

# Unified criterion for security of secret sharing in terms of violation of Bell inequalities

Aditi Sen(De), Ujjwal Sen, and Marek Żukowski

*Instytut Fizyki Teoretycznej i Astrofizyki, Uniwersytet Gdański, PL-80-952 Gdańsk, Poland*

In secret sharing protocols, a secret is to be distributed among several partners so that leaving out any number of them, the rest do not have the complete information. Strong multiqubit correlations in the state by which secret sharing is carried out, had been proposed as a criterion for security of such protocols against individual attacks by an eavesdropper. However we show that states with *weak* multiqubit correlations can also be used for secure secret sharing. That our state has *weak* multiqubit correlations, is shown from the perspective of violation of local realism, and also by showing that its higher order correlations are described by lower ones. We then present a unified criterion for security of secret sharing in terms of violation of local realism, which works when the secret sharing state is the Greenberger-Horne-Zeilinger state (with *strong* multiqubit correlations), as well as states of a different class (with *weak* multiqubit correlations).

## I. INTRODUCTION

Violation of Bell inequalities seem to be a signature of what has been called “useful entanglement” [1, 2]. It was shown in Ref. [1] that violation of local realism can be seen as a criterion for security of secret sharing protocols [3, 4]. It was argued that a strong violation of multi-qubit Bell inequalities [5, 6] by the state that acts as the vehicle in secret sharing can be a criterion for security of the secret sharing. Violation of local realism was also shown to be connected with distillability [2] and super-classical communication complexity [7].

In a secret sharing protocol, the holder of a secret (call her Alice) wants to distribute her secret among  $N-1$  separated parties (call them Bobs,  $B_1, B_2, \dots, B_{N-1}$ ) such that leaving out any (non-zero) number of the Bobs, the other Bobs would have no information about Alice’s secret. Such protocols were shown to be possible in Refs. [3, 4], if Alice and the  $N-1$  Bobs share a large number of certain entangled states [8, 9].

Such a protocol could suffer from the onslaught of a possible eavesdropper Evan. Evan could spy (quantum mechanically) on the channels that carry the states from Alice to the Bobs and obtain information about Alice’s secret. In Ref. [1], it was shown that if the secret sharing is carried out by using Greenberger-Horne-Zeilinger (GHZ) states, the secret of Alice is secure from eavesdropping as long as the state between Alice and Evan (after eavesdropping) does not violate any Bell inequality while the state between Alice and the Bobs (after eavesdropping) violates the  $N$ -qubit Bell inequalities [5, 6] very strongly.

In this paper, we show that security of secret sharing in the multi-qubit scenario can be possible even when the state between Alice and the Bobs of the secret sharing protocol contains weak multi-qubit correlations. We argue that the criterion for security of secret sharing is of a different type.

- (i) The state shared between Alice and the Bobs (after eavesdropping), if suitably projected into certain states by all but one Bobs, the remaining Bob would share a state with Alice, which violates a

two-qubit Bell inequality.

- (ii) At the same time, the state between Alice and Evan must satisfy such inequalities.

We show that both for the GHZ state, which contains *strong* multi-qubit correlations (in the sense of strong violation of multi-qubit Bell inequalities), as well as for another state (we call it the  $G$  state), which in several ways (as indicated below) contains *weak* multi-qubit correlations, the security of secret sharing is exactly in the same range in which both the above conditions, (i) and (ii), are met. This criterion can therefore be seen as a *unified* criterion for security of secret sharing. We subsequently show, that the  $N$ -qubit  $G$  state cannot have strong  $N$ -qubit correlations. This is in the sense that

- (a) the  $N$ -qubit correlation functions of the  $G$ -state only weakly violate the multi-qubit Bell inequalities [5, 6] as compared to the GHZ state.
- (b) The  $N$ -qubit correlations of this state are determined by lower-order correlations of the state in contrast to the GHZ state [10].

## II. NEW STATES FOR SECRET SHARING

The state

$$|G_N\rangle = \frac{1}{\sqrt{2}}(|W_N\rangle + |\overline{W}_N\rangle) \quad (1)$$

can be used for secret sharing, where

$$\begin{aligned} |W_N\rangle &= \frac{1}{\sqrt{N}} \sum |10^{\otimes N-1}\rangle, \\ |\overline{W}_N\rangle &= \frac{1}{\sqrt{N}} \sum |01^{\otimes N-1}\rangle, \end{aligned}$$

with  $|0\rangle$  and  $|1\rangle$  being the eigenvectors of  $\sigma_z$ , and e.g.,  $\sum |10^{\otimes N-1}\rangle$  denotes the unnormalised superposition of all  $N$  arrays of  $(N-1)$   $|0\rangle$ s and a single  $|1\rangle$ . Note that

$$|G_4\rangle = \frac{1}{\sqrt{2}}(|+x\rangle^{\otimes 4} - |-x\rangle^{\otimes 4})$$

where

$$|\pm x\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).$$

However, for more qubits, this family is different than the GHZ family, as we would show later.

