# Rates of asymptotic entanglement transformations for bipartite mixed states: Maximally entangled states are not special 

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#### Abstract

We investigate the asymptotic rates of entanglement transformations for bipartite mixed states by local operations and classical communication (LOCC). We analyse the relations between the rates for different transitions and obtain simple lower and upper bound for these transitions. In a transition from one mixed state to another and back, the amount of irreversibility can be different for different target states. Thus in a natural way, we get the concept of "amount" of irreversibility in asymptotic manipulations of entanglement. We investigate the behaviour of these transformation rates for different target states. We show that with respect to asymptotic transition rates under LOCC, the maximally entangled states do not have a special status. In the process, we obtain that the entanglement of formation is additive for all maximally correlated states. This allows us to show irreversibility in asymptotic entanglement manipulations for maximally correlated states in $2 \otimes 2$. We show that the possible nonequality of distillable entanglement under LOCC and that under operations preserving the positivity of partial transposition, is related to the behaviour of the transitions (under LOCC) to separable target states.


## I. INTRODUCTION

Investigations into the emerging science of quantum information has led to the widespread belief that entanglement in states shared between two systems can be used as a resource in nonclassical applications [1, 2]. It is important to stress that such applications are independent of what interpretation one chooses of the Hilbert space formalism of quantum mechanics and it keeps itself clear of the paradoxes that entanglement has been a storehouse of. Given a state shared between two partners, traditionally called Alice and Bob, we will therefore like to know whether it is possible to use it in some communication task, for example in quantum teleportation (3]. However a given state may not immediately lend itself for use in the envisioned communication task. One may have to transform it to another state, suitable for the particular communication task. And since the state is shared between two partners, there will be natural restrictions on the allowed operations on the state, in the sense that Alice and Bob will be able to act on the state only locally. It turns out that it is useful to allow them to share information over a classical channel also. Entanglement being a resource, they will like to do such transformations optimally.

Suppose therefore that Alice and Bob share the state $\rho$, while for their communication task, they require the state $\sigma$. Let $R(\rho \rightarrow \sigma)$ be the optimal asymptotic rate at which this transformation occurs faithfully, under local operations and classical communication (LOCC) between the sharing parties. Throughout this paper, $\rho \rightarrow \sigma$ will imply a transition of a bipartite state $\rho$ into a bipartite state $\sigma$ under LOCC.

A fundamental question is whether Alice and Bob lose anything irreversibly during this transformation. That is, if Alice and Bob now tries to retrieve the state $\rho$, do they lose anything during the cycle from $\rho$ to $\rho$ via $\sigma$.

More precisely, do we have

$$
\begin{equation*}
\rho \leftrightharpoons \sigma \equiv R(\rho \rightarrow \sigma) R(\sigma \rightarrow \rho) \tag{1}
\end{equation*}
$$

equal to unity? We will call the quantity $\rho \leftrightharpoons \sigma$ as the amount of the irreversibility in the transition $\rho \rightarrow \sigma \rightarrow \rho$.

In quantum information, the maximally entangled states have a special significance. For example, they are the only states which one can use in faithful teleportation. It was convenient therefore to have special names for the rate $R(\rho \rightarrow \sigma)$ and the inverse of the rate $R(\sigma \rightarrow \rho)$, when $\sigma$ is a maximally entangled state in $2 \otimes 2$ 4, 55, 6]. They are respectively called the distillable entanglement $D(\rho)$ and entanglement cost $F(\rho)$ of $\rho[6,7,8,9]$.

The above question has been answered in the case when $\sigma$ is a maximally entangled state in $2 \otimes 2$ 10, 11]. In these references, there are states exhibited for which distillable entanglement is strictly less than its entanglement cost 12, 13]. That is, examples of $\rho$ were given for which $\rho \leftrightharpoons \sigma<1$, with $\sigma$ being a maximally entangled state in $2 \otimes 2$.

There is therefore a possible irreversible loss of entanglement when one transforms $\rho$ into a maximally entangled state. Returning back to $\rho$, we may not be able to get back the entanglement with which we had started with. A question related to the above question can now be asked. How will the amount of the irreversibilities, for such return trips of $\rho$, change for different target states $\sigma$ ? Stated in terms of the rates defined above, for given $\rho$, how does $\rho \leftrightharpoons \sigma$ behave for different $\sigma$ ? This question is the main theme of this paper.

In this paper we analyse the conversion rates $R(\rho \rightarrow \sigma)$ themselves, as well as the relations between rates for different transitions. In Section 【I we give simple upper and lower bounds for the optimal rate in a cycle of $\rho$ to $\sigma$ and back in terms of asymptotic entanglement measures. In Section III we show that $\rho \leftrightharpoons \sigma$ is continuous when $\rho$ and $\sigma$ remains in an open set of distillable states 14].

In section IV we will consider irreversibility in transformations of $\rho$ to $\sigma$ and back in the case when $\rho$ and $\sigma$ are mixtures of two Bell states. In section $\mathbb{\square}$ we show that the cycle $\rho \rightarrow \sigma \rightarrow \rho$ may have varying degrees of irreversibility depending on the chosen $\sigma$. And there is nothing special for a cycle of $\rho$ via a maximally entangled state. The irreversibility $\rho \leftrightharpoons \sigma$ of $\rho$ in a cycle via $\sigma$ can be strictly greater or less than its irreversibility in a cycle via a maximally entangled state. In Section VI we show along the lines of Ref. [11] (see also [15]) that the entanglement of formation is additive for the maximally correlated states $\sum a_{i j}|i i\rangle\langle j j|$ in arbitrary dimensions and this leads to the computation of the entanglement cost of such states in $2 \otimes 2$ [16]. For maximally correlated states in arbitrary dimensions, we express their entanglement cost as a simple optimization procedure. As a by-product, we obtain irreversibility (with respect to a maximally entangled state) in asymptotic manipulations for these states in $2 \otimes 2$. Using the value of entanglement cost of maximally correlated states (in $2 \otimes 2$ ), we will discuss in Section VII that the feature of nonextremal nature of maximally entangled states (as studied in Section (V) can also be obtained by considering the class of maximally correlated states in $2 \otimes 2$. We subsequently show in Section VIII that these considerations can also be seen from the perspective of the ratio problem of entanglement measures [17]. And the ratio problem discussed in this paper, is in a sense complementary to the one considered in Ref. 17]. In the last section (Section IX), we present some discussions. Distillable entanglement of a bipartite state under LOCC is no greater than that under operations preserving the positivity of partial transposition 18, 19]. Whether a strict inequality holds is unknown. In the concluding section, we show that such nonequality is related to the behaviour of the transformation rates (under LOCC) to separable target states.

