# Is it possible to clone using an arbitrary blank state? 

Anirban Roy ${ }^{a}$ 亿, Aditi Sen $(\operatorname{De})^{b / 2}$ and Ujjwal Sen ${ }^{b}$<br>${ }^{a}$ Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 BT Road, Kolkata 700035, India<br>${ }^{b}$ Department of Physics, Bose Institute, 93/1 APC Road, Kolkata 700009, India


#### Abstract

We show that in a cloning process, whether deterministic inexact or probabilistic exact, one can take an arbitrary blank state while still using a fixed cloning machine.


Quantum information cannot be cloned. There cannot exist a machine which can produce two (or more) exact copies of an arbitrary state in a deterministic manner [1]. Replication is not allowed even when the input state is taken at random from a given set of two non-orthogonal states [2]. It has been further shown that probabilistic exact cloning is not possible if the input state is from a given linearly dependent set (3]).

Although exact cloning is not possible, one can approximately clone an arbitrary input state [4, 5, 6, 7]. In this scheme a fixed state is taken as the blank state, depending on which and the initial machine state, the cloning operation is constructed. We extend this operation such that any arbitrary (pure or mixed) state, taken as the blank copy, can do the job.

The optimal universal $1 \rightarrow 2$ inexact qubit cloner of Brußet al. $[5]$ takes an arbitrary qubit $|\psi\rangle\langle\psi|=$ $\frac{1}{2}(I+\vec{s} \cdot \vec{\sigma})$ along with a fixed blank state $|b\rangle$ and a machine state $|M\rangle$ as input. An entangled state of the three qubits is produced as the output such that the reduced density matrices of the first two qubits are two similar approximate copies $\rho=\frac{1}{2}(I+\eta \vec{s} \cdot \vec{\sigma})$ of $|\psi\rangle\langle\psi|$ with $\eta=\frac{2}{3}$. The unitary operator realizing this process is defined on the combined Hilbert space of the input qubit, the blank qubit and machine by

$$
\begin{align*}
& U^{\prime}|0\rangle|b\rangle|M\rangle=\sqrt{\frac{2}{3}}|00\rangle|m\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\left|m_{\perp}\right\rangle \\
& U^{\prime}|1\rangle|b\rangle|M\rangle=\sqrt{\frac{2}{3}}|11\rangle\left|m_{\perp}\right\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)|m\rangle \tag{1}
\end{align*}
$$

where $|b\rangle$ is a fixed blank state (in a two-dimensional Hilbert space), $|M\rangle$ is the initial state of the machine, $|m\rangle$ and $\left|m_{\perp}\right\rangle$ being two mutually orthonormal states of the machine Hilbert space. The two clones are to surface at the first and second qubits. Note that the machine has turned out to be a qubit.
As it stands, the unitary operator $U^{\prime}$ depends on the blank state $|b\rangle$ and the machine state $|M\rangle$. And the quality of the clones could be badly affected if $|b\rangle$ gets changed to an unknown state, say by some environment-induced decoherence. We show that by suitably constraining the unitary operator it is possible to keep the clones intact, even in this changed scenario. After dealing with the $1 \rightarrow 2$ qubit cloner, we show that the same is true for the most general cloner, the $N \rightarrow M$ qudit (elements of a d-dimensional Hilbert space) cloner. We carry over these considerations to the case of probabilistic exact cloning.