The property of the  $G$  state that would help us to use it for secret sharing is the following:

$$\langle \sigma_x^{\otimes N} \rangle_{G_N} = 1, \quad \langle \sigma_y^{\otimes N} \rangle_{G_N} = (-1)^{M+1}, \quad (2)$$

whenever  $N = 2M$ . The first equation follows from the fact that

$$\sigma_x^{\otimes N} |W_N\rangle = |\overline{W}_N\rangle, \quad \sigma_x^{\otimes N} |\overline{W}_N\rangle = |W_N\rangle.$$

To derive the second equation in eq. (2), note that

$$\sigma_y^{\otimes N} |G_N\rangle = i^N \frac{1}{\sqrt{2}} (-|\overline{W}_N\rangle + (-1)^{N-1} |W_N\rangle).$$

The state on the right-hand-side is orthogonal to  $|G_N\rangle$  for odd  $N$ . But for even  $N (= 2M)$ , the right-hand-side is  $-i^{2M} |G_N\rangle$ .

Suppose that Alice (A) has some information which she wishes to secretly distribute among  $2M - 1$  Bobs ( $B_1, B_2, \dots, B_{2M-1}$ ). But she wishes to distribute it in such a way that *all* the Bobs must cooperate to obtain the message. Leaving out any nonzero number of Bobs, the remaining Bobs would not be able to gather the complete information about Alice's bit. This is equivalent to the situation when Alice is able to secretly distribute a random sequence of binary digits (bits) to the Bobs with the same property. Let us now show that this is possible by using the  $G_{2M}$ -state, shared between Alice and the  $2M - 1$  Bobs. Suppose that Alice and the Bobs share a large number of the  $G_{2M}$ -state and they randomly choose between the  $\sigma_x$  and  $\sigma_y$  observables to make a measurement on their respective parts of the shared states [11]. Subsequently they publicly share information about the *bases* in which they have made their measurements. (They obviously do not share information about the *results* of the measurements.) They keep only those results in which *all* of them measured in the same basis, that is when either all of them measured in the  $\sigma_x$ -basis or all of them measured in the  $\sigma_y$ -basis. In any run of such a Bell type experiment, if *all* the parties happen to choose  $\sigma_x$  as their observable, then the  $2M$  results at Alice and the  $2M - 1$  Bobs are related by (see eq.(2))

$$r_A r_{B_1} \dots r_{B_{2M-1}} = 1.$$

If all the Bobs cooperate, then they know that

$$r_A = r_{B_1} \dots r_{B_{2M-1}}.$$

If any nonzero number of Bobs refuse to cooperate or are left out, the remaining Bobs cannot gather the complete information about  $r_A$ . This is because irrespective of what results that any  $2M - 2$  Bobs happen to obtain

in their  $\sigma_x$  basis measurements, the state of remaining two qubits is such that the single-qubit density matrix at Alice is mixed. Similarly, if *all* the parties happen to choose  $\sigma_y$  as their observable, then the result of Alice is related to those of the Bobs by

$$s_A = (-1)^{M+1} s_{B_1} \dots s_{B_{2M-1}}.$$

In this way, a sequence of random bits is created at Alice and if there is no eavesdropping, the Bobs (if all of them cooperate) would be able to reproduce the same random sequence.

### III. SECURITY

We now consider the security of the distributed sequence of random bits against a possible eavesdropper, Evan. We work with the assumption that Evan would only be able to make coherent but individual attacks. Thus although Evan may attack coherently on all the qubits sent to the Bobs (from Alice) in a single run of the experiment, he is not able to perform joint operations on qubits from different runs. As has been stressed in a recent review [12], at present, even individual attacks by an eavesdropper would be very challenging in reality.

