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Fractal Relation of a Morphological Skeleton

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Abstract—The morphological skeleton of a structure which possesses a crenellate outline resembles a stream network. The fractal relation of a morphological skeleton network is shown. The fractal dimension of the structure and its morphological skeleton network are computed using the box counting method. These are then compared with the estimated length–area measures and certain morphometric order ratios. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Every geometric structure can be represented as a morphological skeleton which is a simpler form from which inferences can be drawn. In this context, the general term structure is used to denote 'the expression of the external morphology of the objects' such as geomorphic features, pore space, alveolar lung space, basin outline, rock outline, etc. Components of such structures include traditional characteristics of shape, in two dimensions, in addition to outline textural details. Hitherto, fractal relations were shown in various arboreal networks like bronchial trees, river networks. This is the first attempt to show the fractal relation of a morphological skeleton. The term skeleton has been used to describe a thin line caricature of the geometric structure which summarizes its shape, size, orientation and connectivity. Within the last few years several papers have been presented which address the fractal description of arboreal networks, like bronchial trees [1], lung morphogenesis studies [2], stream network [3–5]. The basis for the analysis of a morphological skeleton presented here is drawn from the works of the above mentioned researchers.

The fractal properties of the morphological skeleton, with tree-like structures, extracted from the second order structure generated by a specific generator with a non-random rule, are shown. The plan of this paper is to first present the fractal generation, extraction of a morphological skeleton of generated fractal, morphometric analysis of the morphological skeleton, and then the fractal relation of the morphological skeleton.

Fractal generation

To generate fractal structures, where the fractal dimension ranges from 1 to 2 in 2-dimensional space, one begins with two shapes, an initiator and a generator. The latter is an oriented broken line made up of N equal sides of length r which can be designed at will [6]. Each stage of the construction begins with a broken line and consists of replacing each straight interval with a copy of the generator, reduced and displaced so as to have the same end points as those of the interval being replaced. In all cases, $D = \text{Log}\, N/\text{Log}\, 1/r$ [6]. Step 0 is to draw the segment (0, 1). Step 1 is to draw either of the kinked curves (Fig. 1),

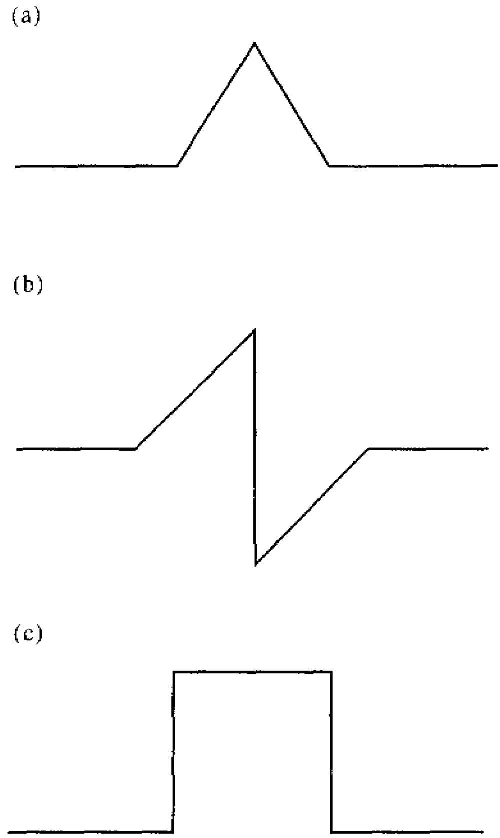

Fig. 1. Different generators.

each made up of N intervals superposable upon the segment (0, 1/3, 1/4 for the generators shown in Fig. 1). Step 2 is to replace each of the N segments used in step 1 by a kinked curve obtained by reducing the curve of Step 1 in the ratio $r(N) = 1/r$. Altogether one obtains N^2 segments of length $1/(r)^2$. Iterating this process adds further detail.

