## Nonergodicity of entanglement and its complementary behavior to magnetization in infinite spin chain

Aditi Sen(De), Ujjwal Sen, and Maciej Lewenstein

Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany

We consider the problem of the validity of a statistical mechanical description of two-site entanglement in an infinite spin chain described by the XY model Hamiltonian. We show that the two-site entanglement of the state, evolved from the initial equilibrium state, after a change of the magnetic field, does not approach its equilibrium value. This suggests that two-site entanglement, like (single-site) magnetization, is a nonergodic quantity in this model. Moreover we show that these two nonergodic quantities behave in a complementary way.

Entanglement plays a key role in the rapidly growing field of quantum information processing [1], where so far, most of the discussions of entanglement concern few body systems. Recently, however, the properties of entanglement has been also used to study and understand behavior of "complex" quantum systems. For instance, entanglement in quantum many body systems such as spin chains or Bose-Einstein condensates were studied (see e.g. [2, 3, 4]). The role of "entanglement length" in quantum phase transitions was pointed out in Ref. [5].

The aim of this paper is to deal with the statistical properties of entanglement, or rather the validity of a statistical mechanical approach to entanglement in realistic many body systems. The relation of entanglement with important notions in statistical mechanics is important to understand the behavior of entanglement in quantum many body systems. We will restrict ourselves to onedimensional infinite spin systems. Entanglement in such systems was considered e.g. in Refs. [2, 3, 6]. These papers deal exclusively with the properties of entanglement of an infinite spin chain, which is either in a ground state or in a thermal equilibrium state. Such states are "static" states of the system, as they do not explicitly depend on time. Exceptions include the recent studies of entanglement in quantum dynamics, which allowed to formulate novel types of unprecedentedly efficient numerical codes for simulation of quantum systems [7] (see also [8, 9]).

The validity of a statistical mechanical description of a quantity, characterizing a physical system, depends on the behavior of that quantity as the system evolves in time. More precisely, a necessary condition for validity of statistical mechanical description of a physical quantity is that it must be "ergodic". A physical quantity is said to be ergodic if the time average of the quantity matches its ensemble average. This is usually a hard question to check, even for classical systems. However, an indication of whether a given quantity is ergodic or not, can be obtained by comparing the time evolved state with the equilibrium state.

To deal with such questions, we will therefore consider real time evolution of an infinite one-dimensional spin chain. We will suppose that the spin chain is described by the so-called XY model (see Eq. (1) below). We will show that for large times, the nearest neighbor entanglement of the evolved states, does not approach to

the corresponding entanglement of the equilibrium state. This suggests that nearest neighbor entanglement in this spin system is *nonergodic* [10], indicating that one cannot describe entanglement in such models by equilibrium statistical mechanics. Temporal dynamics of entanglement in spin systems is important in quantum information and computation tasks. For instance, the temporal dynamics of entanglement in a spin system has been used in Ref. [8] to obtain the one-way quantum computer. For the infinite spin chain in the XY model, it is known that the single-site magnetization is also a nonergodic observable [11]. We will show that (two-site) entanglement has a complementary temporal behavior to (single-site) magnetization in this model. We will also show that the (two-site) entanglement and magnetization of the evolved state saturates for low temperatures, for a given time.

The XY model (see Eq. (1)) that we consider here is integrable [12]. For classical systems, integrable models possess a large number of constants of motion, and are usually not ergodic (see e.g. [13]). However, such observations are valid for the case of systems that have a finite (and usually small) number of degrees of freedom. In our case, we deal with an infinite spin chain, and our considerations are distinctly quantum. Hence, questions about ergodicity, or its absence are nontrivial.