The states that are shared by the Bobs after the measurements by Alice are as follows:

$$\begin{aligned} |\pm x\rangle &: \frac{1}{\sqrt{2}}(|\xi\rangle \pm |\overline{\xi}\rangle) \\ |\pm y\rangle &: \frac{1}{\sqrt{2}}(|\xi\rangle \pm i|\overline{\xi}\rangle), \end{aligned} \quad (3)$$

the left hand column in eq.(3) being the outcome obtained at Alice, while the right hand side gives the corresponding state that is shared by the Bobs. Here  $|\pm x\rangle$  and  $|\pm y\rangle$  are the eigenvectors of  $\sigma_x$  and  $\sigma_y$ , and

$$\begin{aligned} |\xi\rangle &= \frac{1}{\sqrt{2M}} (\sum |10^{\otimes 2M-2}\rangle + |1^{\otimes 2M-1}\rangle) \\ |\overline{\xi}\rangle &= \frac{1}{\sqrt{2M}} (\sum |01^{\otimes 2M-2}\rangle + |0^{\otimes 2M-1}\rangle). \end{aligned} \quad (4)$$

Note that the protocol for secret sharing is exactly equivalent to the BB84 key distribution protocol [13], when we consider the problem of eavesdropping and the eavesdropping is coherent. Consequently the optimal coherent individual attack that Evan ( $E$ ) can perform, is given by the unitary transformation [14]

$$\begin{aligned} U_{BE} |\xi\rangle |0\rangle &= |\xi\rangle |0\rangle \\ U_{BE} |\overline{\xi}\rangle |0\rangle &= \cos \phi |\overline{\xi}\rangle |0\rangle + \sin \phi |\xi\rangle |1\rangle, \end{aligned} \quad (5)$$

where  $B$  denotes  $B_1 B_2 \dots B_{2M-1}$  and  $\phi \in [0, \pi/2]$ . After Evan implements his attack, i.e., after he applies the unitary operation  $U_{BE}$  given by eq.(5), the state  $G_N$  of Alice and the Bobs coupled with the probe  $|0\rangle$  of Evan transforms as

$$\begin{aligned} |G_N\rangle_{AB} |0\rangle_E &\rightarrow |\psi\rangle_{ABE} = \frac{1}{\sqrt{2}} (|0\rangle_A |\xi\rangle_B |0\rangle_E \\ &+ \cos \phi |1\rangle_A |\overline{\xi}\rangle_B |0\rangle_E + \sin \phi |1\rangle_A |\xi\rangle_B |1\rangle_E). \end{aligned} \quad (6)$$

The condition that is needed for security is [15]

$$I(A : B) > I(A : E), \quad (7)$$

where again  $B$  denotes the aggregate of all the Bobs. This security implies that Alice and the Bobs can run a one-way protocol (privacy amplification) if and only if the condition (7) is satisfied. Here the mutual information  $I(X : Y)$  is defined as

$$I(X : Y) = H(X) - H(X|Y),$$

where

$$H(\{p_i\}) = - \sum_i p_i \log_2 p_i$$

is the Shannon entropy of a probability distribution  $\{p_i\}$ .

We would now calculate the mutual information (after Evan's attack), when Alice and the Bobs measure either all of them in the  $\sigma_x$  basis or all of them in the  $\sigma_y$  basis. This happens with equal probability. Therefore in our case,

$$I(A : B) = 1 - \frac{1}{2}(H_x(A|B) + H_y(A|B)).$$

(As Alice chooses her measurements randomly,  $H(A) = 1$ .) The state shared between Alice and the  $2M - 1$  Bobs (after Evan's attack) is (see eq.(6))

$$\rho_{AB} = \frac{1 + \cos^2 \phi}{2} |\alpha\rangle \langle \alpha| + \frac{\sin^2 \phi}{2} |1\rangle \langle 1| \otimes |\xi\rangle \langle \xi|, \quad (8)$$

where

$$|\alpha\rangle = \frac{|0\rangle |\xi\rangle + \cos \phi |1\rangle |\bar{\xi}\rangle}{\sqrt{1 + \cos^2 \phi}}.$$

Let us first evaluate

$$H_x(A|B) \equiv p_x(B = 1)H_x(A|B = 1) + p_x(B = -1)H_x(A|B = -1), \quad (9)$$

where for example,  $p_x(B = 1)$  is the probability of the event "B = 1", when all the Bobs measure in the  $\sigma_x$  basis.

