

# Channel Capacities versus Entanglement Measures in Multiparty Quantum States

Aditi Sen(De) and Ujjwal Sen

*Harish-Chandra Research Institute, Chhatnag Road, Jhansi, Allahabad 211 019, India*

For quantum states of *two* subsystems, entanglement measures are related to capacities of communication tasks – highly entangled states give higher capacity of transmitting classical as well as quantum information. However, we show that this is no more the case in general: quantum capacities of multi-access channels, motivated by communication in quantum networks, do not have any relation with genuine multiparty entanglement measures. Along with revealing the structural richness of multi-access channel capacities, this gives us a tool to classify multiparty quantum states from the perspective of its usefulness in quantum networks, which cannot be visualized by known multiparty entanglement measures.

## I. INTRODUCTION AND MAIN RESULTS

Understanding quantum entanglement [1] has been one of the key features in the development of the science of quantum information [2]. Applications of quantum information had started off in the fields of communication, cryptography, computation, and thermodynamics [2], and has since diffused into diverse areas such as condensed matter physics, ultra-cold gases, and statistical mechanics [3]. Measuring and detecting entanglement of the quantum states appearing in different physical situations has been the cornerstones of the development in these directions. It has therefore been very important to propose entanglement measures of general quantum states of systems consisting of more than one subsystem, and there is a thriving industry of such proposals (see [1, 4] and references therein). However, the main progress in the theory of entanglement measures, and its detection, has been in the case when the physical system consists of only two subsystems. This has been a major handicap in using entanglement as an instrument for handling many-body physics systems like ultra-cold atomic states, where the majority, if not all, of the quantum states involved are of multiparty systems, i.e. a physical system consisting of more than two subsystems. Understanding multiparty quantum entanglement is therefore a distinct necessity to a large portion of physics of our times.

One of the main reasons for the current interest in quantum information is its potential for revolutionizing future communication systems. It is therefore hard to overestimate the importance of capacities of quantum communication channels [5]. Again the main progress in research in this area has been for quantum channels between a single sender and a single receiver, while multi-access channels clearly have more commercial viability. Moreover, a competent functioning of future quantum computers [2] will require efficient communication of quantum information between its different parts.

The archetypical quantum channels are bipartite quantum states used as dense coding [6] and teleportation [7] channels. They are channels respectively for transmitting classical and quantum information, and form the basis of most quantum channels. If a *pure* bipartite quantum

state  $|\Psi\rangle_{AB} (\in \mathbb{C}^d \otimes \mathbb{C}^d)$  is shared between Alice ( $A$ ) and Bob ( $B$ ), it can be used as a quantum channel to perform dense coding, by which classical information can be sent, for example, by Alice to Bob, with the capacity (measured in *bits*) being  $\mathcal{C}_{\text{classical}}(|\Psi\rangle) = \log_2 d + S(\varrho_L)$  [6, 8], where  $\varrho_L$  is the local density matrix of state  $|\Psi\rangle_{AB}$ , and  $S(\cdot)$  is the von Neumann entropy of the argument. Similarly, the same quantum state  $|\Psi\rangle_{AB}$  can be used as a quantum channel to convey quantum information from  $A$  to  $B$ , with the capacity (measured in *qubits*) being  $\mathcal{C}_{\text{quantum}}(|\Psi\rangle) = S(\varrho_L)$  [5, 7, 9]. Entanglement of a bipartite pure quantum state  $|\Psi\rangle_{AB}$  is, for most purposes, the von Neumann entropy of a local subsystem, i.e.  $E(|\Psi\rangle) = S(\varrho_L)$  [9].

Clearly, higher entanglement for a pure quantum state implies higher capacities for both the classical and quantum instances, in the case of a single sender and a single receiver. Here we find that a generalization of this behavior is not mirrored in the multiparty case. More precisely, we find quantum capacities of four-party quantum states that are motivated by considering distillation protocols in multiparty quantum networks, and show that their values are not correlated with those of a measure of genuine four-party entanglement. The measure of genuine four-party entanglement that we use here is a generalization of the “geometric measure of entanglement” (GM) [10], and we call it the “generalized geometric measure” (GGM). As an important by-product, we obtain a *computable* measure of genuine multiparty entanglement, which can potentially have the same usefulness in the multiparty case, as the logarithmic negativity [11] has in the bipartite situation. We also provide bounds on the capacities defined that help us in their understanding as well as their evaluation in a variety of paradigmatic classes of multipartite quantum states.