Morphological skeleton

A connectivity preserving way of erosion called skeletonization is described by Hilditch [7]. The resulting skeleton is one picture element (pixel) thick objects, which have the same connectivity as the original object. Skeletons are of special interest because they reflect the structure of the original objects in their end pixels and vertices. The concept of skeletonization is developed by mathematical morphologists [8–11]. The skeleton or medial axis of a set is the line made up of those points for which the distance to the boundary of the set is reached by at least two points. The skeleton of a geometric structure (Fig. 2(a)) viewed as a subset of R^2 (Euclidean space) is defined as the set of the centres of the maximal disks inscribable inside the structure. A disk is maximal if it is not properly contained in any other disk totally included in the structure. Hence, a maximal disk must touch the boundary of the structure at least at two different points. The combination of centres of the maximal disks inscribable is a skeleton. Figure 2(b) is a morphological skeleton of the structure shown in Fig. 2(a). This concept is being extensively applied in several fields such as biological shape description [12], pattern recognition [7] and metallography [13, 14] with highly promising results. Some examples can be seen in Maragos and Schafer [10].

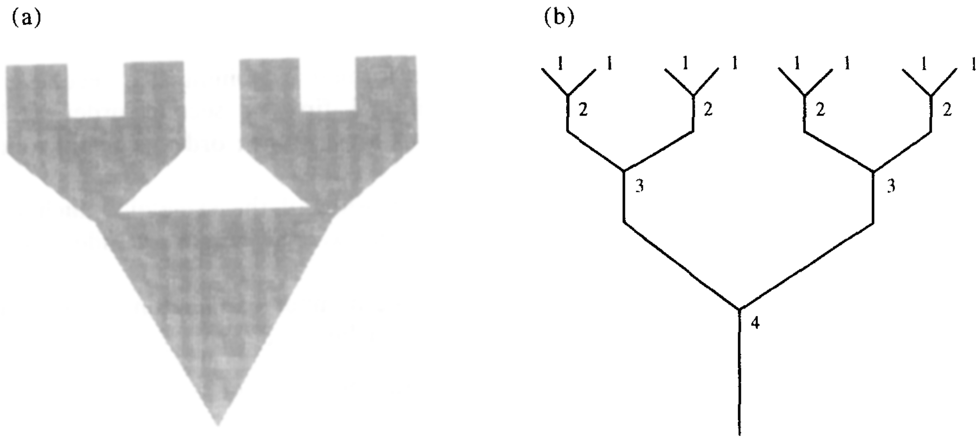

Fig. 2. Shows (a) the structure and (b) the morphological skeleton after designating Strahler's ordering.

MORPHOMETRY OF THE MORPHOLOGICAL SKELETON

The morphological skeleton of a structure with a contorted outline resembles a stream network. Hence, to characterize the morphological skeleton, the morphometric procedures proposed by Horton [15] and Strahler [16] in the context of river network morphometric studies need to be used. Strahler's [16] ordering technique has been followed to designate the orders of morphological skeleton and to further compute certain important dimensionless ratios like the bifurcation ratio (R_B), skeleton length ratio (R_L), and skeleton area ratio (R_A). These three ratios are important parameters in showing relationships. To show the fractal relation of the morphological skeleton, fractal dimensions of the structure (D), morphological skeleton length (D_{TS}), and main skeleton length (d) are computed by various existing methods. These are then compared with computed morphometric order ratios, and length–area measures of the morphological skeleton as shown in the sample study.

To carry out such studies on the morphological skeleton, some new terms are proposed: skeleton network, boundary of the morphological skeleton network, skeleton orders, main morphological skeleton length, total skeleton length, skeleton length of order u and order $u + 1$, skeleton number of order u and $u + 1$, skeleton bifurcation ratio, skeleton length ratio, and skeleton area ratio. For a better understanding these are defined as follows.

List of symbols and nomenclature

- Order of structure or skeleton segment — u
- Number of skeleton segments of order u — N_u
- Total number of skeletons within a skeleton network — $\sum N_u$
- Mean length of skeleton segments of order $u(L_u)$ — L_u/N_u
- Total skeleton length within a network of order u — $\sum L_u$

The boundary of the morphological skeleton is the boundary of the geometric structure to which the morphological skeleton belongs. The boundary of the geometric structure should match exactly with that of the boundary that is reconstructed from its morphological skeleton.