Let us suppose that for the $1 \rightarrow 2$ qubit cloner, the machine state $|M\rangle$ belongs to a four-dimensional Hilbert space. And let the unitary operator $U$ be defined on the combined Hilbert space of the input qubit, blank qubit and the four-dimensional Hilbert space of the machine by

$$
U|0\rangle|b\rangle|M\rangle=\sqrt{\frac{2}{3}}|00\rangle\left|M_{0}\right\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\left|M_{1}\right\rangle
$$

[^0]\[

$$
\begin{aligned}
U|1\rangle|b\rangle|M\rangle & =\sqrt{\frac{2}{3}}|11\rangle\left|M_{1}\right\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\left|M_{0}\right\rangle \\
U|0\rangle\left|b_{\perp}\right\rangle|M\rangle & =\sqrt{\frac{2}{3}}|00\rangle\left|M_{2}\right\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\left|M_{3}\right\rangle \\
U|1\rangle\left|b_{\perp}\right\rangle|M\rangle & =\sqrt{\frac{2}{3}}|11\rangle\left|M_{3}\right\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\left|M_{2}\right\rangle
\end{aligned}
$$
\]

where $\left\langle M_{i} \mid M_{j}\right\rangle=\delta_{i j}(i, j=0,1,2,3)$ and $\left\langle b \mid b_{\perp}\right\rangle=0$.
Let $|B\rangle=c|b\rangle+d\left|b_{\perp}\right\rangle$ be an arbitrary pure state of the Hilbert space of the blank qubit. Then

$$
\begin{align*}
& U|0\rangle|B\rangle|M\rangle=\sqrt{\frac{2}{3}}|00\rangle|X\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\left|X^{\prime}\right\rangle \\
& U|1\rangle|B\rangle|M\rangle=\sqrt{\frac{2}{3}}|11\rangle\left|X^{\prime}\right\rangle+\sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)|X\rangle \tag{2}
\end{align*}
$$

where

$$
\begin{aligned}
|X\rangle & =c\left|M_{0}\right\rangle+d\left|M_{2}\right\rangle \\
\left|X^{\prime}\right\rangle & =c\left|M_{1}\right\rangle+d\left|M_{3}\right\rangle
\end{aligned}
$$

are orthogonal. This form is exactly the same as in equation (1). Thus an arbitrary input qubit $|\psi\rangle$ would be just as well cloned by equation (2) as it would be through equation (1).
We now consider the most general cloning machine, the one that produces $M$ approximate copies of the input, when $N(<M) d$-dimensional inputs are provided [7]. The corresponding unitary operator need only be defined on the symmetric subspace ${ }^{-1}$ of the $d^{N}$-dimensional Hilbert space of the $N$ input qudits. It is defined by [7]

$$
\begin{equation*}
U_{N M}^{\prime}|\vec{n}\rangle \otimes|R\rangle=\sum_{\vec{j}=0}^{M-N} \alpha_{\vec{n}} \vec{j}|\vec{n}+\vec{j}\rangle \otimes|M \vec{j}\rangle \tag{3}
\end{equation*}
$$

where $\vec{n}=\left(n_{1}, n_{2}, \ldots, n_{d}\right),|\vec{n}\rangle$ is a completely symmetric and normalised state with $n_{i}$ systems in $|i\rangle$ with $\sum_{i=1}^{d} n_{i}=N, \vec{j}=\left(j_{1}, j_{2}, \ldots, j_{d}\right)$ with $\sum_{k=1}^{d} j_{k}=M-N,|R\rangle$ denoting the combined state of the $M-N$ fixed pure $d$-dimensional blank states $\left|b_{d}\right\rangle$ and the initial state $|M\rangle$ of the cloning machine and $|M \vec{j}\rangle$ denoting the orthonormal states of the cloning machine. And

$$
\alpha_{\vec{n} \vec{j}}=\sqrt{\frac{(M-N)!(N+d-1)!}{(M+d-1)!}} \sqrt{\prod_{k=1}^{d} \frac{\left(n_{k}+j_{k}\right)!}{n_{k}!j_{k}!}}
$$