A one dimensional spin system (a one-dimensional array (lattice) of spin-1/2 particles) with nearest neighbor interactions is described by a (dimensionless) Hamiltonian of the form  $H_{int} = \sum_i (\mathcal{A}S_i^x S_{i+1}^x + \mathcal{B}S_i^y S_{i+1}^y + \mathcal{C}S_i^z S_{i+1}^z)$ , where the  $S_i^x, S_i^y, S_i^z$  are one-half of the Pauli spin matrices  $\sigma^x, \sigma^y, \sigma^z$  at the *i*-th site of the array, and  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  are coupling constants. Here, we take  $\mathcal{A} \neq \mathcal{B}$  and  $\mathcal{C} = 0$ . We introduce also an external field into the Hamiltonian, so that the total Hamiltonian of the system is  $H(t) = H_{int} - h(t)H_{mag}$ . To ensure that this external field has nontrivial effects on the evolution, we must have  $[H_{int}, H_{mag}] \neq 0$ . The simplest way in which this can be effected is by choosing  $H_{mag} = \sum_i S_i^z$ , and  $\mathcal{A} = 1 + \gamma$ ,  $\mathcal{B} = 1 - \gamma, \gamma \neq 0$ . In this way we arrive at the XY model in the transverse field

$$H(t) = \sum_{i} \left[ (1+\gamma)S_{i}^{x}S_{i+1}^{x} + (1-\gamma)S_{i}^{y}S_{i+1}^{y} - h(t)S_{i}^{z} \right],$$
(1)

where  $\gamma \neq 0$ . In the following, we set  $\hbar = 1$ . A spin chain whose dynamics is described by H(t), is said to

be described by the "XY model". For a finite number, N, of spins, we assume a periodic boundary condition,  $\vec{S}_{N+1} = \vec{S}_1$ . Ultimately, we will be interested in the thermodynamic limit  $N \to \infty$ . Such systems can be realized in atomic gas in an optical lattice (see e.g. [14]).

At a given time t, the (thermal) equilibrium state is given by  $\rho_{\beta}^{eq}(t) = \exp(-\beta H(t))/Z$ , where Z = $\operatorname{tr}(\exp(-\beta H(t)))$ . Here  $\beta = 1/kT$ , where k is the Boltzmann constant, and T denotes the (absolute) temperature. To consider questions about ergodicity, we will be interested in the behavior of the evolved state. The evolution is governed by the Hamiltonian H(t), from a given initial state. Since we will compare the properties of the evolved state, with those of the equilibrium state, it is natural to suppose that the initial state (at t = 0), is the equilibrium state at t = 0. We denote the evolved state by  $\rho_{\alpha}(t)$ , where the suffix corresponds to the temperature of the initial equilibrium state  $\rho_{\alpha}^{eq}(0) = \rho_{\alpha}(0)$ . For later times, properties of the evolved state  $\rho_{\alpha}(t)$  will be compared to those of the equilibrium state  $\rho_{\beta}^{eq}(t)$ , so that they have the same energies:

$$\operatorname{tr}(H(t)\rho_{\alpha}(t)) = \operatorname{tr}(H(t)\rho_{\beta}^{eq}(t)).$$
(2)

For simplicity, we will consider the case of sudden switch of the field h(t), as h(t) = a, for  $t \le 0$ , and = b, for t > 0.

Both  $\rho_{\alpha}(t)$  and  $\rho_{\beta}^{eq}(t)$  are states of an infinite number of spin-1/2 particles. For our purposes, it will be sufficient to consider single-site and two-site density matrices. Let us first consider the single-site density matrix for the state  $\rho_{\beta}^{eq}(t)$ . By symmetry, the single-site density matrices of the chain are all the same. We will denote it by  $\rho_1^{eq}(t)$  (hiding the suffix  $\beta$ ). Now  $\rho_1^{eq*}(t) = \rho_1^{eq}(t)$ , when the complex conjugation is taken in the computational basis, which (for each site) is the eigenbasis of Pauli matrix  $\sigma_z$ . Therefore  $\operatorname{tr}(S^y \rho_1^{eq}(t)) = 0$ . Moreover the Hamiltonian H(t) has the global phase flip symmetry  $([H, \Pi_i S_i^z] = 0)$ , from which it follows that  $\operatorname{tr}(S^x \rho_1^{eq}(t)) =$ 0. Consequently, the single-site density matrix of the equilibrium state is of the form  $\rho_1^{eq}(t) = \frac{1}{2}I + 2M_z^{eq}(t)S^z$ , where I is the  $2 \times 2$  identity matrix. The evolved state does not necessarily have the property of being equal to its complex conjugation, and consideration of the global phase flip symmetry is complicated by the fact that the Hamiltonian is explicitly dependent on time. However, using the Wick's theorem, as in [11, 12, 15], the singlesite density of the evolved state turns out to be of the form  $\rho_1(t) = \frac{1}{2}I + 2M_z(t)S^z$ .

