DISORDERED COMPLEX SYSTEMS USING COLD GASES AND TRAPPED IONS*

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*This work is partially supported by Deutsches Forchunggemainshaft SFB407, SPP1078 and SPP1116, the Spanish MCYT BFM-2002-02588 and MAT2002-00699, the Alexander von Humboldt Foundation, the EC Program QUPRODIS, the ESF Program QUDEDIS, and EU IP SCALA.
†Also at Institució Catalana de Recerca i Estudis Avançats

We report our research on disordered complex systems using cold gases and trapped ions, and address the possibility of using complex systems for quantum information processing. Two simple paradigmatic models of disordered complex systems are revisited here. The first one corresponds to a short range disordered Ising Hamiltonian (spin glasses), which can be implemented with a Bose-Fermi (Bose-Bose) mixture in a disordered optical lattice. The second model we address here is a long range disordered Hamiltonian, characteristic of neural networks (Hopfield model), which can be implemented in a chain of trapped ions with appropriately designed interactions.

1. Introduction

Complex many body systems are often characterized by structurally simple interactions, but complexity arises because the different terms or constraints appearing in the Hamiltonian compete one with another. If the system presents disorder, the Hamiltonian is no longer translational invariant and depends locally on random parameters. When the system is not able to accommodate to all the constraints present in the Hamiltonian, it exhibits frustration. This leads to the appearance of exotic phenomena, e.g. fractal, hierarchic, or ultrametric structures, distinct quantum phase transitions, etc[1]. Over the last 40 years, disordered and frustrated systems have played a central role in condensed matter physics and have posed some of the most challenging open questions of many body systems. Quenched disorder (i.e., frozen disorder) determines the physics of various phenomena, from transport and conductivity through localization, to percolation, spin glasses, neural networks, high Tc-superconductivity, etc. The description of such systems is, however, extremely difficult, because it normally requires the averaging over each particular realization of the disorder. Systems which are not disordered but frustrated, lead very often to similar difficulties because, at low temperatures, they are often characterized by an enormously large number of low energy excitations.

Recently, it has been shown that one can introduce local disorder and/or frustration in ultracold quantum gases in a **controlled** way, using various experimentally feasible methods (for details see e.g.[2], and references therein), ranging from using several incomensurable optical lattices to trap the atoms, or superimposing a speckle pattern in a regular optical lattice, or taking advantage of Feshbach resonances in fluctuating or inhomogeneous magnetic fields in order to induce a novel type of disorder that corresponds to random, or at least inhomogeneous nonlinear interaction couplings. Thus, different disorder and/or frustrated systems can be conveniently prepared to study e.g. Bose glass[3], Anderson localization[4,5], fermionic spin

glasses[6] or quantum percolation[6], kagomé lattices[7] among others. We have also recently investigated the possibilities offered by trapped ions with engineered interactions[8,9] to model neural networks[10]. A review of the different phases displayed by ultracold atomic gases in disordered optical lattices can be found in[2].

In this contribution, we present our approach to the study of both, short and long range, disordered systems. In the former case, we focus on a spin glasses model[11], i.e. short-range disordered magnetic systems which can be simulated by Bose-Fermi mixtures in random potentials. In the case of long range interactions, we study a neural network model simulated by a chain of trapped ions with appropriately designed interactions. In both cases, we examine the possibilities offered by those systems for quantum information tasks. In spite of the fact that using disordered systems to perform quantum information processing seems to be an impossible task, at least two possible advantages arise immediately. First, these systems have typically a large number of different metastable (free) energy minima, as it happens in spin glasses (SG)[12]. Such states might be used to store information distributed over the whole system, similarly as in neural network (NN) models[13]. The information is thus naturally stored in a redundant way, like in error correcting schemes[14]. Second, in disordered systems with long range interactions, the stored information is robust: metastable states have quite large basins of attraction in the thermodynamical sense. We have shown[15] that in both models, short and long range, it is possible to generate entanglement that survives over long times. Moreover, in the neural network model, it is possible to store patterns that can be used as distributed qubits over the whole system. Since the patterns are robust and act as attractor points in the energy diagram, these qubits can be partially destroyed by noise or any other non-desired effect. The free evolution of the systems, however, retrieves the patterns back and thus makes the qubits very robust.