## II. THE MULTI-ACCESS CAPACITIES

Let us begin by defining the multi-access capacities that we will deal with, and by considering their quantum computational significance. We will define two such quantities, both of which are given from the perspective of quantum networks. Although the definitions, and

the subsequent propositions, will be given only for four-party systems, their generalizations to more parties (or for three parties) are straightforward. The first quantity is *maximal assisted remote singlet production* ( $\mathcal{C}_a$ ), and defined for a single copy of a four-party pure quantum state,  $|\psi\rangle$ , shared between Alice ( $A$ ), Bob ( $B$ ), Claire ( $C$ ), and Danny ( $D$ ), as the maximal probability with which a single copy of the singlet state,  $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$  ( $|0\rangle$  and  $|1\rangle$  are mutually orthonormal), can be prepared at  $CD$ , by using an additional resource of a singlet state shared between Alice and Bob, and by using local quantum operations and classical communication (LOCC) between Alice, Bob, Claire, and Danny.  $\mathcal{C}_a$  therefore measures the amount of entanglement that can be transferred from Alice and Bob to Claire and Danny, when Alice and Bob are assisted by an additional singlet. It is therefore natural to multiply this quantity by the entanglement ( $E$ ) value of 1 ebit of the singlet state, and express the capacity in ebits. If the state is not symmetric with respect to interchange of the parties, we define  $\mathcal{C}_a$  as the maximum of the transfer probabilities corresponding to all possible permutations of the parties.

The other quantity is *maximal unassisted remote singlet production* ( $\mathcal{C}_{ua}$ ), and has exactly the same definition as  $\mathcal{C}_a$ , but without the additional singlet assistance. These quantities, or their generalized versions for large quantum networks, are important elements in quantum computational setups, e.g. in the Knill-Laflamme-Milburn model of quantum computation [12], or in the cluster state model of quantum computation [13] (see also [14]).

The multi-access capacities  $\mathcal{C}_{ua}$  and  $\mathcal{C}_a$  can be shown to be monotonically decreasing under LOCC between the four observers. More importantly, we have the following results.

**Proposition C1.**  $\mathcal{C}_{ua} \leq \mathcal{C}_a \leq p_{\max\min}^s$ , where  $p_{\max\min}^s$  is defined as follows. Consider the set of four quantities  $\{p_{\max}^{i:\text{rest}} | i = A, B, C, D\}$ , where e.g.  $p_{\max}^{C:ABD}$  is the maximum probability of obtaining a singlet between Claire and the other observers (who are at the same location), and where  $A, B, C$ , and  $D$  share the quantum state  $|\psi\rangle$ . Choose all six pairs from the set  $\{p_{\max}^{i:\text{rest}}\}$ , find the minimum for each pair, and then the maximum of these six minima is  $p_{\max\min}^s$ .

*Proof.* The definitions of  $\mathcal{C}_{ua}$  and  $\mathcal{C}_a$  imply the first inequality. Now,  $p_{\max}^{C:ABD}(|\psi\rangle_{ABCD}) = p_{\max}^{C:ABD}(|\psi\rangle_{ABCD} \otimes |\psi^-\rangle_{AB})$ , as adding a local ancilla (local with respect to the  $C : ABD$  split) cannot change an LOCC monotone  $p_{\max}$ . Further,  $p_{\max}^{C:ABD}(|\psi\rangle_{ABCD} \otimes |\psi^-\rangle_{AB}) \geq p_{\max}^{AB \rightarrow CD}(|\psi\rangle_{ABCD} \otimes |\psi^-\rangle_{AB})$ , where  $p_{\max}^{AB \rightarrow CD}$  is the probability that  $A$  and  $B$  can create a singlet between  $C$  and  $D$ , when all four parties, at separated locations, share the quantum state in the argument. This is because the probability of a singlet being prepared between  $C$  and  $D$  by LOCC between all four observers, cannot exceed the corresponding probability when  $A, B$ , and  $D$  are together. Similar relations hold when Claire is replaced by