The skeleton network, and its order designation, is the pattern formed by the skeletal branches that are determined by the inequalities of the outline of the geometric structure.

The order of the skeleton ranges from 1 to n (any finite number). All finger tips are designated as first order skeletons. Two first order skeletons unite to form a second order skeletal segment. A third order skeletal segment is formed by joining two second order skeletal segments which may be joined by additional first or second order skeletal segments. Two third order skeletal segments join to form a fourth order segment, and so on (Fig. 2(b)).

The main morphological skeleton length (l) is a measure of the skeletal branch which follows the longest axis of the structure. The main skeletal length includes skeletal segments of all orders.

The skeleton bifurcation ratio (R_B) is the ratio of the number of skeleton segments of a given order N_u to the number of skeletons of the next highest order, N_{u+1}.

$$\text{Bifurcation ratio } (R_B) = \frac{\text{No. of skeleton branches of order } u}{\text{No. of skeleton branches of order } u+1} = \frac{N_u}{N_{u+1}}. \quad (1)$$

Skeleton length ratio (R_L) is the ratio of the mean length, L, of segments of order u to the mean length of segments of the next lower order (L_{u+1}).

$$\text{Skeleton length ratio } (R_L) = \frac{\text{Mean length of skeleton branches of order } u+1}{\text{Mean length of skeleton branches of order } u} = \frac{L_{u+1}}{L_u}. \quad (2)$$

Skeleton area ratio (R_A) is the ratio of the mean skeleton segment area of order u, A_u, to the mean skeleton segment area of the next lower order, A_{u-1}.

$$\text{Skeleton area ratio } (R_A) = \frac{\text{Mean skeleton branch area of order } u}{\text{Mean skeleton branch area of order } u-1} = \frac{A_u}{A_{u-1}}. \quad (3)$$

SAMPLE STUDY

Considering the square block as an initiator (Fig. 3(a)), a second order structure (Fig. 3(c)) is generated using the designed generator (Fig. 3(b)). The fractal dimension for the fractal generated by the generator, shown in Fig. 3(b)), is 1.5.

$$D = \text{Log}(N)/\text{Log}(1/r) = \text{Log}\,8/\text{Log}\,4 = 1.5. \quad (4)$$

(a) (b)

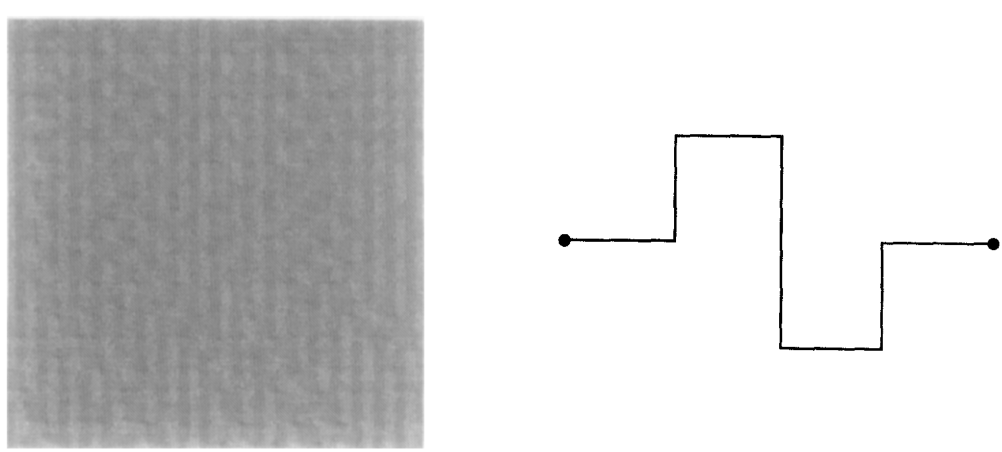

Fig. 3(a) and (b). *Caption opposite.*

(c)