## II. DEFINITIONS AND SOME BOUNDS

Let us first fix our notations. We use both the notations $R(\rho \rightarrow \sigma)$ and $D_{\sigma}(\rho)$ for the optimal rate of the transformation $\rho \rightarrow \sigma$ for bipartite states $\rho$ and $\sigma$ under LOCC. Similarly we use $1 / R(\sigma \rightarrow \rho)$ and $F_{\sigma}(\rho)$ interchangeably. It will be also useful to introduce a notation for rates of transitions of the general form

$$
\rho_{1} \rightarrow \rho_{2} \rightarrow \ldots \rightarrow \rho_{n} \rightarrow \ldots \rightarrow \rho_{2} \rightarrow \rho_{1}
$$

The rate of such transitions is the optimal ratio of final number copies of the state $\rho_{1}$ to the initial number of copies of $\rho_{1}$, in the route specified above, and it will be denoted by

$$
\rho_{1} \leftrightharpoons \rho_{2} \leftrightharpoons \ldots \leftrightharpoons \rho_{n}
$$

We have

$$
\begin{equation*}
(\rho \leftrightharpoons \sigma)=R(\rho \rightarrow \sigma) R(\sigma \rightarrow \rho)=\frac{D_{\sigma}(\rho)}{F_{\sigma}(\rho)} \tag{2}
\end{equation*}
$$

(we will sometimes put brackets, in order not to confuse between "=" and "乙"). By definition we obtain

$$
\begin{equation*}
(\rho \leftrightharpoons \sigma)=(\sigma \leftrightharpoons \rho) \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
(\rho \leftrightharpoons \sigma \leftrightharpoons \omega)=(\rho \leftrightharpoons \sigma)(\sigma \leftrightharpoons \omega) \tag{4}
\end{equation*}
$$

Since there are more possibilities for going from $\rho$ to $\omega$ directly, rather than via the intermediate state $\sigma$ we obtain the inequality

$$
\begin{equation*}
\rho \leftrightharpoons \sigma \leftrightharpoons \omega \leq \rho \leftrightharpoons \omega \tag{5}
\end{equation*}
$$

The above properties will help us to establish some bounds for $\rho \leftrightharpoons \sigma$ in terms of the quantities $\rho \leftrightharpoons \psi^{-}$ and $\sigma \leftrightharpoons \psi^{-}$, where we take the singlet

$$
\left|\psi^{-}\right\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)
$$

as our canonical maximally entangled state in $2 \otimes 2$. (We will use $\psi^{-}$to denote $\left|\psi^{-}\right\rangle\left\langle\psi^{-}\right|$.) First, due to (5), we have

$$
\psi^{-} \leftrightharpoons \rho \leftrightharpoons \sigma \leq \psi^{-} \leftrightharpoons \sigma
$$

which, in view of (4), gives

$$
\begin{equation*}
\rho \leftrightharpoons \sigma \leq \frac{\sigma \leftrightharpoons \psi^{-}}{\rho \leftrightharpoons \psi^{-}} \tag{6}
\end{equation*}
$$

On the other hand, from (5) we can also get

$$
\begin{equation*}
\rho \leftrightharpoons \psi^{-} \leftrightharpoons \sigma \leq \rho \leftrightharpoons \sigma \tag{7}
\end{equation*}
$$

Joining relations (6) and (7) and exchanging the roles of $\rho$ and $\sigma$ in (5) we finally obtain

$$
\begin{gather*}
\left(\rho \leftrightharpoons \psi^{-}\right)\left(\sigma \leftrightharpoons \psi^{-}\right) \leq \\
(\rho \leftrightharpoons \sigma)  \tag{8}\\
\leq \min \left\{\frac{\sigma \leftrightharpoons \psi^{-}}{\rho \leftrightharpoons \psi^{-}}, \frac{\rho \leftrightharpoons \psi^{-}}{\sigma \leftrightharpoons \psi^{-}}\right\}
\end{gather*}
$$

So far we have related the operational quantity $\rho \leftrightharpoons \sigma$ to other operational quantities (rates involving singlets). There arises therefore the question whether one can improve the inequalities by use of the results on asymptotic entanglement monotones [20, 21]. For example, it is known that entanglement measures satisfying some assumptions give upper bounds for $D$ (distillable entanglement) and lower bounds for $F$ (entanglement cost). Thus to obtain suitable bounds, one does not have to go into the very difficult issue of optimizing distillation or formation task, but rather one can choose a function with the needed properties. On the other hand, to have for example a lower bound for $D$, an operational approach usually cannot be overcome: one needs to point out a specific protocol of distillation. And similarly for obtaining an upper bound of entanglement cost. It turns out that in our case, there is a similar issue. We will be able to
prove an upper bound that refers to entanglement monotones, rather than to conversion rates. To this end let us recall 22] that if a function $E$ is (i) nonincreasing under LOCC, and (ii) asymptotically continuous, then we have

$$
\begin{equation*}
R(\rho \rightarrow \sigma) \leq \frac{E^{\infty}(\rho)}{E^{\infty}(\sigma)} \tag{9}
\end{equation*}
$$