Tracing out  $A$  from  $\rho_{AB}$ , the state  $\rho_B$  shared between the  $2M - 1$  Bobs is (see eq.(8))

$$\frac{1 + \sin^2 \phi}{2} |\xi\rangle \langle \xi| + \frac{\cos^2 \phi}{2} |\bar{\xi}\rangle \langle \bar{\xi}|.$$

Consequently,

$$p_x(B = \pm 1) = \frac{1}{2}$$

and from eq.(8),

$$p_x(A = 1, B = \pm 1) = p_x(A = -1, B = \mp 1) = \frac{1 \pm \cos \phi}{4}$$

Therefore

$$H_x(A|B = 1) = H\left(\frac{1 + \cos \phi}{2}\right),$$

where  $H(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$  is the binary entropy function. By symmetry, the expression for  $H_x(A|B = -1)$  is exactly the same. So from eq.(9), one obtains

$$H_x(A|B) = H\left(\frac{1 + \cos \phi}{2}\right)$$

By symmetry, the expression for  $H_y(A|B)$  is exactly the same as that for  $H_x(A|B)$ . So finally,

$$I(A : B) = 1 - H\left(\frac{1 + \cos \phi}{2}\right). \quad (10)$$

$I(A : E)$  is obtained by replacing  $\phi$  by  $\pi/2 - \phi$  in  $I(A : B)$ . With these expressions, one can see that the security condition (7) holds if and only if  $\phi < \pi/4$ .

Consider now the state  $\rho_{AB}$  (given by eq.(8)), obtained by tracing out the eavesdropper  $E$ , after the eavesdropping. If any  $2M - 2$  of the  $2M - 1$  Bobs perform measurements in the  $\sigma_z$  basis and obtains either  $|0\rangle^{\otimes 2M-2}$  or  $|1\rangle^{\otimes 2M-2}$  (cf. [16, 17]), then the remaining Bob (say  $B_k$ ) shares with Alice the (two-qubit) state

$$\rho_{AB_k} = \frac{1 + \cos^2 \phi}{2} |\beta\rangle \langle \beta| + \frac{\sin^2 \phi}{2} |11\rangle \langle 11|,$$

where

$$|\beta\rangle = \frac{|01\rangle + \cos \phi |10\rangle}{\sqrt{1 + \cos^2 \phi}}.$$

The two-qubit state  $\rho_{AB_k}$  violates local realism if and only if  $\phi < \pi/4$  [18]. Note that there are no sequential measurements involved. The parties follow the ordinary Bell-type experiment and classical communication is needed only to share this data. We consider violation of local realism exhibited by a subset of this data. Measurements at the parties do not depend on results of measurements at the other parties.

On the other hand, the state obtained after we trace out the Bobs (from the state  $|\psi\rangle_{ABE}$ , given by eq. (6)), is

$$\rho_{AE} = \frac{1 + \sin^2 \phi}{2} |\gamma\rangle \langle \gamma| + \frac{\cos^2 \phi}{2} |10\rangle \langle 10|,$$

where

$$|\gamma\rangle = \frac{|00\rangle + \sin \phi |11\rangle}{\sqrt{1 + \sin^2 \phi}}.$$

The state  $\rho_{AE}$  violates local realism if and only if  $\phi > \pi/4$  [18].

Therefore for  $\phi \in [0, \pi/4]$ , the state shared (after eavesdropping by Evan) between Alice and the Bobs violate local realism in the sense described above, while the Alice-Evan state does not violate local realism. And the secret sharing is secure in exactly the same range.

#### IV. COMPARISON WITH THE GHZ STATE

A related feature was obtained in [1], where secret sharing was investigated by using the GHZ state [19]

$$|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle).$$

However in this case, it was shown that the secret sharing protocol is secure as long as after eavesdropping by Evan, strong  $N = 2M$  qubits correlations are still present in the state shared by Alice and Bobs. If secret sharing is carried out by a shared GHZ state, as has been considered in [1], the shared state (after optimal eavesdropping by Evan) between Alice, the  $2M - 1$  Bobs and Evan is just the same as displayed on the right hand side of eq.(6) with the replacement

$$\begin{aligned} |\xi\rangle &\rightarrow |0^{\otimes 2M-1}\rangle \\ |\bar{\xi}\rangle &\rightarrow |1^{\otimes 2M-1}\rangle. \end{aligned}$$

Consider the state between Alice and any one of the Bobs, after measurements in the  $\sigma_x$ -basis at the other Bobs, and suppose that either  $|+x\rangle$  clicks at all those  $2M - 2$  Bobs, or  $|-x\rangle$  clicks at all of them. As can be checked, this state again violates local realism if and only if  $\phi < \pi/4$ . The state shared by Alice and Evan does not violate local realism in that range. This is the range in which secret sharing is secure.