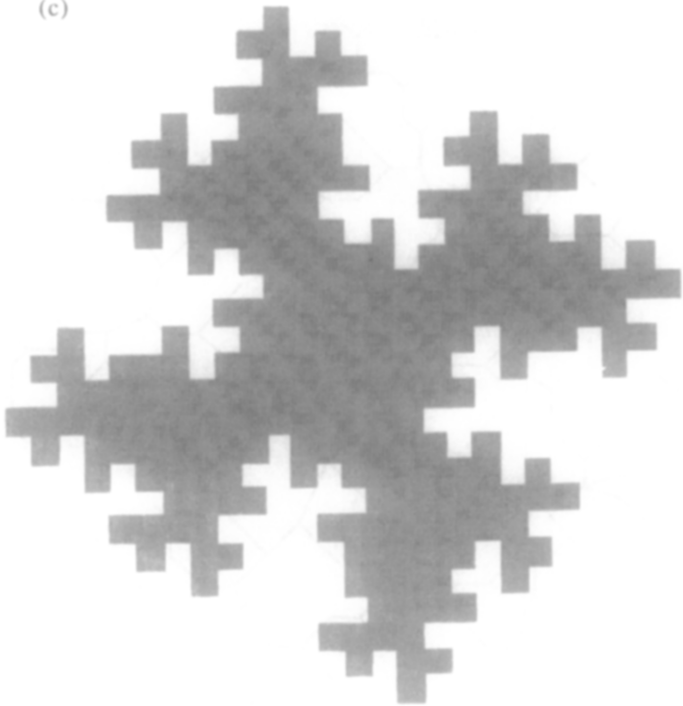

Fig. 3. Shows (a) the initiator; (b) the generator; and (c) the second order structure generated by the generator shown in (b) of which the fractal dimension is 1.5.

The morphological skeleton of the second order structure (Fig. 3(c)) is extracted (Fig. 4).

Following Strahler's [16] ordering technique, the entire extracted morphological skeleton can be designated with orders. The basic measures such as area, perimeter of the structure, order wise skeleton lengths and number, main skeleton length, and total skeleton length, are computed. Bifurcation, length, and area ratios of the morphological skeleton are computed (Table 1) using equations (1), (2), and (3).

Fractal dimension measurements and their relation to morphological skeletons

Bifurcation, length, and area ratios of a morphological skeleton are computed as 2.33, 1.72, and 2.385, respectively, (Table 2). Through these dimensionless parameters, the fractal dimension of the skeleton network (D_{TS}) is computed as 1.56 using equation (5).

$$D = \text{Log}\, R_B/\text{Log}\, R_L. \tag{5}$$

The fractal dimension of the main skeleton length (d) is computed as 1.25 using the equation

$$d = 2\,\text{Log}\, R_L/\text{Log}\, R_A. \tag{6}$$

The fractal dimension of the total skeleton length (D_{TS}) is computed as 1.96 using the equation

$$D_{TS} = 2\,\text{Log}\, R_B/\text{Log}\, R_A. \tag{7}$$

The fractal dimensions computed using morphometric parameters are found to be valid and related to that of the fractal dimensions computed by the box counting method proposed elsewhere [17]. The fractal dimensions of the structure, the extracted morphological