2. Short range disordered systems: Spin glasses

Spin glasses are random disordered systems with competing ferromagnetic and antiferromagnetic interactions, which in dimensions $d > 1$ present frustration, since it is not possible to simultaneously accommodate all pairs of spins connected by a ferromagnetic (antiferromagnetic) bond. In the early 70's, Edwards and Anderson realized that the essential physics of a spin glass does not lay in the details of their microscopic interactions, but

rather in the competition between quenched ferro- and antiferro-magnetic interactions. To study the nature of spin glasses, they proposed a very simple short range disordered Ising Hamiltonian, nowadays known as the Edwards-Anderson (E-A) model of spin glasses[11]:

$$H_{E-A} = -\sum_{\langle ij \rangle} J_{ij}\sigma_i^z\sigma_j^z - h^z \sum_i \sigma_i^z. \tag{1}$$

Here σ_k^z denotes an Ising spin (± 1) at the k-th site, the J_{ij}'s describe nearest neighbor couplings for an arbitrary lattice and h^z is a magnetic field along the z-direction. In the E-A model, the J_{ij} couplings are given by independent random variables, which have Gaussian probability distributions with mean $\bar{J} = 0$ and variance Δ^2. Since interactions are short range, a mean field theory cannot be used[16] and, traditionally, one has to rely on replica tricks[12], to do the appropriate average over the quenched disorder, in order to obtain the free energy F of the system, and derive the thermodynamical properties of the system from F. A formally identical Hamiltonian as the one of Eq. (1) can be derived from the Bose-Fermi (Bose-Bose) Hubbard Hamiltonian[1] (BFH, BH) describing a Bose-Fermi (Bose-Bose) mixture in an optical lattice with random disorder:

$$H_{\rm BFH} = -\sum_{\langle ij \rangle}(T_B b_i^\dagger b_j + T_F f_i^\dagger f_j + \text{h.c.}) \tag{2}$$
$$+\sum_i \left[\frac{1}{2}V n_i(n_i - 1) + U n_i m_i - \mu_i^B n_i - \mu_i^F m_i\right].$$

Here $b_i^\dagger$, b_j, $f_i^\dagger$, f_j are the bosonic and fermionic creation-annihilation operators, $n_i = b_i^\dagger b_i$, $m_i = f_i^\dagger f_i$ are the number operators, and μ_i^B and μ_i^F are the bosonic and fermionic local chemical potentials, respectively. The BFH model describes: i) nearest neighbor (n.n.) boson (fermion) hopping, with an associated negative energy $-T_B$ ($-T_F$); ii) on-site repulsive boson-boson interactions with an energy V; iii) on-site boson-fermion interactions with an energy U, which is positive (negative) for repulsive (attractive) interactions, and finally, iv) interactions with the external inhomogeneous potential, with energies μ_i^B and μ_i^F. In the limit of equal tunneling for bosons and fermions ($T_B = T_F = T$) and a strong coupling regime ($T \ll U, V$), using a quasi-degenerate perturbation theory up to the second order, an effective Hamiltonian can be derived, which describes the dynamics of the Bose-Fermi mixture in terms of composite fermions[17,6] made of one fermion plus -s bosons or one fermion plus s bosonic holes. The annihilation operators

of the composite fermions are given by[17]:

$$F_i = \sqrt{\frac{(\tilde{n}-s)!}{\tilde{n}!}} \left(b_i^\dagger\right)^s f_i \quad \text{for } s \text{ bosonic holes} \tag{3}$$

$$F_i = \sqrt{\frac{\tilde{n}!}{(\tilde{n}-s)!}} \left(b_i\right)^{-s} f_i \quad \text{for } -s \text{ bosons.} \tag{4}$$

Using the above notation, the effective Hamiltonian reads:

$$H_{\text{eff}} = -\sum_{\langle i,j\rangle} t_{ij}(F_i^\dagger F_j + h.c.) + \sum_{\langle i,j\rangle} K_{ij} M_i M_j - \sum_i \bar{\mu_i} M_i \tag{5}$$

where $M_i = F_i^\dagger F_i$. In the limit of negligible tunneling between composites ($t_{ij} \approx 0$), the Hamiltonian reduces to the E-A Hamiltonian (Eq.(1)) with an effective inhomegenous magentic field given by $\bar{\mu_i}$.