In [1], it was numerically shown that the complete set of Bell inequalities in multi-qubit systems [5, 6] is violated by the state shared between Alice and the  $2M - 1$  Bobs (after eavesdropping by Evan) by a magnitude of more than  $2^{\frac{2M-1}{2}}$  if and only if  $\phi \in [0, \pi/4)$ . In this range, the Alice and Evan state does not violate any Bell inequality. It was therefore proposed that this form of strong violation of local realism for the  $2M$ -qubit correlations is a criterion for security in secret sharing.

However we have shown that the range in which security of secret sharing by a GHZ state is obtained, is also concurrent with violation of local realism in another form; that of satisfying items (i) and (ii) in the Introduction. This feature is also shared by our  $G$  states. The proposed criterion is therefore a unified criterion for security of secret sharing.

#### V. STRENGTH OF MULTIQUBIT CORRELATIONS

The GHZ states have strong multiqubit correlations. In contrast, the family  $|G_{2M}\rangle$ , which can also be considered as the vehicle of the secret sharing, is unlikely to produce a strong violation of local realism for  $2M$ -qubit correlations. On the contrary, we will show that it has a large amount of its entanglement concentrated in correlations of lower number of parties. In the following, we drop the restriction that the number of parties is even.

#### A. Violation of local realism

Consider the white noise admixed  $N$ -qubit state  $G_N$ ,

$$\rho^{G_N} = p_N |G_N\rangle \langle G_N| + (1 - p_N)\rho_{noise}^N \quad (11)$$

where  $\rho_{noise}^N = \frac{1}{2^N}I$  is the maximally mixed state of  $N$  qubits. If  $N - 2$  parties make measurements in the  $\sigma_z$  basis, and if either  $|0\rangle$  clicks at all the parties, or  $|1\rangle$  clicks at all the parties, the collapsed state is given respectively by

$$\rho^{G_2} \otimes (\otimes_{i=3}^N |0\rangle_{ii} \langle 0|)$$

or

$$\rho^{G_2} \otimes (\otimes_{i=3}^N |1\rangle_{ii} \langle 1|),$$

where  $\rho^{G_2}$  is the Werner state

$$\rho^{G_2} = p_{(N \rightarrow 2)} |G_2\rangle \langle G_2| + (1 - p_{(N \rightarrow 2)})\rho_{noise}^2$$

with

$$p_{(N \rightarrow 2)} = \frac{1}{1 + \frac{(1-p_N)N}{p_N 2^{N-2}}},$$

and

$$|G_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

The state  $\rho^{G_2}$  has no local realistic description for  $p_{(N \rightarrow 2)} > 1/\sqrt{2}$ . This would imply that the state  $\rho^{G_N}$  cannot have a local realistic model for

$$p_N > p_{G_N}^{crit} \equiv \frac{N}{N + (\sqrt{2} - 1)2^{N-2}}.$$

A GHZ state of  $N$  qubits maximally violates the multi-qubit Bell inequalities [5, 6] and the amount of violation is  $\sqrt{2^{N+1}}$ . Consequently the noisy GHZ state

$$\rho^{GHZ_N} = q_N P_{|\text{GHZ}_N\rangle} + (1 - q_N)\rho_{noise}^N \quad (12)$$

violates local realism for

$$q_N > q_{GHZ_N}^{crit} \equiv 1/\sqrt{2^{N-1}}.$$

But for  $N \geq 13$ ,

$$p_{G_N}^{crit} > q_{GHZ_N}^{crit}.$$

Thus for  $N \geq 13$ , the nonclassicality of the correlations in  $G_N$  is more robust to “white noise” admixture than the one for the corresponding GHZ state. A similar method was used in Ref. [17] (cf. [16]) to show that the nonclassical behavior of the  $W$  states is stronger than that of the GHZ state for sufficiently large number of parties. Note here that the robustness to white noise admixture of the nonclassicality of correlations in the  $G_N$  states obtained by the above method is weaker than the one that is obtained for the corresponding  $W$  state.

