Cloning of orthogonal mixed states entails irreversibility

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Orthogonal pure states can be cloned as well as deleted. However if there is an initial disorder in the system, that is for orthogonal mixed states, one cannot perform deletion. And cloning, in such cases, necessarily produces an irreversibility, in the form of leakage of information into the environment.

Nonorthogonal states cannot be cloned [1]. And orthogonal states can be cloned. However we will show that if there is an initial disorder in the system, one necessarily produces "irreversibility", even when cloning orthogonal states.

Consider two arbitrary mixed orthogonal states. For definiteness, we take them to be of rank two each. This restriction does not change the generality of the statements that are made below. Suppose therefore that the two orthogonal mixed states are

$$\begin{aligned} \varrho_0 &= p\,|0\rangle\langle 0| + q\,|2\rangle\langle 2|\,, \\ \varrho_1 &= r\,|1\rangle\langle 1| + s\,|3\rangle\langle 3|\,, \end{aligned} \tag{1}$$

where of course p, q, r, s are nonnegative numbers such that $p+q=1$, and $r+s=1$, and $|0\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$ are a set of mutually orthonormal states.

Consider now the task of cloning the two states, ϱ_0 and ϱ_1, if any one is given. So we want to create $\varrho_i \otimes \varrho_i$ from ϱ_i coupled with a blank state (that is, a state that has no information about i, $i = 0, 1$). There is a trivial way to do that. One simply makes a measurement onto the (rank-two) projection operators

$$\begin{aligned} P_0 &= |0\rangle\langle 0| + |2\rangle\langle 2|\,, \\ P_1 &= |1\rangle\langle 1| + |3\rangle\langle 3|\,. \end{aligned}$$

If P_0 clicks, then the conclusion is that the given state was ϱ_0. Otherwise, the state was ϱ_1. After finding out what the state is, one can just prepare the required extra copy of the state that is indicated by the measurement. Note that one may deliberately perform the finer measurement onto the orthonormal basis consisting of the states $|0\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$. That would destroy the information about the mixing probabilities (p, q, and r, s) in the states ϱ_0 and ϱ_1. But we assume that these mixing probabilities are known. So if, for example, $|2\rangle$ clicks in such a finer measurement, the conclusion is that the given state was ϱ_0. One can then prepare two copies of ϱ_0.

However, for the case when $q = s = 0$, that is when ϱ_0 and ϱ_1 are pure, and are respectively $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, there is another way to clone. It is by using a gate that takes

$$\begin{aligned} |0\rangle_S\,|0\rangle_B &\to |0\rangle_S\,|0\rangle_B\,, \\ |1\rangle_S\,|0\rangle_B &\to |1\rangle_S\,|1\rangle_B\,. \end{aligned} \tag{2}$$

This could for example be the CNOT gate, which additionally takes

$$\begin{aligned} |0\rangle\,|1\rangle &\to |0\rangle\,|1\rangle\,, \\ |1\rangle\,|1\rangle &\to |1\rangle\,|0\rangle\,. \end{aligned}$$

To see the cloning of the states, consider the states marked by S in eq. (2) as the original copy, and those marked by B as the blank copy. One then obtains the two copies of the original on the right-hand-side of eq. (2).

Note an important difference in the two ways of cloning. In the first method, a measurement step is involved. This measurement, results in making the process "open". After the whole process of cloning has been completed, there is information about the system (precisely, the result of the measurement) left in the environment. Let us call this method of cloning orthogonal states as "open cloning".

Now contrast this method of cloning with the second method of cloning orthogonal *pure* states, by using CNOT-type gates. In this case, after the cloning process has been completed, there is no information about the system that is left in the environment. Let us call this method of cloning orthogonal *pure* states as "closed cloning".

Open cloning is thus an irreversible process. It leads to production of some "garbage" in the environment. Closed cloning, on the other hand, is a "clean" process. There is no left out garbage in this case. We will show that cloning of orthogonal mixed states will always produce garbage. Such states cannot be cloned by closed cloning.

Consider the state ϱ_i ($i = 0, 1$). We want to produce two copies of ϱ_i. That is, we want to produce $\varrho_i \otimes \varrho_i$. At the input, we take a blank copy ϱ_b (along with ϱ_i). Therefore our input is $\varrho_i \otimes \varrho_b$, $i = 0, 1$. And we want $\varrho_i \otimes \varrho_i$ at the output. If we do not want the information about i to get leaked into the environment, this evolution must be done unitarily. Clearly, this cannot be done. To see this, just note that unitary evolution preserves the spectrum. But $\varrho_i \otimes \varrho_b$ cannot have the same spectrum as $\varrho_i \otimes \varrho_i$ for both $i = 0$ *and* 1. Therefore cloning of two orthogonal mixed states is not possible in a closed system. To implement this cloning, one is bound to do a measurement, so that the cloning is an open cloning.

Let us now consider the deleting process. In deleting [2], one requires to have $\varrho_i \otimes \varrho_b$ at the output, when $\varrho_i \otimes \varrho_i$ is fed at the input. And we know that deleting must necessarily be considered in a closed system (see [3, 4] in this regard). So the same reasoning as above, renders such deleting impossible by a unitary evolution. Thus we have that deleting of orthogonal mixed states is not possible.

In the case of orthogonal pure states ($|0\rangle$ and $|1\rangle$), one can clone (as in eq. (2)), and by the inverse operation, one can delete. To see this, note that the inverse operation takes

$$|0\rangle\,|0\rangle \to |0\rangle\,|0\rangle\,,$$
$$|1\rangle\,|1\rangle \to |1\rangle\,|0\rangle\,.$$

However in the case of orthogonal mixed states, considered in this paper, one cannot delete, while cloning is possible if one considers an open system. Therefore for orthogonal *mixed* states, cloning and deleting are not inverse processes of each other.

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