Let us now address the generation and evolution of nearest neighbors (n.n.) entanglement in this model. To deal with quantum information processing in the E-A model, we consider now a quantum Ising model. Therefore, in the following σ_k^z denotes the Pauli z-operator. In the short range Ising model without disorder, it is possible to create cluster and graph states (i.e. entanglement) starting from an appropriate initial product state[18]. Here we show that, while the disorder averaged density matrix of two neighboring spins remains always separable, the disorder averaged entanglement (quantified by the logarithmic negativity[19]) converges with time to a finite value[15]. The generation of entanglement as well as its evolution for arbitrary times in an Ising model without disorder but with long-range interactions, has also been addressed in Ref.[18]. Starting from a pure product state of the form $|\Psi\rangle = \prod_i |+\rangle_i$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, we evaluate first the density matrix of the system after a finite time: $\rho(t, \{J_{ij}\}) = \exp\{-iH_{E-A}t\}|\Psi\rangle\langle\Psi|\exp\{+iH_{E-A}t\}$. To calculate n.n. entanglement with respect the pair (i,j) we calculate first the reduced density matrix tracing over all spins except (i,j), and then we use logarithmic negativity as a measure of entanglement of the remaining two-qubit system. Finally we average over the disorder.

Our results show that (i) after a finite time, entanglement converges to a finite amount (see Fig. 1) independently of the mean $\bar{J}$ of the Gaussian distribution, although the short-time dependece does depend on the mean, and (ii) n.n. entanglement decays exponentially with the number of neighbors of a given site, which in turn depends on the configuration of the lattice we consider. For example, for a 1D chain, any pair of neighboring

lattice sites has 2 neighbors, a 2D honey comb lattice has 4, a 2D square lattice has 6, and a 3D square lattice has 10.

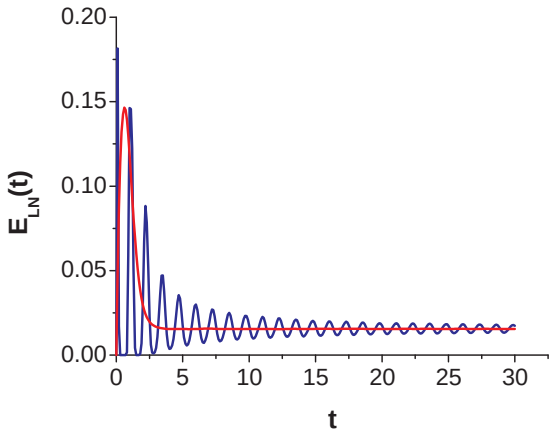

Figure 1. Temporal behavior of n.n. averaged entanglement in a 2D square lattice spin glass model. For a model with $\bar{J} = 0$, the logarithmic negativity $E_{LN}(t)$ quickly converges to a constant value. For the case $\bar{J} = 5$, $E_{LN}(t)$ exhibits damped oscillations, but again converging to the same value ≈ 0.0154 as reached by the frustrated case. It is interesting to note that the dynamical behavior of the entanglement depends on $\bar{J}$, although at large times, it converges to a fixed value, independent of $\bar{J}$.

3. Long range disordered systems: Neural Networks

Neural networks are paradigmatic models of parallel distributed information processing[13,12], and have been intensively studied by physicists since the famous paper by Hopfield[20]. Following the models of Hopfield[20] and Little[21], a neuron can be viewed as an Ising spin with two possible states: an "up" position ($S = +1$) and a "down" position ($S = -1$), depending on whether the neuron has or has not fired an electromagnetic signal, in a given interval of time. The state of the network of N neurons at a certain time is defined by the instantaneous configuration of all the spin variables $\{S_i\}$ at this time. The dynamic evolution of these states is determined by the interactions, J_{ij}, among neurons. The interaction are symmetric, so that for any pair of neurons, $J_{ij} = J_{ji}$. Moreover, full connectivity is as-