where $E^{\infty}(\rho)=\lim _{n \rightarrow \infty} \frac{1}{n} E\left(\rho^{\otimes n}\right)$ is the regularization of $E$. Note that the conditions are on the function $E$, while the bound is with its regularization. Examples of such measures are entanglement of formation and relative entropy distance from separable states 23] or the so called PPT states 19]. Now, recalling that $\rho \leftrightharpoons \sigma=R(\rho \rightarrow \sigma) R(\sigma \rightarrow \rho)$ and applying the inequality (9) with different entanglement measures to each factor we obtain

$$
\begin{equation*}
\rho \leftrightharpoons \sigma \leq \frac{E_{1}(\rho) E_{2}(\sigma)}{E_{2}(\rho) E_{1}(\sigma)} \tag{10}
\end{equation*}
$$

where $E_{i}$ are regularizations of any two chosen measures satisfying (i) and (ii). Putting $E_{1}=F, E_{2}=D$ we can recover the right-hand-side bound of formula (6) (even though we do not know if $D$ satisfies (ii) or if it is a regularization of some measure which satisfies (i) and (ii)). It can be written by means of $F$ and $D$ as

$$
\begin{equation*}
\rho \leftrightharpoons \sigma \leq \frac{F(\rho) D(\sigma)}{F(\sigma) D(\rho)} \tag{11}
\end{equation*}
$$

Note that eq. (11) (which is the same as eq. (6)) is not obtained from eq. (10). It is obtained (as shown just before eq. (6)) from general considerations on the rates of transformations.

To make the results transparent, we introduce the following quantity:

$$
\begin{equation*}
R_{D i f f}=R_{D i f f}(\rho, \sigma)=\left(\rho \leftrightharpoons \psi^{-}\right)-(\rho \leftrightharpoons \sigma) \tag{12}
\end{equation*}
$$

whose continuity properties we will consider next.

## III. CONTINUITY

In this section we present some continuity arguments for $R_{D i f f}$ that we will use later on. This follows from the results in Ref. [14]. The only requirement that is needed to be imposed on an asymptotic measure of entanglement $E$, for it to be continuous in any open set of distillable states, is that

$$
E\left(\eta_{1}\right) \geq \frac{y}{x} E\left(\eta_{2}\right)
$$

whenever the transformation

$$
x \times \eta_{1} \rightarrow y \times \eta_{2} \quad(x, y \geq 0)
$$

is achievable in the asymptotic limit by LOCC for two bipartite states $\eta_{1}$ and $\eta_{2}$. Here by $x \times \eta$, we mean $x$
copies of $\eta$, with suitable changes when $x$ is not a positive integer. We now show that $F_{\sigma}($.$) satisfies this condition.$

By definition of $F_{\sigma}(),. x \times \eta_{1} \rightarrow y \times \eta_{2}$ implies

$$
\begin{array}{ll} 
& F_{\sigma}\left(\eta_{1}\right) \times \sigma \rightarrow \eta_{1} \rightarrow \frac{y}{x} \eta_{2} \\
\Leftrightarrow & \frac{x}{y} F_{\sigma}\left(\eta_{1}\right) \times \sigma \rightarrow \eta_{2} \\
\Leftrightarrow & F_{\sigma}\left(\eta_{2}\right) \leq \frac{x}{y} F_{\sigma}\left(\eta_{1}\right) \\
\text { i.e. } & F_{\sigma}\left(\eta_{1}\right) \geq \frac{y}{x} F_{\sigma}\left(\eta_{2}\right) .
\end{array}
$$

The proof that $D_{\sigma}($.$) also satisfies the condition required$ for continuity, is similar. And therefore we have the continuity of $R_{D i f f}$ for arbitrary $\rho$ and $\sigma$ in any open set of distillable states. We stress that this proof is for arbitrary $\rho$ and $\sigma$ in arbitrary dimensions. We will use this continuity later on, to understand the nature of $R_{D i f f}$ in general.

## IV. IRREVERSIBILITY OF THE CYCLE $\rho \rightarrow \sigma \rightarrow \rho$ FOR DIFFERENT $\sigma$

We will now use the bounds obtained in Section II to tackle the problem of irreversibility of the cycle $\rho \rightarrow \sigma \rightarrow$ $\rho$ for different $\sigma$. We will like to ask as to when the strict inequality of the following form is possible:

$$
\begin{equation*}
\rho \leftrightharpoons \sigma<\rho \leftrightharpoons \psi^{-} \tag{13}
\end{equation*}
$$

Definitely it is the case for states for which we can show

$$
\begin{equation*}
\frac{D(\sigma)}{F(\sigma)}<\left(\frac{D(\rho)}{F(\rho)}\right)^{2} \tag{14}
\end{equation*}
$$

This follows from inequality (11). Below we show that the inequality (14) is indeed satisfied for some choices of $\rho$ and $\sigma$ as mixtures of two Bell states. More examples will be reported in Section VII

We take $\rho$ and $\sigma$ as mixtures of two Bell states with different mixing parameters. Let

$$
\begin{equation*}
\rho=(1-p)\left|\phi^{+}\right\rangle\left\langle\phi^{+}\right|+p\left|\phi^{-}\right\rangle\left\langle\phi^{-}\right|, \quad p \in\left[\frac{1}{2}, 1\right] \tag{15}
\end{equation*}
$$

where

$$
\begin{aligned}
\left|\phi^{+}\right\rangle & =\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) \\
\left|\phi^{-}\right\rangle & =\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)
\end{aligned}
$$

And let $\sigma$ be another mixture of the same Bell states:

$$
\begin{equation*}
\sigma=(1-q)\left|\phi^{+}\right\rangle\left\langle\phi^{+}\right|+q\left|\phi^{-}\right\rangle\left\langle\phi^{-}\right|, \quad q \in\left[\frac{1}{2}, 1\right] . \tag{16}
\end{equation*}
$$