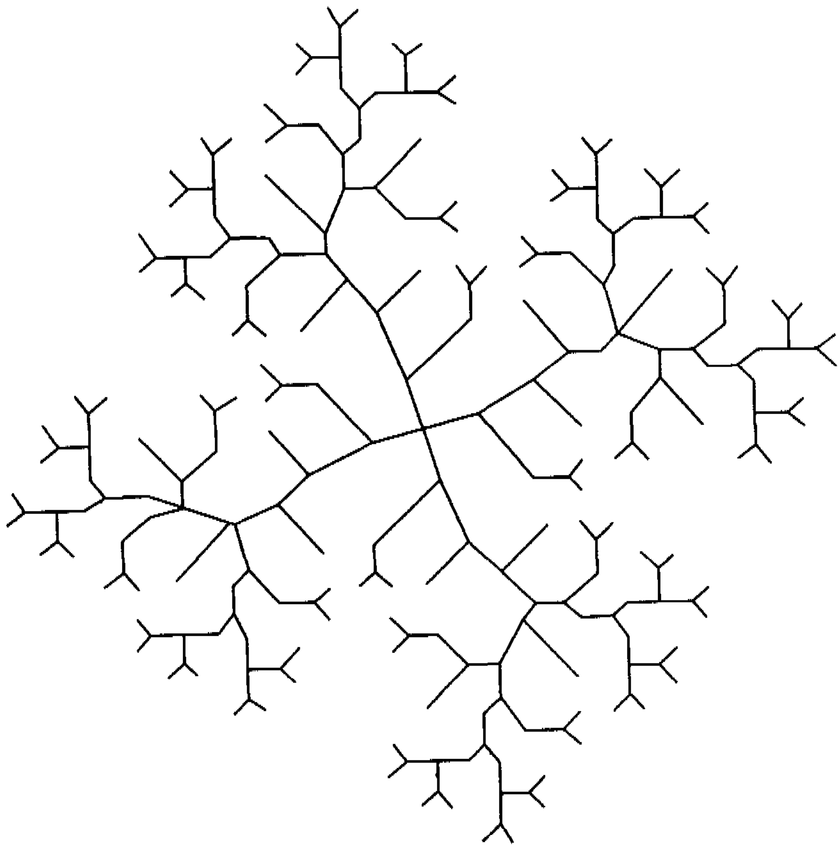

Fig. 4. Morphological skeleton of the structure shown in Fig. 3(c).

Table 1. Fractal dimensions for the morphological skeleton network: comparison between length–area measures and the estimated values from order ratios

Parameter	Estimated values	Equation No.
	Order ratio	
Bifurcation ratio	2.33	1
Skeleton length ratio	1.725	2
Skeleton area ratio	2.385	3
	Total skeleton length vs area	
Exponent β	0.98	9
Fractal dimension $(D_{TS}) = 2\beta$	1.96	
	Main skeleton length vs area	
Exponent α	0.612	10
Fractal dimension $d = 2\alpha$	1.224	
	Estimation of fractal dimensions from order ratios	
$D = \text{Log}\, R_B/\text{Log}\, R_L$	1.56	5
$d = 2\,\text{Log}\, R_L/\text{Log}\, R_A$	1.25	6
$D_{TS} = 2\,\text{Log}\, R_B/\text{Log}\, R_A$	1.92	7
$D_{TS} = 2\,\text{Log}\, R_L/\text{Log}\, R_B$	1.23	8

Table 2. Fractal dimensions of the structure and its morphological skeleton length: comparison between box-counting measures and the estimated values from morphometric order ratios

Measured fractal dimensions through the box counting method		
D (Generated fractal)	D_{TS} (Morphological skeleton)	d (Main skeleton length)
1.5	1.56	1.23

skeleton, and main skeleton length are computed, following the box counting method, as 1.5, 1.56 and 1.23, respectively, (Table 2). This method provides the values of D, D_{TS}, and d as the slopes of the straight lines which are fitted to the log-transformed pairs of observed box numbers and box size values (Fig. 5(a)–(c)). These results have been compared with the fractal dimensions arrived by order ratios and length–area measures which have been obtained from morphometric analysis of the morphological skeleton.

Fractal dimension of the skeleton network

The scaling properties of the morphological skeleton network, like a tree network, as a whole can be viewed as the product of the structural composition of the skeleton system and the effect of small irregularities reflected by d. Just as in the case of computing the fractal dimension of a river network, which depends on its R_B and R_L, the fractal dimension of the morphological skeleton is computed using equation (8). Equation (8) was originally proposed in the context of a stream network by Feder [17].