Therefore strong nonclassical properties of the  $G_N$  states emerge after local projection measurements in some parties. This strong nonclassicality for lower number of parties seems to imply that the  $N$ -party correlations are weak. One can see this more directly in the following considerations for the 6-qubit  $G$  state, the lowest number of qubits in which secret sharing is possible with a  $G$  state and which is not simply a GHZ state. Consider the correlation tensor  $\hat{T}^{G_6}$  of the state  $G_6$ , the elements of which are given by

$$T_{x_1 \dots x_6} = \text{tr}(P_{|G_6\rangle} \sigma_{x_1}^{(1)} \dots \sigma_{x_6}^{(6)}), \quad (x_i = x, y, z).$$

Explicit calculation reveals the following structure of the tensor:

$$\begin{aligned} \hat{T}^{G_6} = & \otimes_{i=1}^6 \vec{x}_i + \otimes_{i=1}^6 \vec{y}_i - \otimes_{i=1}^6 \vec{z}_i \\ & + \frac{1}{3} \{ \sum (\otimes_{i=1}^2 \vec{x}_i) \otimes (\otimes_{j=3}^6 \vec{z}_j) - \sum (\otimes_{i=1}^4 \vec{x}_i) \otimes (\otimes_{j=5}^6 \vec{z}_j) \\ & - \sum (\otimes_{i=1}^4 \vec{y}_i) \otimes (\otimes_{j=5}^6 \vec{z}_j) + \sum (\otimes_{i=1}^2 \vec{y}_i) \otimes (\otimes_{j=3}^6 \vec{z}_j) \\ & - \sum (\otimes_{i=1}^2 \vec{x}_i) \otimes (\otimes_{j=3}^6 \vec{y}_j) - \sum (\otimes_{i=1}^4 \vec{x}_i) \otimes (\otimes_{j=5}^6 \vec{y}_j) \\ & + \sum (\otimes_{i=1}^2 \vec{x}_i) \otimes (\otimes_{j=3}^4 \vec{y}_j) \otimes (\otimes_{k=5}^6 \vec{z}_k) \}, \end{aligned}$$

where for example  $\sum \vec{x}_1 \otimes \vec{x}_2 \otimes \vec{z}_3 \otimes \vec{z}_4 \otimes \vec{z}_5 \otimes \vec{z}_6$  contains all the 15 combinations of two  $\vec{x}$ s and four  $\vec{z}$ s.

A sufficient condition for an  $N$ -qubit state  $\rho$  to have a local realistic model for  $N$ -qubit correlations (of two-settings Bell experiments) is that

$$\sum_{x_1, \dots, x_N = x, y} T_{x_1 \dots x_N}^2 \leq 1$$

for *any* set of local coordinate systems [6]. Here  $x$  and  $y$  denote any two arbitrary orthogonal directions, which can be separately defined for each observer.

For the noisy  $G_6$  state  $\rho^{G_6}$  given by the eq.(11),

$$\sum_{x_1, \dots, x_6 = x, y} T_{x_1 \dots x_6}^2 = \frac{16p_6^2}{3},$$

for the *fixed* local coordinate system in which the secret sharing is considered. Thus an explicit local realistic description for the  $(x, y)$  correlations for the state  $\rho^{G_6}$  exists for

$$p_6 \leq \sqrt{3/16} \approx 0.433012$$

whereas (for the correlations in the same sector) the noisy 6-qubit GHZ state (eq.(12)) has no local realistic description for

$$q_6 > 1/\sqrt{32} \approx 0.176777.$$

The latter follows from the fact that the 6-qubit GHZ state violates the multiqubit Bell inequalities maximally, and the amount of violation is  $\sqrt{2^5}$ . This shows that the  $N$ -qubit correlations of the GHZ state are stronger in comparison to that of the  $G$ -state. Since the scheme of the secret sharing protocol is for  $\sigma_x$  and  $\sigma_y$  measurements in a fixed local coordinate system, it is probably enough to check the sufficiency for local realism in the

$(x, y)$  sector. However we show below that the stronger  $N$ -qubit correlations of the GHZ state as compared to the  $G$  state persists even when *any* local coordinate systems are considered.

For *any* set of local coordinate systems,

$$\sum_{x_1, \dots, x_6 = x, y} T_{x_1 \dots x_6}^2 \leq \sum_{x_1, \dots, x_6 = x, y, z} T_{x_1 \dots x_6}^2 = 23,$$

for the state  $G_6$ . So the noisy  $G_6$ -state has for sure a local realistic description for 6-qubit correlations, for

$$p_6 \leq 1/\sqrt{23} \approx 0.208514.$$

Therefore the 6-qubit GHZ state is definitely more robust to white noise admixture than the state  $G_6$  *when 6-qubit correlations are considered*.