In this case, we know the values of $D$ and $F$ exactly 11], and in certain regions on the $(p, q)$-plane, the inequality (14) is satisfied. In Fig. 11 we plot the function

$$
f=D^{2}(\rho) F(\sigma)-F^{2}(\rho) D(\sigma)
$$

over the $(p, q)$-plane. For any $q$, there exists a nonzero range of $p$ near $p=1$, for which $f$ is positive. The


FIG. 1: Plot indicating nonzero value of $R_{D i f f}$. The function $f=D^{2}(\rho) F(\sigma)-F^{2}(\rho) D(\sigma)$, with $\rho$ and $\sigma$ given by eqs. (15) and (16), is plotted on the $(p, q)$-plane. A positive value of $f$ indicates a positive value of $R_{D i f f}$.
inequality (14) is therefore satisfied in those regions of the $(p, q)$-plane. Consequently, in those regions of the $(p, q)$-plane, we have $(\rho \leftrightharpoons \sigma)<\left(\rho \leftrightharpoons \psi^{-}\right)$. Note that this region cannot have a nonempty intersection with the line $p=q$. We will discuss in the next section (Section (V), the inequality (13) cannot be satisfied in this case.

## V. THERE'S NOTHING SPECIAL ABOUT A SINGLET

As singlets are useful in many quantum information processes, one may tend to believe that the amount of the irreversibility in the transition $\rho \rightarrow \sigma \rightarrow \rho$, that is $\rho \leftrightharpoons \sigma$, will somehow acquire an extreme value when the target $\sigma$ is a singlet. But in this section, we will show by an explicit example that this is not the case. (See Refs. [24, 25] in this regard.)

First note that in the previous section, we have already shown that

$$
(\rho \leftrightharpoons \sigma)<\left(\rho \leftrightharpoons \psi^{-}\right)
$$

holds for certain choices of the states $\rho$ and $\sigma$. More such examples will be exhibited in Section VII

In this section, we will show that this inequality is not true for all combinations of $\rho$ and $\sigma$. We will present cases where this inequality is strictly reversed, that is where

$$
\begin{equation*}
(\rho \leftrightharpoons \sigma)>\left(\rho \leftrightharpoons \psi^{-}\right) \tag{17}
\end{equation*}
$$

is true.
Indeed for arbitrary states $\rho$ and $\sigma,(\rho \leftrightharpoons \sigma)=1$ for $\rho=\sigma$, while

$$
\left(\rho \leftrightharpoons \psi^{-}\right)=R\left(\rho \rightarrow \psi^{-}\right) R\left(\psi^{-} \rightarrow \rho\right)
$$

being the rate at which we get back the state $\rho$ in a return journey via the target state $\psi^{-}$, must remain $\leq 1$. This is just another way of stating that the entanglement cost of a state cannot be less than its distillable entanglement. Consequently, we have

$$
(\rho \leftrightharpoons \sigma) \geq\left(\rho \leftrightharpoons \psi^{-}\right)
$$

whenever $\rho=\sigma$. We will also have a strict inequality, that is the inequality (17) will hold (for $\rho=\sigma$ ), once we have $\rho \leftrightharpoons \psi^{-}$strictly less than unity, for some $\rho$. It may seem that this is true for any nondistillable state, i.e., states $\rho$ for which $R\left(\rho \rightarrow \psi^{-}\right)=0$ (this includes separable as well as bound entangled states 13]). However for the right hand side of the inequality (17) to vanish for any nondistillable state $\rho$, one must also have finite $R\left(\psi^{-} \rightarrow \rho\right)$. If $R\left(\psi^{-} \rightarrow \rho\right)$ is arbitrarily high (as we know is true at least for any separable state), one must consider some kind of limiting procedure. It is a nontrivial question as to what limiting procedure one must consider in such a case. We will come back to this question in Section IX However for some bound entangled states $\rho$, it was shown that $R\left(\psi^{-} \rightarrow \rho\right)$ is finite so that the inequality (17) is satisfied for such states (the left hand side being unity and the right hand side vanishing) 10]. The inequality (17) is satisfied even for some distillable states (i.e. states $\rho$ for which $\left.R\left(\rho \rightarrow \psi^{-}\right)>0\right)$ as was shown in Refs. [10, 11].

## A. A case study: $\rho$ and $\sigma$ are mixtures of two Bell states

Let us take $\rho$ and $\sigma$ as in eqs. (15) and (16). It was shown in Ref. 11] that $D(\rho)$ is strictly less than $F(\rho)$ for $1 / 2<p<1$. Consequently we have

$$
(\rho \leftrightharpoons \sigma)>\left(\rho \leftrightharpoons \psi^{-}\right)
$$

whenever $p=q \neq 1 / 2,1$. The opposite inequality holds, i.e.

$$
(\rho \leftrightharpoons \sigma)<\left(\rho \leftrightharpoons \psi^{-}\right)
$$

is true for any $q$ and a sufficiently high $p$, as was shown in the previous section (see also Fig 1). Therefore it seems that with respect to the transition rates, maximally entangled states do not have a special status. Related points were made in Refs. [24, 25]. However in their cases, the nonmaximally entangled "extreme" state was a nonmaximally entangled pure state and the considerations were in the non-asymptotic regime.

To get a more clear picture of what is going on, let us try to estimate the behavior of the difference

$$
\begin{equation*}
R_{D i f f}=\left(\rho \leftrightharpoons \psi^{-}\right)-(\rho \leftrightharpoons \sigma) \tag{18}
\end{equation*}
$$

in the case when $\rho$ and $\sigma$ are given by eqs. (15) and (16), for $p \in(1 / 2,1)$ with a fixed $q \neq 1 / 2,1$. From Fig. 1, it
is clear that $R_{\text {Diff }}$ is positive for $p \in(1-\varepsilon, 1)$ for some $1-\varepsilon>q$.