$$D_{TS} = 2\,\mathrm{Log}\,R_L/\mathrm{Log}\,R_B. \tag{8}$$

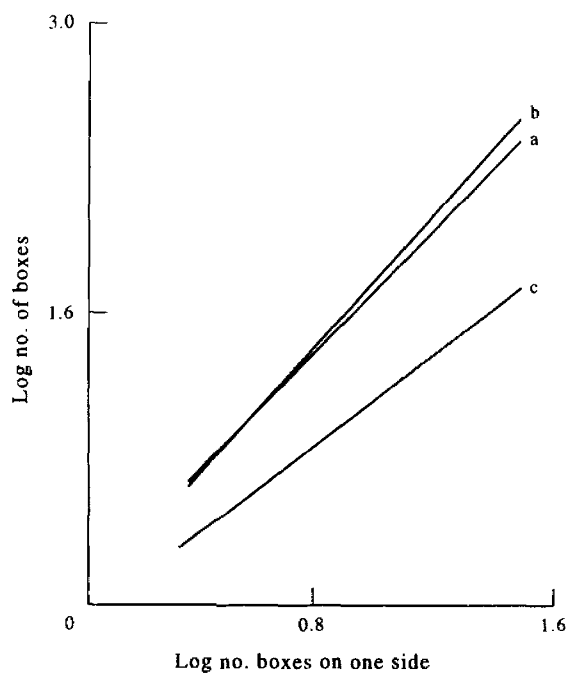

Fig. 5. Fractal plots of (a) fractal structure; (b) total morphological skeleton length; and (c) main skeleton length through the box counting method.

For the extracted morphological skeleton, the estimate of $D_{TS} = 2\log 1.73/\log 2.33 = 1.30$ (Table 1). This result can be compared with the direct estimate of d, which can be obtained by means of the main skeleton length (l)–area (A) relationship shown in equation (10).

The result obtained by equation (7) is compared with the direct estimate of D which can be obtained by means of the total skeleton length (L)–area (A) relationship in the form of

$$L \sim (A)^{\beta} \tag{9}$$

where L denotes the total length of the skeleton, in the structure of area A, $\beta = D_{TS}/2$ is a fitted exponent, where D_{TS} is $2\log R_B/\log R_A$. β is computed through equation (9) as 0.98. The value of 2β, i.e. 1.96 is very close to the estimate obtained (1.92) from equation (7). Variation either in the generating mechanism or in the initiator shows difference in both the fractal generated, and in its morphological skeletons. But the fractal dimension of the morphological skeleton computed through the box counting method is directly proportional to that of the fractal dimension of the structure. It can be observed that the fractal dimensions of morphological skeleton networks seem to vary from one structure to another, and it is therefore quite arbitrary to assign an invariant fractal dimension to the total length of skeleton of a structure.

In the morphological skeleton, the maximum skeleton length (l) is computed along the longest axis. The fractal dimension of the main skeletal length is also computed using length–area measures as shown in the equation (10).

$$d = 2(\mathrm{Log}\, l/\mathrm{Log}\, A) = 1.224 \tag{10}$$

for this sample study. This result is close to the direct estimate of d computed using equation (6) and also to the dimension computed through the box counting method (Table 1). Considering the obtained results, the following fractal relationship of a morphological skeleton is proposed.

$$D_{TS} = d\,\mathrm{Log}\, R_B/\mathrm{Log}\, R_L = 1.92 + 0.04 \tag{11}$$

with $\log R_B/\log R_L = 1.56$, and $d = 1.23$, $D_{TS} = 1.92$, which is very close to 1.96 arrived at by equation (7). The fractal dimension of the morphological skeleton network reaches 2 for $d = 1.23$ in the relation of $R_B = R_L{}^{2/d} = R_L{}^{1.63}$.

CONCLUSION

In this paper, an eight-sided (N) generator, with r as 1/4, is selected to generate the fractal structure of the initiator, i.e. square, with deterministic rule. The morphological skeleton of the generated structure is extracted. It is worth generating several features by changing the initiator, generator and the rules to further extract their morphological skeletons. This approach enables the reader to simulate natural features with more appropriateness and accuracy. A priority for further work on this topic is to simulate the structures in such a way that they fit well with the nature of data contained for instance in digital elevation maps. It is very clear that certain morphometric parameters, in particular order ratios and length–area measures, can be directly related to the fractal measures of tree-like structures. For instance, a morphological skeleton of the structure.

The accuracy in the simulation of the basin evolution process depends on the selection of generator, which controls the mechanism, random rule to be followed while generating fractal structures, and the initial structure on which the generator shows impact. The selection of generator and the rule can be done through studying the structure temporally. The contorted outline of the structure, for instance a basin, determines the morphological skeleton which largely resembles the stream network of the basin. Another possible application is modelling of the alteration process of the pore structure by the flow of fluids through the pore space, by considering the pore structure at different time periods. The generating mechanism can be better predicted if the pore structure across time periods is considered.

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