Danny, and so we have  $p_{\max}^{AB \rightarrow CD}(|\psi\rangle_{ABCD} \otimes |\psi^-\rangle_{AB}) \leq \min\{p_{\max}^{j:ABk}(|\psi\rangle_{ABCD}) | j, k = C, D; j \neq k\}$ . Taking the maximum, over all possible permutations of the four parties, in the preceding inequality, we obtain the second inequality in the proposition. ■

**Proposition C2.**  $\mathcal{C}_{ua} \leq p_{\max\min}^d$ , where  $p_{\max\min}^d$  is defined as follows. Consider the set of three quantities  $\{p_{\max}^{AC:BD}, p_{\max}^{AD:BC}, p_{\max}^{AB:CD}\}$ , where e.g.  $p_{\max}^{AC:BD}$  is the maximum probability of obtaining a singlet in the  $AC : BD$  bipartite split, and where  $A, B, C$ , and  $D$  share the quantum state  $|\psi\rangle$ . Choose all three pairs from the set, and find the minimum for each set.  $p_{\max\min}^d$  is the maximum of these minima.

*Proof.* Suppose that Alice and Claire are together, and so are Bob and Danny. The probability of preparing a singlet state in the  $AC : BD$  bipartite split must be greater than or equal to the corresponding quantity in the situation when all four parties are at separate locations, and the singlet is to be prepared between Claire and Danny. That is,  $p_{\max}^{AC:BD}(|\psi\rangle_{ABCD}) \geq p_{\max}^{AB \rightarrow CD}(|\psi\rangle_{ABCD})$ . A similar inequality holds when Alice and Bob change sides, i.e.  $p_{\max}^{AD:BC}(|\psi\rangle_{ABCD}) \geq p_{\max}^{AB \rightarrow CD}(|\psi\rangle_{ABCD})$ , so that we have  $p_{\max}^{AB \rightarrow CD}(|\psi\rangle_{ABCD}) \leq \min\{p_{\max}^{AC:BD}(|\psi\rangle_{ABCD}), p_{\max}^{AD:BC}(|\psi\rangle_{ABCD})\}$ . Taking a maximum, over all possible permutations of the four observers, of the preceding inequality proves the inequality in the proposition. ■

### III. THE GENERALIZED GEOMETRIC MEASURE

These capacities, motivated by quantum networks, will be compared with a measure of genuine four-party entanglement measure, GGM, which we now define. Consider a four-party pure quantum state  $|\psi\rangle$ , and let  $\Lambda_{\max}(|\psi\rangle) = \max|\langle\phi|\psi\rangle|$ , where the maximum is over all four-party pure quantum states  $|\phi\rangle$  that are not genuinely four-party entangled. An  $n$ -party pure quantum state is said to be genuinely  $n$ -party entangled, if it is not a product across any bipartite partition. The GGM of  $|\psi\rangle$  is defined as  $\mathcal{E}(|\psi\rangle) = 1 - \Lambda_{\max}^2(|\psi\rangle)$ . Note that  $\Lambda_{\max}$  quantifies the closeness of the state  $|\psi\rangle$  to all pure quantum states that are *not* genuinely multipartite entangled. Generalization to arbitrary number of parties is straightforward. The definition is motivated by the GM, introduced in [10], in which the maximization in  $\Lambda_{\max}$  is only over pure states that are product over every bipartite partition. [We denote the GM of a quantum state  $|\psi\rangle$  as  $\mathcal{E}_G(|\psi\rangle)$ .] Clearly,  $\mathcal{E}$  is vanishing for all pure states that are not genuine multipartite entangled, and non-vanishing for others. We will now show that this measure is computable (for an arbitrary number of parties), and that it is indeed a monotonically decreasing quantity under LOCC.

**Proposition E1.** The generalized geometric measure can be written in closed (computable) form for all multipartite pure quantum states.