There are $s$ equations required to define $U_{N M}^{\prime}$, where $s$ is the dimension of the symmetric subspace [8]. The required dimension of the machine is $D=\frac{(M-N+d-1)!}{(M-N)!(d-1)!}$.
Now we proceed as we had done for the $1 \rightarrow 2$ qubit cloner. If we want to allow an arbitrary pure state (possibly entangled) of the $d^{M-N}$-dimensional Hilbert space of the blank states to act as the new blank state and still produce the same outputs, we have to use a ( $D \times d^{M-N}$ )-dimensional machine. The new unitary operator $U_{N M}$ satisfies, along with the $s$ equations in $(3),\left(d^{M-N}-1\right) s$ more equations corresponding to the $d^{M-N}-1$ more blank states on which the new operator is to be defined. Then by linearity, for an arbitrary pure blank state $\left|B_{1}\right\rangle$, we would have

$$
\begin{equation*}
U_{N M}|\vec{n}\rangle|B\rangle|M\rangle=\sum_{\vec{j}=0}^{M-N} \alpha_{\vec{n} \vec{j}}|\vec{n}+\vec{j}\rangle \otimes|X \vec{j}\rangle \tag{4}
\end{equation*}
$$

${ }^{4}$ The symmetric subspace is defined by the linear span of the set of all tensor product states $|\psi\rangle \otimes|\psi\rangle \otimes \ldots N$ times, $|\psi\rangle$ being any $d$-dimensional state.
where $X \vec{j}$ are orthonormal states of the machine. This has the same form as eq. (3) and thus would be equally efficient in producing the requisite approximate copies.

Similar considerations carry over to the case of probabilistic exact cloning. Although we only consider the case qubits, the considerations carry over to higher dimensions. In the case of qubits, instead of the $U_{1}^{\prime}$ defined by [3]

$$
\begin{aligned}
& U_{1}^{\prime}\left|\psi_{0}\right\rangle|b\rangle|M\rangle=\sqrt{\gamma}\left|\psi_{0}\right\rangle\left|\psi_{0}\right\rangle|m\rangle+\sqrt{1-\gamma}|\Phi\rangle \\
& U_{1}^{\prime}\left|\psi_{1}\right\rangle|b\rangle|M\rangle=\sqrt{\gamma}\left|\psi_{1}\right\rangle\left|\psi_{1}\right\rangle|m\rangle+\sqrt{1-\gamma}|\Phi\rangle
\end{aligned}
$$

with $\gamma=1 /\left(1+\left|\left\langle\psi_{0} \mid \psi_{1}\right\rangle\right|\right)\left(|m\rangle\right.$ and $|\Phi\rangle$ are orthogonal), $\left|\psi_{0}\right\rangle,\left|\psi_{1}\right\rangle$ being two non-orthogonal states which are to be probabilistically cloned, we define $U_{1}$ as

$$
\begin{aligned}
U_{1}\left|\psi_{0}\right\rangle|b\rangle|M\rangle & =\sqrt{\gamma}\left|\psi_{0}\right\rangle\left|\psi_{0}\right\rangle\left|M_{0}\right\rangle+\sqrt{1-\gamma}|\Phi\rangle \\
U_{1}\left|\psi_{1}\right\rangle|b\rangle|M\rangle & =\sqrt{\gamma}\left|\psi_{1}\right\rangle\left|\psi_{1}\right\rangle\left|M_{0}\right\rangle+\sqrt{1-\gamma}|\Phi\rangle \\
U_{1}\left|\psi_{0}\right\rangle\left|b_{\perp}\right\rangle|M\rangle & =\sqrt{\gamma}\left|\psi_{0}\right\rangle\left|\psi_{0}\right\rangle\left|M_{1}\right\rangle+\sqrt{1-\gamma}\left|\Phi^{\prime}\right\rangle \\
U_{1}\left|\psi_{1}\right\rangle\left|b_{\perp}\right\rangle|M\rangle & =\sqrt{\gamma}\left|\psi_{1}\right\rangle\left|\psi_{1}\right\rangle\left|M_{1}\right\rangle+\sqrt{1-\gamma}\left|\Phi^{\prime}\right\rangle
\end{aligned}
$$