We have already shown that there are states for which

$$
(\rho \leftrightharpoons \sigma)>\left(\rho \leftrightharpoons \psi^{-}\right)
$$

for $\rho=\sigma$. From the continuity of $R_{D i f f}$ (SectionIII) and the fact that the set of distillable states is an open set [26], it follows that this inequality holds also for unequal $\rho$ and $\sigma$. The above argument using continuity shows that for some states, it is better not to return back via the singlet but via some other states.

Coming back to estimating the behavior of $R_{D i f f}$ in the case when $\rho$ and $\sigma$ are given by eqs. (15) and (16), for $p \in(1 / 2,1)$ with a fixed $q \neq 1 / 2,1$, it follows that $R_{\text {Diff }}$ has a negative value around the point $p=q$. Note that we are using the fact that mixtures of two Bell states, if not mixed in equal proportions (when it is separable), are distillable states and as the set of distillable states is an open set [26], the class of all mixtures (excepting equal mixture case) of two Bell states are in an open set of distillable states, so that the considerations in Section III become applicable.

In Fig 2 we try to plot $R_{D i f f}$ as a function of $p$, in the case when $\rho$ and $\sigma$ are given by eqs. (15) and (16), for $p \in(1 / 2,1)$ with a fixed $q \neq 1 / 2,1$. In the figure, we take $q=2 / 3$. All we know is that $R_{D i f f}=1$ for $p=1$ and $R_{D i f f}=-1$ for $p=q$. Also we know that near the point $p=1$, there is a certain neighborhood $(1-\varepsilon, 1)$, in which $R_{D i f f}$ is positive (see Fig. [1). Due to continuity, these two regions must meet. And in the process, $R_{\text {Diff }}$ must cross the $R_{D i f f}=0$ line at least once. We do not know if there are more than one crossing. It will be very interesting to find some general properties of the points where $R_{D i f f}$ vanishes. The set of all pairs of states $\{\rho, \sigma\}$, for which $R_{\text {Diff }}$ vanishes probably has some interesting properties, as the transformation $\rho \leftrightharpoons \sigma$ behaves like a transformation to a singlet. This is another way to see that maximally entangled states have no speciality with respect to transformation rates. In the figure, we join the two portions near $p=1$ and near $p=q$ by a monotonic curve. This monotonicity is by no means known. On the left of the point $p=q(=2 / 3)$, we draw the $R_{\text {Diff }}$ curve as monotonically reaching the value 0 as $p \rightarrow 1 / 2$. Neither this monotonicity nor the limiting value are known. Note that for $p=1 / 2, \rho$ is a separable state. We will come back to the issue of the limiting value of the transformation rates, near a separable state, in the concluding section.

## VI. ENTANGLEMENT OF FORMATION FOR MAXIMALLY CORRELATED STATES

The class of states for which we have proved that the inequality (eq. (13))

$$
(\rho \leftrightharpoons \sigma)>\left(\rho \leftrightharpoons \psi^{-}\right)
$$



FIG. 2: Hypothetical shape of $R_{\text {Diff }}$ in the case when $\rho$ and $\sigma$ are given by eqs. (15) and (16), for $p \in(1 / 2,1)$ with a fixed $q \neq 1 / 2,1$. We take $q=2 / 3$ in the figure. Let us warn the reader that we compose facts and imagination in drawing the curve (see text for full details). Notice that there is (at least) one point where $R_{D i f f}=0$. The set of pairs of states $\{\rho, \sigma\}$, for which this happens, probably has some special properties, as in the transformation $\rho \leftrightharpoons \sigma$ behaves like a transformation to a singlet. This is another way to see that, with respect to transformation rates, there is nothing special about a singlet.
holds (in Section (IV), is a relatively small class of states. Hence the nonextremal nature of the singlet (in terms of asymptotic LOCC transformation rates) is depicted using this small class. Can the same considerations be extended to a larger class? In the next Section, we try to make the extension to the class of maximally correlated states 19]

$$
\begin{equation*}
\rho^{m c}=\sum a_{i j}|i i\rangle\langle j j| \tag{19}
\end{equation*}
$$

There are a number of obstacles in such an enterprise. We deal with these obstacles in this Section. The distillable entanglement for the class of maximally correlated states is not known. The PPT-distillable entanglement (the optimal asymptotic fraction of faithful maximally entangled states obtainable by any superoperator preserving the positivity of partial transposition [18]) for any $\rho_{A B}^{m c}$ from this class is known to be [7, 19] (see also [27])

$$
S\left(\rho_{A}^{m c}\right)-S\left(\rho_{A B}^{m c}\right)
$$

where $S(\rho)$ is the von Neumann entropy of $\rho$ and $\rho_{A}^{m c}=$ $\operatorname{tr}_{B}\left(\rho_{A B}^{m c}\right)$.

It has been conjectured in [19] that the PPT-distillable entanglement $\left(D_{\Gamma}\right)$ is equal to the distillable entanglement under LOCC operations for any $\rho^{m c}$. In general it is not known whether there are states for which PPTdistillable entanglement and LOCC-distillable entanglement are provably different. In Section IX we will discuss this open problem from the perspective of the results obtained in this paper. Among the maximally correlated states, for mixtures of two Bell states as also for all pure states, these two quantities are equal. This is also true
for certain other states of the class of maximally correlated states, as has been checked in Refs. 28]. Our further steps may therefore be restricted by this problem. Note however that to check whether the inequality (13) is satisfied by using inequality (14), we may bypass this problem by choosing $\rho$ to be a state whose (LOCC-) distillable entanglement is known while $\sigma$ to be a state whose PPT-distillable entanglement is known. This is due to the fact that the class of PPT superoperators is strictly larger than the LOCC class ("strictly", because of the existence of bound entangled states [13]). Consequently the (LOCC-) distillable entanglement $D(\sigma)$ of $\sigma$ is smaller than or equal to its PPT-distillable entanglement $D_{\Gamma}(\sigma)$. Thus we will have inequality (14) and hence inequality (13), once we have

$$
\frac{D_{\Gamma}(\sigma)}{F(\sigma)}<\left(\frac{D(\rho)}{F(\rho)}\right)^{2}
$$

