angles (θ^*s) at these regulatory parameters are computed (Table 1) by considering the initial sandpile specification with $\alpha = 0.00001$ or $\theta = 179.57334°$. The corresponding attractor sandpile profiles at respective regulatory parameters are arrived through numerical simulations that are performed by iterating 3×10^4 time steps. Figure 4(a) shows the initial profile of the sandpile. This profile is allowed to undergo morphological changes with different threshold regulatory parameters. The Fig. 4(b–e) show attractor sandpile profiles under the threshold regulatory parameters. During the evolution, variations in the length of slip faces may be seen. In Fig. 4(b), the sandpile crest was progressively raised. Periodically changing morphologies of sandpile under dynamics with regulatory parameter 3.46 is shown in Fig. 4(c). Similarly period 4 and 8 are also shown in Fig. 4(d) and (e) respectively.

1.7. Computation of metric universality considering θ^*s

Feignbaum [10] proposed the metric universality constant, i.e., 2.5029. for the celebrated nonlinear first order difference eqn (1) by considering the distance between the openings of attractors at respective threshold regulatory parameters. The δ can be computed for the sandpile under dynamics by considering the *AISFA*s by eqn (6). This equation is derived by computing

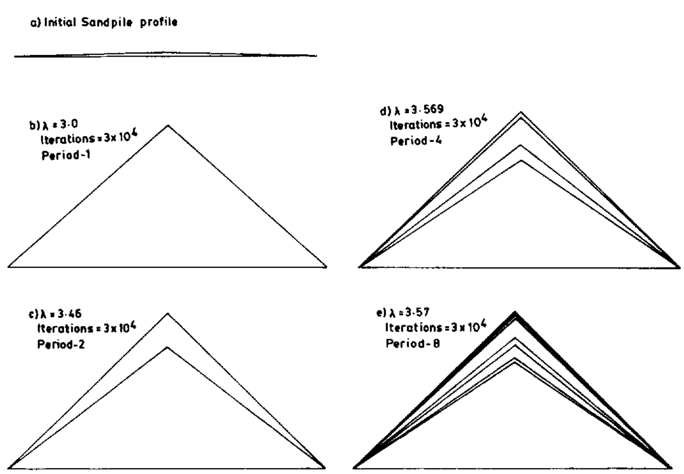

Fig. 4. (a) Initial sandpile profile with $\alpha = 0.00001$ or $\theta = 179.57334$. The attractor sandpile profiles at various threshold regulatory parameters (b) $\lambda = 3$, fixed point attractor sandpile (c) $\lambda = 3.46$, period two attractor sandpiles (d) $\lambda = 3.569$, period four attractor sandpiles and (e) $\lambda = 3.57$, period eight attractor sandpiles. The attractor sandpile profiles shown in (b), (c), (d), and (e) are arrived after simulations performed by iterating 3×10^4 time steps. The attractor interslip face angles of these attractor sandpile profiles were shown in Table 1.

the differences at respective attractor normalised fractal dimensions. Instead of attractor *NFD*s, their corresponding *AISFA*s are considered to compute δ.

$$\delta \sim \frac{\{\mathrm{Log}[2\mathrm{Sin}(\theta^*_{N+1}/2)] - \mathrm{Log}[2\mathrm{Sin}(\theta^*_N/2)]\}\mathrm{Log}[2\mathrm{Sin}(\theta^*_{2N+2}/2)]\mathrm{Log}[2\mathrm{Sin}(\theta^*_{2N+3}/2)]}{\mathrm{Log}[2\mathrm{Sin}(\theta^*_N/2)]\mathrm{Log}[2\mathrm{Sin}(\theta^*_{N+1}/2)]\{\mathrm{Log}[2\mathrm{Sin}(\theta^*_{2N+3}/2)] - \mathrm{Log}[2\mathrm{Sin}(\theta^*_{2N+2}/2)]\}}, \quad (6)$$

where:
$N = 2, 4, 8, 16, .,$
θ^* = Attractor interslipface angle.

2. CONCLUSIONS

Certain possible morphological behaviours with respective critical states represented by inter-slip face angles of a sandpile under the influence of non systematic processes are qualitatively illustrated by considering the first order difference equation that has the physical relevance to model the morphological dynamics of the sandpile evolution as the basis. It is deduced that the critical state of a sandpile under dynamics depends on the regulatory parameter that encompasses exodyanmic processes of random nature and the morphological configuration of sandpile. With the aid of the regulatory parameter, and the specifications of initial state of sandpile, morphological history of the sandpile evolution can be investigated. As an attempt to furnish the interplay between numerical experiments and theory of morphological evolution, the process of dynamical changes in the sandpile with a change in threshold regulatory parameter is modelled qualitatively for a better understanding. An equation to compute metric universality by considering θ^*s is also proposed.

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