where $\left|M_{0}\right\rangle,\left|M_{1}\right\rangle,|\Phi\rangle,\left|\Phi^{\prime}\right\rangle$ are mutually orthogonal. Then

$$
\begin{aligned}
& U_{1}\left|\psi_{0}\right\rangle|B\rangle|M\rangle=\sqrt{\gamma}\left|\psi_{0}\right\rangle\left|\psi_{0}\right\rangle\left|m^{\prime}\right\rangle+\sqrt{1-\gamma}\left|\Phi^{\prime \prime}\right\rangle \\
& U_{1}\left|\psi_{1}\right\rangle|B\rangle|M\rangle=\sqrt{\gamma}\left|\psi_{1}\right\rangle\left|\psi_{1}\right\rangle\left|m^{\prime}\right\rangle+\sqrt{1-\gamma}\left|\Phi^{\prime \prime}\right\rangle
\end{aligned}
$$

for an arbitrary blank qubit $|B\rangle=c|b\rangle+d\left|b_{\perp}\right\rangle$ so that $\left|m^{\prime}\right\rangle=c\left|M_{0}\right\rangle+d\left|M_{1}\right\rangle$ and $\left|\Phi^{\prime \prime}\right\rangle=$ $c|\Phi\rangle+d\left|\Phi^{\prime}\right\rangle$ are orthogonal states. Consequently, the probabilistic cloning goes through with the same optimal efficiency even if we use an arbitrary blank pure qubit.
As we have mentioned, the main motivation behind consideration of an arbitrary blank state was decoherence. Decoherence, however, usually produces mixed states. But in our discussion we have only considered pure states. We now show that the input blank state could as well be an arbitrary mixed state. For definiteness, let us consider only mixed qubits. Any such mixed state can be written as $\rho=a_{1}\left|a_{1}\right\rangle\left\langle a_{1}\right|+a_{2}\left|a_{2}\right\rangle\left\langle a_{2}\right|$ where $a_{1}, a_{2} \geq 0, a_{1}+a_{2}=1$ and $\left\langle a_{1} \mid a_{2}\right\rangle=0$. Since $\left|a_{1}\right\rangle$ and $\left|a_{2}\right\rangle$ are pure states, by linearity it follows from (2) that the same unitary operator that allowed an arbitrary pure blank state, would just as well clone even when the blank state is a mixed qubit.
To summarize, we have shown that an unknown blank state can be used for cloning, whether it is deterministic inexact or probabilistic exact.

We thank the anonymous referee for making useful suggestions to revise our earlier manuscript. We acknowledge Guruprasad Kar, Sibasish Ghosh, Debasis Sarkar, Pinaki Pal and Mridula Kanoria for helpful discussions. A.S. and U.S. thanks Dipankar Home for encouragement and U.S. acknowledges partial support by the Council of Scientific and Industrial Research, Government of India, New Delhi.

## References

[1] W. K. Wootters and W. H. Zurek, Nature 299 (1982); D. Dieks, Phys. Lett. A 92 (1982) 271
[2] H. P. Yuen, Phys. Lett. A 113 (1986) 405
[3] L.-M. Duan and G.-C. Guo, Phys. Lett. A 243 (1998) 261; L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. 80 (1998) 4999
[4] V. Bužek and M. Hillery, Phys. Rev. A 54 (1996) 1844
[5] D. Bruß, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello and J. A. Smolin, Phys. Rev. A 57 (1998) 2368
[6] N. Gisin and S. Massar, Phys. Rev. Lett. 79 (1997) 2153
[7] H. Fan, K. Matsumoto and M. Wadati, Quantum Cloning Machines of d-level systems, quantph/0103053
[8] The dimension $s$ of the symmetric subspace of $N d$-level systems is $s=(N+d-1)!/ N!(d-1)$ !


[^0]:    ${ }^{1}$ res9708@isical.ac.in
    ${ }^{2}$ aditisendein@yahoo.co.in
    ${ }^{3}$ ujjwalsen@yahoo.co.in