The next obstacle is that entanglement cost is also not known for the maximally correlated states. In general, entanglement cost is equal to regularization of entanglement of formation $E_{f}$ [9]. If, for a given state $\rho$ we have

$$
E_{f}(\rho \otimes \rho)=2 E_{f}(\rho)
$$

then the two quantities are equal. Below we will show, on the lines of [11] (see also [15]), that the entanglement of formation is additive for maximally correlated states. Consequently, the entanglement cost is obtained for these states in $2 \otimes 2$, as entanglement of formation is known for all states of two qubits [16]. Also we supply an optimization procedure for calculating the entanglement of formation for maximally correlated states in arbitrary dimensions. The procedure is simpler than that is contained in the very definition of entanglement of formation. Having obtained the entanglement cost for maximally correlated states in $2 \otimes 2$ and as the PPT-distillable entanglement is known for all maximally correlated states [7, 19], we show that all nonpure entangled maximally correlated states in $2 \otimes 2$ have irreversibility in asymptotic LOCC manipulations of entanglement.
A. Additivity of entanglement of formation for maximally correlated states in arbitrary dimensions

In this subsection, we provide some formula for $E_{f}$ for maximally correlated states in $d \otimes d$ and we will show that their entanglement of formation is additive. Entanglement of formation $E_{f}$ of a bipartite state $\rho$ is defined as

$$
E_{f}(\rho)=\inf \sum_{i} p_{i} S_{A}\left(\psi_{i}\right)
$$

where $S_{A}(\psi)$ denotes entropy of reduction of the bipartite state $\psi$ to a single party and the optimization is taken over all decompositons $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$ of $\rho$ into pure
bipartite states. The maximally correlated states (19) are determined by the matrix $a_{i j}$. One finds that the matrix has to be positive semidefinite and of unit trace. Thus with any maximally correlated state $\rho^{m c}$ in $C^{d} \otimes C^{d}$, one can associate the state $\rho^{\prime}$ of a single system of the form

$$
\begin{equation*}
\rho^{\prime}=\sum_{i j} a_{i j}|i\rangle\langle j| \tag{20}
\end{equation*}
$$

One notes that the support of $\rho^{m c}$ is spanned by the vectors of the form

$$
\begin{equation*}
\psi=\sum_{i} c_{i}|i\rangle|i\rangle \tag{21}
\end{equation*}
$$

where $\{\mid i>\}$ is the same basis as the one used in the definition of the maximally correlated state in eq. (19). As this is a subspace, any decomposition of $\rho^{m c}$ will consist of vectors of the above form. Consider then any decomposition $\left\{p_{k}, \psi_{k}\right\}$ with $\psi_{k}=\sum_{i} c_{i}^{k}|i\rangle|i\rangle$. The coefficients have to satisfy the constraints

$$
\sum_{k} p_{k} c_{i}^{k} \overline{c_{j}^{k}}=a_{i j}
$$

Treating $\left\{c_{i}^{k}\right\}_{i}$ as vectors $x_{k}$ belonging to the Hilbert space $C^{d}$ of a single system, we obtain

$$
\begin{equation*}
\sum_{k} p_{k}\left|x_{k}\right\rangle\left\langle x_{k}\right|=\varrho^{\prime} \tag{22}
\end{equation*}
$$

The entropy $S_{A}\left(\psi_{k}\right)$ is equal to Shannon entropy of the diagonal elements of the state $\left|x_{k}\right\rangle\left\langle x_{k}\right|$ in the basis $|i\rangle$ :

$$
\begin{equation*}
S_{A}\left(\psi_{k}\right)=-\sum_{i}\left|\left\langle i \mid x_{k}\right\rangle\right|^{2} \log _{2}\left|\left\langle i \mid x_{k}\right\rangle\right|^{2} \equiv H\left(x_{k}\right) \tag{23}
\end{equation*}
$$

We then obtain the following formula for entanglement of formation of the maximally correlated state $\rho^{m c}$ :

$$
\begin{equation*}
E_{f}\left(\varrho_{m c}\right)=\inf \sum_{k} p_{k} H\left(x_{k}\right) \tag{24}
\end{equation*}
$$

where the infimum is taken over all decompositions of the state $\rho^{\prime}$ (defined above) into pure states $x_{k}$. Similarly as in the definition of entanglement of formation, we can take infimum over all decompositions of $\rho^{\prime}$ (including mixed states as members of decomposition). The obtained formula is simpler than the original optimization procedure because it involves state of one system (not a compound one).

Let us now pass to the problem of whether $E_{f}=F$ for maximally correlated states. In Ref. [11] it was shown that any state with support within subspace $V \subset$ $\mathcal{H}_{A} \otimes \mathcal{H}_{B}$ has $E_{f}=F$ if the subspace has the following property: the map $\Lambda: \mathcal{B}(V) \rightarrow \mathcal{B}\left(\mathcal{H}_{A} \otimes \mathcal{H}_{B}\right)$ given by partial trace is so-called entanglement breaking map (using such a map as a channel, one cannot share an entangled state). By extending Example 1 of Ref. [11] one easily finds that the subspace spanned by vectors of the form $|i\rangle|i\rangle$ has such a property. The class of states with support lying within the supspace coincides with maximally correlated ones, so that $F=E_{f}$ for those states.