For the case of the two-site density matrix  $\rho_{12}^{eq}(t)$  of the equilibrium state  $\rho_{\beta}^{eq}(t)$ , we can again use the global phase flip symmetry and the fact that it is equal to its complex conjugate, so that it is of the form

$$\rho_{12}^{eq}(t) = \frac{1}{4}I \otimes I + M_z^{eq}(t)(S^z \otimes I + I \otimes S^z) + \sum_{j=x,y,z} T_{jj}^{eq}(t)S^j \otimes S^j, \qquad (3)$$

where the correlation functions,  $T_{jj}^{eq}(t)$ , are defined as  $T_{jj}^{eq}(t) = 4 \operatorname{tr}(S^j \otimes S^j \rho_{12}^{eq}(t)), \quad j = x, y, z.$  In the case

of the two-site density matrix of the evolved state, the yz and zx correlations are absent (via use of the Wick's theorem). However the xy correlations does not vanish, just as the xx, yy, and zz correlations. Thus the two-site density matrix of the evolved state  $\rho_{\alpha}(t)$  is of the form

$$\rho_{12}(t) = \frac{1}{4}I \otimes I + M_z(t)(S^z \otimes I + I \otimes S^z)$$
$$+ T_{xy}(t)(S^x \otimes S^y + S^y \otimes S^x) + \sum_{j=x,y,z} T_{jj}(t)S^j \otimes S^j, (4)$$

where the correlation functions,  $T_{jk}(t)$ , are defined as  $T_{jk}(t) = 4 \operatorname{tr}(S^j \otimes S^k \rho_{12}(t)), \quad j, k = x, y, z.$ 

Let us consider the (single-site) magnetization and the (two-site) entanglement in an infinite chain for the two states of interest. The magnetization of the equilibrium state is  $M_z^{eq}(t) = \operatorname{tr}(S^z \rho_1^{eq}(t))$ , while that for the evolved state is  $M_z(t) = \operatorname{tr}(S^z \rho_1(t))$ . For studying entanglement [16, 17], we will use logarithmic negativity (LN) [18] as our measure of entanglement. LN of a bipartite state  $\rho_{AB}$  is defined as  $E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_A}\|_1$ , where  $\|.\|_1$  is the trace norm, and  $\rho_{AB}^{T_A}$  denotes the partial transpose of  $\rho_{AB}$  with respect to the A-part [19]. Note that the two-site density matrices in our case acts on  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . Consequently, a positive value of the LN implies that the state is entangled and distillable [19, 20], while  $E_N = 0$  implies that the state is separable [19].

The (single-site) magnetizations of the equilibrium state and the evolved state are respectively given by [11]

$$M_z^{eq}(t) = \frac{1}{2\pi} \int_0^\pi d\phi \frac{\tanh\left(\frac{1}{2}\beta\Lambda(h(t))\right)}{\Lambda(h(t))} (h(t) - \cos\phi), \quad (5)$$

and

$$M_{z}(t) = \frac{1}{2\pi} \int_{0}^{\pi} d\phi \frac{\tanh(\frac{1}{2}\beta\Lambda(a))}{\Lambda(a)\Lambda^{2}(b)} \\ \times \left[\cos(2\Lambda(b)t)\gamma^{2}(a-b)\sin^{2}\phi\right] \\ - (\cos\phi - b)\left[(\cos\phi - a)(\cos\phi - b) + \gamma^{2}\sin^{2}\phi\right],$$
(6)

where  $\Lambda(a)$  and  $\Lambda(b)$  can be obtained from  $\Lambda(h(t)) = [\gamma^2 \sin^2 \phi + (h(t) - \cos \phi)^2]^{\frac{1}{2}}$ .