sumed, that is, every neuron can receive an input from any other one, and can send an output to it. The Hamiltonian of the system can be written as

$$H_{NN} = -\frac{1}{2}\sum_{i,j} J_{ij} S_i S_j. \tag{6}$$

The interactions, J_{ij}, among neurons are calculated *a posteriori* by first fixing the configurations or patterns of spins to be stored in the network. In this way, the learning process is Hebbian, meaning that learning adjusts the network's weights such that its output reflects its familiarity with an input. The more probable an input, the larger the output will become (on average). Physically that means that these patterns will be learned if the system is able to accommodate them as attractors, so that they are stable in front of any single-spin flips and present a significant basin of attraction. Therefore, the interactions, J_{ij}, are defined in such a way that the local minima of the Hamiltonian are correlated with these configurations:

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu. \tag{7}$$

Here $i \neq j$. The p sets of $\{\xi_i^\mu\}$ (where each ξ_i^μ can be ± 1) are the patterns that we wish to be fixed by the learning process. Despite the fact that the interactions have been constructed to guarantee that certain specified patterns are fixed points of the dynamics, the non-linearity of the dynamical process induces additional attractors, the so-called spurious states.

Recently it has been demonstrated that a linear chain of harmonically trapped ions can be appropiately designed, by applying either an external magnetic field[8] or external lasers[9], to describe a spin sytem with long range interactions:

$$H = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j + \sum_i B'\sigma_i, \tag{8}$$

where

$$J_{ij} = \frac{F^2}{m}\sum_n \frac{M_{i,n}M^{j,n}}{\omega_n^2}. \tag{9}$$

The M's are the unitary matrices that diagonalize the vibrational Hamiltonian. With these assumptions, Eq. (6) and Eq. (8) have the same form, and the possibility of implementing a classical neural network using a linear chain of ions arises. Also, comparison between Eq. (7) and Eq. (9), indicates that the network configuration in the ions trap case will be given by the vibrational modes of the chain ($M_{i,n}$) and their eigenvalues (ω_n).

The vibrational modes are determined through the harmonic displacements of the ions around their equilibrium positions, when the trapping potential is balanced by the Coulomb interactions between the ions. Thus, in our model, the sign of the displacement of each ion with respect to its equilibrium position, is associated with an Ising spin. To reproduce the NN model (Eq. (7)), and thus to be able to store different patterns, the vibrational modes should be, ideally, almost degenerate in energy and possess large basins of attraction. In other words, the patterns should correspond to sufficiently different configurations of the spins, so that each configuration is stable in front of random spin flips of several of its components. To achieve the above situation, either the vibrational spectrum is modified by changing the shape of the trapping potential, or an external longitudinal magnetic field (along the axial frequency) is used. We obtain the best possible results concerning the number of stored patterns, if we use for the ions' trap, a confining potential of the form $V(x) = A|x|^{0.5}$, without any additional external magnetic field. In this case, we can store up to 4 patterns (2 patterns plus their reverse ones) in a 20 ions chain (for details see[10]). Notice that due to the fact that the interactions are now given by the vibrational modes, our system does not correspond exactly to the Hopfield model of neural networks, and therefore, the system is not able to learn the same number of patterns as the Hopfield one.

Let us now move to the entanglement properties of a quantun neural network. As before, we replace the classical Ising spins, $\{S_i\} = \pm 1$, by Pauli operators σ_i^z. We apply here the same procedure as in Section 2 and consider as initial state a pure product state of the form $|\Psi\rangle = \prod_i |+\rangle_i$. The entanglement of any two spins is calculated by evaluating first the time evolved density matrix $\rho(t, \{J_{ij}\}) = \exp\{-iH_{NN}t\}|\Psi\rangle\langle\Psi|\exp\{+iH_{NN}t\}$, and then tracing over all subsystems except i, j, and finally performing the proper average over the disorder (for details see[15]). We have shown that (i) there is an efficient way to calculate bipartite as well as multipartite states in this model, and (ii) entanglement displays collapses and revivals as a function of time and number of ions.

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