FIG. 3: Plot for $f=F-D_{\Gamma}$ on the $\left(q,|a|^{2}\right)$-plane for maximally correlated states $\sigma$ given by (25) in $2 \otimes 2$.
B. Irreversibility in asymptotic manipulations of entanglement for maximally correlated states in $2 \otimes 2$

In the previous subsection, we have shown that entanglement of formation is additive for maximally correlated states (given by eq. (19) ) in $d \otimes d$. As entanglement of formation is known for all two qubit states 16], we therefore are able to calculate the entanglement cost $F$ for all maximally correlated states of two qubits. Any such state in $2 \otimes 2$ can be written as

$$
\begin{equation*}
\sigma=(1-q)|\phi\rangle\langle\phi|+q|\psi\rangle\langle\psi|, \quad q \in\left[\frac{1}{2}, 1\right] \tag{25}
\end{equation*}
$$

where

$$
|\phi\rangle=a|00\rangle+b|11\rangle, \quad|\psi\rangle=\bar{b}|00\rangle-\bar{a}|11\rangle
$$

with $|a|^{2}+|b|^{2}=1$. Consequently, one finds that 16]

$$
F=h\left(\frac{1}{2}+\frac{1}{2} \sqrt{1-4(2 q-1)^{2}|a|^{2}|b|^{2}}\right)
$$

for the maximally correlated state $\sigma$ (given by eq. (25)), where $h(x)=-x \log _{2} x-(1-x) \log _{2}(1-x)$ is the binary entropy function. Now the PPT-distillable entanglement $D_{\Gamma}$ of such states is also known, as we noted earlier. In Fig. 3] we plot $F-D_{\Gamma}$ for $\sigma$ on the $\left(q,|a|^{2}\right)$-plane. We see that the value of $F-D_{\Gamma}$ is strictly positive for all $q \in(1 / 2,1)$ and $|a|^{2} \in(0,1 / 2]$. As $D_{\Gamma}$ is greater than or equal to $D, F-D$ is also strictly positive for these ranges. We therefore have irreversibility in asymptotic manipulations of entanglement (i.e. $D<F$ ) for all nonpure entangled maximally correlated states in $2 \otimes 2$.

## VII. MORE CASES TO SHOW THAT THE SINGLET IS NOT SPECIAL WITH RESPECT TO ASYMPTOTIC LOCC TRANSFORMATION RATES

Having calculated the entanglement cost of the maximally correlated states in $2 \otimes 2$, we will now be able to find more examples where the inequality (eq. (13))

$$
\rho \leftrightharpoons \sigma<\rho \leftrightharpoons \psi^{-}
$$

is satisfied. Let us consider the case when $\rho$ is a mixture of two Bell states and $\sigma$ is a maximally correlated state of two qubits. So let $\rho$ be given by eq. (15), i.e.

$$
\rho=(1-p)\left|\phi^{+}\right\rangle\left\langle\phi^{+}\right|+p\left|\phi^{-}\right\rangle\left\langle\phi^{-}\right|
$$

and let $\sigma$ be given by (25), i.e.

$$
\sigma=(1-q)|\phi\rangle\langle\phi|+q|\psi\rangle\langle\psi|, \quad q \in\left[\frac{1}{2}, 1\right]
$$

where

$$
|\phi\rangle=a|00\rangle+b|11\rangle, \quad|\psi\rangle=\bar{b}|00\rangle-\bar{a}|11\rangle
$$

with $|a|^{2}+|b|^{2}=1$.
As is easily checked, for every value of $|a|$, we get a similar surface for the function $D^{2}(\rho) F(\sigma)-F^{2}(\rho) D(\sigma)$ over the $(p, q)$-plane as in Fig. 1 Therefore for a fixed $q$ (and $|a|$ ), there is always a region $(1-\varepsilon, 1)$ (with $1-\varepsilon<$ $q)$ for which $\rho$ (for $p \in(1-\varepsilon, 1)$ ) and $\sigma$ satisfies the inequality (13). That is, $R_{D i f f}$ is positive for $p \in(1-\varepsilon, 1)$ (for some $1-\varepsilon<q$ ), for all fixed values of $q$ and $|a|$.

We have $\rho=\sigma$ on the intersection of $|a|=1 / \sqrt{2}$ and $p=q$, whereby $R_{D i f f}$ is negative on that intersection. Via continuity of $R_{D i f f}$ as well as due to the fact that the set of distillable states is open [26],$R_{\text {Diff }}$ will be negative for unequal $\rho$ and $\sigma$ also, provided they are sufficiently close to each other as well as to the intersection of $|a|=$ $1 / \sqrt{2}$ and $p=q$.

Exactly similar results are obtained even when both $\rho$ and $\sigma$ are from the class of maximally correlated states, if we accept the conjecture of Ref. [19] discussed above.

## VIII. RATIO PROBLEM

In this section, we will view our results from the perspective of the ratio problem of entanglement measures 17]. Let us first briefly recall what is already known about the problem. This will provide a better setting for the aspect of the problem that we want to discuss. The distillable entanglement $D(\rho)$ of a bipartite state $\rho$ is defined as an (optimal) asymptotic ratio. It is the optimal asymptotic fraction of the number of faithful singlets ( $\psi^{-}$) obtainable via an LOCC protocol. It is therefore defined with the maximally entangled states as unit. As distillable entanglement is a measure of a physical quantity (indicating the potential of $\rho$ to teleport, for example), the ratio of the distillable entanglements of different
states, may be hoped to be independent of the chosen unit. The heights of two persons must have the same ratio whether their heights are measured in centimetres or inches. However it turned out that it is not true. In Ref. 17], examples were cited for which

$$
\frac{D_{\sigma}\left(\rho_{1}\right)}{D_{\sigma}\left(\rho_{2}\right)} \neq \frac{D\left(\rho_{1}\right)}{D\left(\rho_{2}\right)}
$$

The same problem arises for entanglement cost as well.
Let us now consider a complementary aspect of the ratio problem. The ratios discussed in Ref. 17] were between different states. Can we not have such a ratio for a single state? It should not be "different" properties of the same state. The height to weight ratio of a person may vary in different unit systems. But the ratio of the height of an individual to the length of her/his arm must remain the same in any unit system. Is this the case with distillable entanglement and entanglement cost of a state? That is, are the ratios $\frac{D_{\sigma}(\rho)}{F_{\sigma}(\rho)}$ and $\frac{D(\rho)}{F(\rho)}$ equal?