For definiteness, let us consider the case where  $\gamma = 0.5$ ,  $\alpha = 200$ , a = 0.5, b = 0. For these values of the parameters, Eq. (2) gives  $\beta \approx 3.9$ . The equilibrium magnetization  $M_z^{eq}(t)$  has an initial value of  $\approx 0.148328$  and then jumps down to zero for all later times. The magnetization  $M_z(t)$  of the evolved state, on the other hand, is an oscillating function (see Fig. 1), which of course starts at the same initial value [21] as  $M_z^{eq}(t)$ , but remains positive for long times. This suggests that the magnetization of the spin system in the XY model is nonergodic [11] (cf. [22]).

The nearest neighbor correlations for the equilibrium state are given by [15, 23]  $T_{xx}^{eq}(t) = -G^{eq}(-1, t), T_{yy}^{eq}(t) = -G^{eq}(1, t), T_{zz}^{eq}(t) = 4[M_z(t)]^2 - G^{eq}(1, t)G^{eq}(-1, t),$ 

where  $G^{eq}(R,t)$ , for  $R = \pm 1$ , are given by

$$G^{eq}(R,t) = \frac{\gamma}{\pi} \int_0^{\pi} d\phi \cos(\phi R) \sin\phi \frac{\tanh\left(\frac{1}{2}\beta\Lambda(h(t))\right)}{\Lambda(h(t))} -\frac{1}{\pi} \int_0^{\pi} d\phi \cos(\phi R) (\cos\phi - h(t)) \frac{\tanh\left(\frac{1}{2}\beta\Lambda(h(t))\right)}{\Lambda(h(t))}.$$
 (7)

We can now calculate the equilibrium LN  $E_N^{eq}(t)$  of the two-site density matrix  $\rho_{12}^{eq}(t)$  of the equilibrium state  $\rho^{eq}(t)$  via the prescription in Eq. (3). For the same values of the parameters as above, so that  $\beta \approx 3.9$  via Eq. (2), the equilibrium LN of the two-site equilibrium state has an initial value of  $\approx 0.132635$ . Then  $E_N^{eq}(t)$  jumps upto  $\approx 0.157188$  for all t > 0. Note that the equilibrium magnetization  $M_z^{eq}(t)$  jumps down in the same situation. So with more entanglement, we have more local disorder (cf. [24]).

Our interest is to compare this equilibrium two-site entanglement with the two-site entanglement of the evolved state. The nearest neighbor correlations of the evolved state are given by [15]  $T_{xy} = S(1,t)/i$ ,  $T_{xx}(t) =$ -G(-1,t),  $T_{yy}(t) = -G(1,t)$ ,  $T_{zz}(t) = 4[M_z(t)]^2 -$ G(1,t)G(-1,t) + S(1,t)S(-1,t), where G(R,t) and S(R,t), for  $R = \pm 1$ , are given by

$$G(R,t) = \frac{\gamma}{\pi} \int_0^{\pi} d\phi \sin(\phi R) \sin\phi \frac{\tanh\left(\frac{1}{2}\beta\Lambda(a)\right)}{\Lambda(a)\Lambda^2(b)}$$

$$\times \left[\gamma^2 \sin^2\phi + (\cos\phi - a)(\cos\phi - b) + (a - b)(\cos\phi - b)\cos(2\Lambda(b)t)\right]$$

$$- \frac{1}{\pi} \int_0^{\pi} d\phi \cos(\phi R) \frac{\tanh\left(\frac{1}{2}\beta\Lambda(a)\right)}{\Lambda(a)\Lambda^2(b)}$$

$$\times \left[\{\gamma^2 \sin^2\phi + (\cos\phi - a)(\cos\phi - b)\}(\cos\phi - b) - (a - b)\gamma^2 \sin^2\phi\cos(2\Lambda(b)t)\right], \quad (8)$$

$$S(R,t) = \frac{\gamma(a - b)i}{\pi} \int_0^{\pi} d\phi \sin(\phi R) \sin\phi \frac{\sin\left(2t\Lambda(b)\right)}{\Lambda(a)\Lambda(b)}.$$

$$(9)$$

The LN  $E_N(t)$  of the two-site density matrix  $\rho_{12}(t)$ of the evolved state  $\rho(t)$  can now be calculated by using Eq. (4). The behavior of LN of the evolved state for the same parameters as above, is shown in Fig. 1. With time, the entanglement converges to a fixed value. As  $\gamma$ approaches 1, the time taken for convergence is longer, until for  $\gamma = 1$ , the revivals occur for all times. Henceforth, we consider  $\gamma \neq 1$ .