### B. Higher order correlations are determined by lower order ones for the $|G_N\rangle$ state

We would now give yet another argument to show that the  $N$ -qubit state  $|G_N\rangle$  has weaker  $N$ -qubit correlations than the  $N$ -qubit GHZ state. Note that for the GHZ state, information regarding the reduced density matrices of lower number of parties is insufficient to describe the whole  $N$ -party state. Indeed the state

$$\frac{1}{2}(|0^{\otimes N}\rangle\langle 0^{\otimes N}| + |1^{\otimes N}\rangle\langle 1^{\otimes N}|),$$

has the same single-party, 2-party,  $\dots$ ,  $(N-1)$ -party reduced density matrices as the state  $|GHZ_N\rangle$ . Consequently, high order correlations of the GHZ states are not determined by lower order correlations. However, as shown in [10], this feature is rare. For almost all states, actually the opposite is true.

On the lines of [10], we will now show that for the states  $|G_N\rangle$ , for  $N \geq 5$ , there is no other state (pure or mixed) whose  $(N-1)$ -party reduced density matrices match those of the  $G_N$  state.

For convenience of notation, let us call the parties sharing the  $N$ -qubit  $G_N$  state as  $1, 2, \dots, N$ . A state whose  $(N-1)$ -party reduced density matrices agree with those of  $G_N$ , may be a mixed state. We therefore allow for an environment  $E$ . Any such state, whose  $(N-1)$ -party reduced density matrix of the parties  $1, 2, \dots, (N-1)$  agree with the state  $G_N$ , can be written as

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|v_0\rangle_{12\dots(N-1)} |E_0\rangle_{NE} + |v_1\rangle_{12\dots(N-1)} |E_1\rangle_{NE}), \quad (13)$$

where

$$\begin{aligned} |v_0\rangle &= \frac{1}{\sqrt{N}}(\sum |10^{\otimes N-2}\rangle + |1^{\otimes N-1}\rangle) \\ |v_1\rangle &= \frac{1}{\sqrt{N}}(\sum |01^{\otimes N-2}\rangle + |0^{\otimes N-1}\rangle) \end{aligned} \quad (14)$$

and  $|E_0\rangle$  and  $|E_1\rangle$  are orthonormal. Note that  $|v_0\rangle$  and  $|v_1\rangle$  are just the states  $|\xi\rangle$  and  $|\bar{\xi}\rangle$  extended also to the

case of even  $N$  (see eq. (4)). Since  $|E_0\rangle$  and  $|E_1\rangle$  are orthonormal, they can be written as

$$\begin{aligned} |E_0\rangle &= |0\rangle_N |e_{00}\rangle_E + |1\rangle_N |e_{01}\rangle_E \\ |E_1\rangle &= |0\rangle_N |e_{10}\rangle_E + |1\rangle_N |e_{11}\rangle_E. \end{aligned} \quad (15)$$

Let us now try to match the  $(N-1)$ -party reduced density matrix of the parties  $2, 3, \dots, N$  of the state  $G_N$  with that of  $|\chi\rangle$ . Consequently, it should be possible to write  $|\chi\rangle$  as

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|v_0\rangle_{23\dots N} |F_0\rangle_{1E} + |v_1\rangle_{23\dots N} |F_1\rangle_{1E}), \quad (16)$$

where the states  $|v_0\rangle$  and  $|v_1\rangle$  are given by eq. (14), and the orthonormal environment states  $|F_0\rangle$  and  $|F_1\rangle$  are given by

$$\begin{aligned} |F_0\rangle &= |0\rangle_1 |f_{00}\rangle_E + |1\rangle_1 |f_{01}\rangle_E \\ |F_1\rangle &= |0\rangle_1 |f_{10}\rangle_E + |1\rangle_1 |f_{11}\rangle_E. \end{aligned} \quad (17)$$

The states on the right hand sides of the equations (13) and (16) are actually the same states. Comparing the different terms, it is not hard to see that for  $N \geq 5$ ,

$$\begin{aligned} |e_{00}\rangle &= |e_{11}\rangle \\ |e_{01}\rangle &= |e_{10}\rangle = 0. \end{aligned} \quad (18)$$

Therefore the state  $|\chi\rangle$ , for  $N \geq 5$ , is a product state of  $|G_N\rangle$  with a state of the environment, if  $|\chi\rangle$  has its  $(N-1)$ -party reduced density matrices equal to the corresponding ones for  $|G_N\rangle$ . Thus for  $N \geq 5$ , there are no other  $N$ -qubit states whose  $(N-1)$ -qubit reduced density matrices are equal to those of  $|G_N\rangle$ . Therefore, contrary to the GHZ states, given the  $(N-1)$ -qubit reduced density matrices, the state  $|G_N\rangle$  is already completely specified. This again underlines the fact that the state  $|G_N\rangle$  does not possess strong  $N$ -qubit correlations. This also explicitly shows that the states  $|G_N\rangle$ , for  $N \geq 5$ , are not from the GHZ family.