*Proof.* We provide the proof for four-party states,

the other cases being similar. The maximization in  $\Lambda_{\max}(|\psi\rangle_{ABCD}) = \max_{|\phi\rangle_{ABCD}} |\langle\phi|\psi\rangle|$  is over all pure quantum states  $|\phi\rangle_{ABCD}$  that are not genuinely multipartly entangled. The square of  $\Lambda_{\max}(|\psi\rangle_{ABCD})$  can therefore be interpreted as the Born probability of some outcome in a quantum measurement on the state  $|\psi\rangle$ . However, since entangled measurements cannot be worse than the product ones for any set of subsystems, we have that the only measurements that we need to consider for the maximization are the ones in the single-party versus rest, and in the two-parties versus remaining-two splittings. Next we note that, if e.g. the maximization in  $\Lambda_{\max}(|\psi\rangle_{ABCD})$  is performed over all states that are product in the  $A : BCD$  split, the result is the maximal Schmidt coefficient,  $\lambda_{A:BCD}$ , of the state  $|\psi\rangle_{ABCD}$ , when written in the  $A : BCD$  split. Similar expressions hold for the other splittings. Therefore, we have that the GGM of  $|\psi\rangle$  is given by  $\mathcal{E}(|\psi\rangle) = 1 - \max\{\lambda_{i:\text{rest}}^2, \lambda_{ij:\text{rest}}^2 | i, j = A, B, C, D; i \neq j\}$ . ■

**Remark.** The closed form of the GGM may induce one to redefine the GGM in terms of Rényi entropies of the Schmidt coefficients, especially for statistical mechanical applications. The current definition is akin to the case when the Rényi parameter tends to infinity, sometimes referred to as the min-entropy.

**Proposition E2.** The generalized geometric measure is monotonically decreasing under LOCC.

*Proof.* The proof follows from the fact that the  $\lambda$ 's involved in the closed form of the GGM, as derived in the proof of Proposition E1, are all increasing under LOCC [15]. ■

#### IV. APPLICATIONS AND ESTABLISHING THE REMAINING RESULTS

With these results in hand, we now move to apply them to different classes of quantum states. As stated in the introduction, our main motivation is to study the defined quantum capacities from the perspective of quantum computational networks. In line with that, we begin by considering two classes of multipart quantum states that have been found to be useful in several quantum informational and computational tasks.

##### A. Case I: Generalized GHZ

A very important class of states, with several informational and computational applications, is that of generalized Greenberger-Horne-Zeilinger states [16],  $|\text{GHZ}_\alpha\rangle_{ABCD} = \alpha|0\rangle^{\otimes 4} + \beta|1\rangle^{\otimes 4}$  (with  $|\alpha| \geq |\beta|$ ), shared between the four observers. By Proposition C1,  $\mathcal{C}_{ua} \leq \mathcal{C}_a \leq 2|\beta|^2$ . Supposing now that measurements in the  $\{|+\rangle, |-\rangle\}$  basis, where  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ , are carried out at both  $A$  and  $B$ , and the resulting pure state at  $CD$ , corresponding to each set of measurement results at  $A$  and  $B$ , is LOCC-transformed to the singlet state, we ob-

tain that  $\mathcal{C}_{ua} \geq 2|\beta|^2$  [15]. Therefore,  $\mathcal{C}_{ua} = \mathcal{C}_a = 2|\beta|^2$ . For the generalized GHZ state  $|\text{GHZ}_\alpha\rangle_{ABCD}$ , the GGM and GM coincide and are equal to  $|\beta|^2$ . The GM is found by some algebra, while the GGM is found by using Proposition E1. Therefore, for the GHZ state (generalized GHZ state for  $\alpha = \beta = 1/\sqrt{2}$ ), the capacities are both unit ebits, and the GGM and GM are both equal to one-half.

##### B. Case II: Cluster states

From the point of view of quantum computational networks, the cluster states have acquired great significance [13]. The cluster state for four observers is  $|C\rangle_{ABCD} = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$ . It is a non-symmetric state. A trivial upper bound on the capacities is  $\mathcal{C}_{ua} \leq \mathcal{C}_a \leq 1$ . However, Alice and Bob can make measurements in the  $\{|0\rangle, |1\rangle\}$  basis, and corresponding to every outcome, the state at the remaining parties turns out to be local unitarily equivalent to the singlet state. Therefore, we have that the unassisted capacity  $\mathcal{C}_{ua}$  is 1 ebit, so that  $\mathcal{C}_{ua} = \mathcal{C}_a = 1$ . By explicit algebra, the GM for this state is 3/4, while the GGM is 1/2. [Let us mention here that the state  $|X\rangle_{ABCD} = \frac{1}{2\sqrt{2}}(|00\rangle(|00\rangle - |11\rangle) + |11\rangle(|00\rangle + |11\rangle) - |01\rangle(|01\rangle - |10\rangle) + |10\rangle(|01\rangle + |10\rangle))$  of Ref. [17] has exactly the same values for  $\mathcal{C}_{ua}$ ,  $\mathcal{C}_a$ ,  $\mathcal{E}$ , and  $\mathcal{E}_G$ , as the cluster state. However, the states are different, as can be seen by looking at their entanglements in the  $AB : CD$  split.]