As we see, the two-site entanglement of the state obtained from the time evolution (from an initial equilibrium state) is not approaching to the two-site entanglement of the equilibrium state for large times. This suggests that, just like the (single-site) magnetization, the two-site entanglement of an infinite one-dimensional spin system in the XY model is *nonergodic*.

We will now compare these two nonergodic quantities of the XY chain. We will show that their temporal behaviors are in a sense complementary. First note that the



FIG. 1: Comparison of temporal behavior of the nearest neighbor entanglement to the magnetization, with time, of the evolved state of the infinite spin chain in XY model: We plot  $E_N(t)$  and  $M_z(t)$  for  $\gamma = 0.5$ ,  $\alpha = 200$ , a = 0.5, b = 0.

magnetization  $M_z(t)$  of the evolved state is a damped oscillatory function. For our choice of parameters, the damping decreases the amplitude of the oscillation, but the mean value of the oscillation is more or less fixed (see Fig. 1). Therefore, long time average of  $M_z(t)$  can essentially be considered to be a constant at  $\approx 0.02$ , after starting with an initial *higher* value of  $\approx 0.148328$ . Similar feature was seen before for  $M_z^{eq}(t)$ .

The opposite is true when we compare the two-site entanglements of the equilibrium state and the evolved states. The two-site LN  $E_N(t)$  of the evolved state converges to  $\approx 0.18$ , after starting from an initial *lower* value of  $\approx 0.132635$  (see Fig. 1). Again a similar behavior was seen in the equilibrium state. It is in this sense that we say that the two-site entanglement and (single-site) magnetization has a complementary behavior in the infinite spin chain in the XY model.

The magnetization and nearest neighbor entanglement in the evolved state are plotted in Fig. 1. We note here that the next-nearest neighbor correlation functions are also known [15]. Using them, we have calculated the LN for the next-nearest neighbor of the evolved state. It turns out that it is smaller than the nearest neighbor LN. For example, for the parameters as above, the nextnearest neighbor LN is vanishing already at t = 0.8 (and never going over  $\approx 0.0071$ ), while the nearest neighbor LN is about 0.18, on average, for large times.

Let us now consider the the behavior of the magnetization and nearest neighbor entanglement of the evolved state, for different values of the temperature, but for a fixed time t = 1. (See Fig. 2.) As expected, for high temperatures, the (nearest neighbor) entanglement is vanishing. Note, that both magnetization and nearest neighbor entanglement ultimately saturate, with decreasing T.

Summarizing, we have presented evidence that in the thermodynamic limit of an infinite one-dimensional chain of spin-1/2 particles described by the XY model Hamiltonian, the two-site entanglement is nonergodic. The two-site entanglement of the evolved state does not ap-



FIG. 2: The behavior in the evolved state, with temperature T, of nearest neighbor entanglement and magnetization of the infinite spin chain in XY model are compared for a fixed time t = 1 and for  $\gamma = 0.5$ , a = 10, b = 0. We actually plot the curves with respect to  $\beta$ , where  $\beta = 1/kT$ .

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proach its equilibrium value. That is, entanglement in

such systems does not, by itself (i.e. without contact with external reservoirs), relax to its equilibrium value, after a change of the external magnetic field. This indicates that entanglement in such systems cannot be described by equilibrium statistical mechanics. We also show that the entanglement has a complementary temporal behavior with respect to magnetization. We believe that such studies of the dynamics of entanglement in spin systems will help us to implement quantum information processing tasks in such systems (cf. [8]).

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