By what we have already shown in the previous sections, these ratios can be shown to be unequal in certain cases. Indeed in the above case-studies one needs to substitute $R(\rho \rightarrow \sigma)$ by $D_{\sigma}(\rho)$ and $R(\sigma \rightarrow \rho)$ by $\frac{1}{F_{\sigma}(\rho)}$ as also $R\left(\rho \rightarrow \psi^{-}\right)$by $D(\rho)$ and $R\left(\psi^{-} \rightarrow \rho\right)$ by $\frac{1}{F(\rho)}$ to obtain the envisioned nonequality

$$
\frac{D_{\sigma}(\rho)}{F_{\sigma}(\rho)} \neq \frac{D(\rho)}{F(\rho)}
$$

## IX. DISCUSSIONS

We have shown that the optimal asymptotic rate at which $\rho$ is retrieved (under LOCC) in a cycle $\rho \rightarrow \rho$ via $\sigma$ can be different for different $\sigma$. This gives a new dimension to the fundamental irreversibility in asymptotic manipulations of entanglement in that it provides in a natural way, a notion of the "amount" of the irreversibility in local asymptotic manipulations of a state.

For the considerations made in this paper, we take $\rho$ and $\sigma$ as two mixtures of two Bell states as also $\rho$ as a mixture of two Bell states and $\sigma$ as a maximally correlated state.

The considerations naturally led us to consider the quantity

$$
R_{D i f f}=\left(\rho \leftrightharpoons \psi^{-}\right)-(\rho \leftrightharpoons \sigma)
$$

whose behavior we have tried to judge in certain simple cases. To judge the character of $R_{\text {Diff }}$, we have used either certain bounds on the transformation (derived in Section III) or we have used certain continuity arguments (Section IIII). The exact value of the quantity is not known for a single nontrivial case. Nevertheless, one can have a feel for the general behavior of this difference. Consider a distillable state $\eta$ in $m \otimes n$ for which $D(\eta)$ is strictly less than $F(\eta)$. Then

$$
\left(\eta \leftrightharpoons \psi^{-}\right)-1 \equiv \frac{D(\eta)}{F(\eta)}-1<0
$$

Therefore $R_{D i f f}$, for $\rho=\sigma=\eta$, is strictly less than zero. From the continuity of $R_{D i f f}$, which we proved earlier, and as the set of distillable states is open [26], it follows that negativity (of $R_{D i f f}$ ) remains for $\rho$ and $\sigma$ sufficiently close to $\eta$. On the other hand,

$$
1-\left(\psi^{-} \leftrightharpoons \eta\right) \equiv 1-\frac{D(\eta)}{F(\eta)}>0
$$

Therefore $R_{D i f f}$ is strictly positive for $\rho=\psi^{-}$and $\sigma=\eta$. And again by continuity, $R_{D i f f}$ is positive for $\rho$ and $\sigma$ sufficiently close to $\psi^{-}$and $\eta$ respectively.

We tried to plot $R_{D i f f}$ in a simple case in Fig. 2 Related to this, there are some interesting open questions which we have discussed in Section $\nabla$

Here we will like to discuss the issue of the behaviour of the quantity $R_{D i f f}$ near separable states. For example, consider the case when $\rho$ and $\sigma$ are mixtures of two Bell states (given by eqs. (15) and (16)). Take a fixed $q \neq$ $1 / 2,0$. We are interested to find the value of $R_{\text {Diff }}$ as we approach the point $p=1 / 2$. Note that at $p=1 / 2, \rho$ is a separable state. It may seem that in general, the rate $\rho \leftrightharpoons \sigma$ at which $\rho$ is retrieved in a return journey via $\sigma$ is vanishing in the limit when $\sigma$ approaches to a separable state while $\rho$ remains distillable. In the case when $\rho$ is given by eqs. (15), we have checked that $\rho \leftrightharpoons \psi^{-} \rightarrow 0$ as $p \rightarrow 1 / 2$ [11]. In Fig. 2] we have plotted $R_{\text {Diff }} \rightarrow 0$ as $p \rightarrow 1 / 2$ with this intuition.

We will now see that if we assume that $\rho \leftrightharpoons \sigma \rightarrow 0$ as $\sigma$ approaches to a separable state (with $\rho$ remaining distillable), then one can arrive at examples of states for which the PPT-distillable entanglement is strictly greater than its LOCC distillable entanglement. The example is against the conjecture given in [19] that for maximally correlated states, $D_{\Gamma}=D$. Let us mention however that we are not in a position to give a counterexample to this conjecture. We merely show that a counterexample exists if we believe that $\rho \leftrightharpoons \sigma \rightarrow 0$ as $\sigma$ approaches to a separable state (with $\rho$ remaining distillable). Consider the state

$$
\rho(p)=p|00\rangle\langle 00|+(1-p)\left|\phi^{+}\right\rangle\left\langle\phi^{+}\right|
$$

and the rate

$$
\rho \leftrightharpoons \psi^{-}=\frac{D(\rho)}{F(\rho)}
$$

Note here that the state is product when $p=1$. Therefore according to our assumption, $\frac{D(\rho)}{F(\rho)} \rightarrow 0$ as $p \rightarrow 1$. For the state $\rho$, we know the values of PPT-distillable entanglement [7, 19, 27] as well as its entanglement cost (Section VI). One can easily check that the quantity $\frac{D_{\Gamma}(\rho)}{F(\rho)}$ tends to $1 / 2$ as $p$ tends to 1 . Thus, modulo our assumption, we have that $D_{\Gamma}$ is strictly greater than $D$ for states $\rho(p)$ which are sufficiently close to $\rho(1)$.

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