## VI. SUMMARY

We have given a criterion on security of secret sharing protocols. The criterion is based on violation in a specific way, of Bell inequalities (as given by items (i) and (ii) in the Introduction). A criterion for security of secret sharing based on violation of Bell inequalities was provided in [1]. Let us mention here that we do not simply extend this previously known criterion of Ref. [1]. The criterion mentioned in Ref. [1] (call it C1), is not satisfied by the G-states considered in our paper. And yet secret sharing is possible by using the G-states. We have put forward an independent criterion. Our criterion (call it C2), although again based on violation of Bell inequalities, is drastically different from the criterion mentioned in Ref. [1]. And we find that both the GHZ state (which was shown in Ref. [1] to satisfy C1 and useful for secret sharing) and the G-state (which is shown in this paper to violate C1 and still useful for secret sharing) are satisfying our criterion C2.

We believe that these considerations would be useful in the general discussions as well as as for applications of quantum cryptography [12].

## Acknowledgments

AS and US acknowledges the University of Gdańsk, Grant No. BW/5400-5-0236-2 and BW/5400-5-0256-3. MZ is supported by Professorial Subsidy of Foundation for Polish Science.

- 
- [1] V. Scarani and N. Gisin, Phys. Rev. Lett. **87**, 117901 (2001); *ibid.*, Phys. Rev. A **65**, 012311 (2001), and references therein.
- [2] A. Acín, Phys. Rev. Lett. **88**, 027901 (2002); A. Acín, V. Scarani, and M.M. Wolf, quant-ph/0112102; *ibid.*, Phys. Rev. A **66**, 042323 (2002).
- [3] M. Żukowski, A. Zeilinger, M. Horne, and H. Weinfurter, Acta Phys. Pol. A **93**, 187 (1998).
- [4] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).
- [5] H. Weinfurter and M. Żukowski, Phys. Rev. A **64**, 010102 (2001); R.F. Werner and M.M. Wolf, Phys. Rev. A **64**, 032112 (2001).
- [6] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
- [7] Č. Brukner, M. Żukowski and A. Zeilinger, Phys. Rev. Lett. **89**, 197901 (2002); Č. Brukner, M. Żukowski, J.-W. Pan, and A. Zeilinger, quant-ph/0210114.
- [8] Note here that the task that we deal with is to secretly share a *classical* bit, in the way mentioned in the text. More general tasks of secretly sharing a quantum bit has been considered (see [9]). However these general tasks do not concern us here.
- [9] R. Cleve, D. Gottesman, and H.-K. Lo, Phys. Rev. Lett. **83**, 648 (1999); D. Gottesman, Phys. Rev. A **61**, 042311 (2000); D.P. DiVincenzo, P. Hayden, and B.M. Terhal, quant-ph/0207147.
- [10] N. Linden, S. Popescu, and W.K. Wootters, Phys. Rev. Lett. **89**, 207901 (2002); N. Linden and W.K. Wootters, *ibid.* **89**, 277906 (2002).
- [11] A.K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
- [12] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).

- [13] C.H. Bennett and G. Brassard, in *Proceedings of the International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, NY, 1984); C.H. Bennett, G. Brassard, and N. D. Mermin, *Phys. Rev. Lett.* **68**, 557 (1992).
- [14] C.A. Fuchs, N. Gisin, R.B. Griffiths, C.-S. Niu, and A. Peres, *Phys. Rev. A* **56**, 1163 (1997).
- [15] I. Csiszár and J. Körner, *IEEE Trans. Inf. Theory* **IT-24**, 339 (1978).
- [16] S. Popescu and D. Rohrlich, *Phys. Lett. A* **166**, 293 (1992).
- [17] A. Sen(De), U. Sen, M. Wieśniak, D. Kaszlikowski, and M. Żukowski, quant-ph/0211023 (*Phys. Rev. A*, in press).
- [18] R. Horodecki, P. Horodecki and M. Horodecki, *Phys. Lett. A* **200**, 340 (1995).
- [19] Here one must note that secret sharing by the GHZ state has the added feature that if any one of the Bobs are left out, the remaining Bobs are with strictly no information. We however do not deal with this extreme case. In our protocol, if any of the Bobs are left out, the remaining Bobs cannot obtain the complete information about Alice's bit.