##### C. Bipartite versus multipartite

In the case of a single sender and a single receiver, we have seen that the channel capacities are consistently correlated with entanglement measures. Precisely, for two bipartite pure states  $|\Psi\rangle$  and  $|\Phi\rangle$ , if  $E(|\Psi\rangle) = E(|\Phi\rangle) + \epsilon$  for some positive  $\epsilon$ , then  $\mathcal{C}(|\Psi\rangle) = \mathcal{C}(|\Phi\rangle) + \delta$  for some positive  $\delta$ , where  $\mathcal{C}$  is either  $\mathcal{C}_{\text{classical}}$  or  $\mathcal{C}_{\text{quantum}}$ . In the case of multi-access quantum channels, we see that both the assisted and unassisted quantum capacities ( $\mathcal{C}_a$  and  $\mathcal{C}_{ua}$  respectively) are unity for the GHZ state as well as for the cluster state, while the geometric measure of entanglement returns different values for the two states. The GM however is not a measure of *genuine* multipart entanglement. [Indeed, it was defined by its inventors from quite a different perspective.] The GGM, which *is* a measure of genuine multipart entanglement, of these two states are however equal, and so we regain the rather comfortable picture that is true in the case of a single sender and a single receiver. We will soon find that this simple picture to not hold in the case of multi-access channels. In any case, the fact that the picture does hold in certain cases also in the multi-access domain, especially in instances that are important from a quantum networks perspective, is still satisfying. Let us now continue with our case studies.

### D. Case III: Generalized W

The next class of states that we handle is that of generalized W [18] (see also [19]), defined as  $|W_{abcd}\rangle_{ABCD} = a|0001\rangle + b|0010\rangle + c|0100\rangle + d|1000\rangle$  (with  $|a| \geq |b| \geq |c| \geq |d|$ ), and is another class of states with interesting conceptual and practical utilities in quantum information. This is also a non-symmetric state. By Proposition C1,  $\mathcal{C}_{ua} \leq \mathcal{C}_a \leq p_{\max\min}^s = 2|b|^2$ . Measuring in the  $\{|0\rangle, |1\rangle\}$  basis at both  $A$  and  $B$ , we find that  $|00\rangle_{AB}$  clicks with probability  $p_{00}^{W_{abcd}} = |a|^2 + |b|^2$ , and creates the state  $(a|01\rangle + b|10\rangle) / \sqrt{|a|^2 + |b|^2}$  at  $CD$ . The latter can be locally transformed to the singlet state with probability  $2|b|^2 / \sqrt{|a|^2 + |b|^2}$  [15]. The other clicks at  $A$  and  $B$  always produces a product state. Therefore,  $\mathcal{C}_{ua} \geq p_{00}^{W_{abcd}} \times 2|b|^2 / \sqrt{|a|^2 + |b|^2} = 2|b|^2$ , whereby  $\mathcal{C}_{ua} = \mathcal{C}_a = 2|b|^2$ . By using Proposition E1, the GGM of the generalized W state is given by  $\mathcal{E}(|W_{abcd}\rangle) = |d|^2$ . So for the W state (generalized W state for  $a = b = c = d = 1/2$ ), the capacities are both one-half ebits, and  $\mathcal{E}(|W\rangle) = 1/4$ . [Also,  $\mathcal{E}_G(|W\rangle) = 37/64 = 0.578125$ .]

### E. Case IV: $W_2$

Let us now consider the state  $|W_2\rangle_{ABCD} = (|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle + |0101\rangle + |1010\rangle) / \sqrt{6}$ . Of course we have  $\mathcal{C}_{ua} \leq \mathcal{C}_a \leq 1$ . Now, in the assisted case, we can use the singlet assistance to teleport [7, 20] the  $B$ -part of the  $W_2$  state to  $A$  (or vice-versa). Subsequently, we make a measurement in the Bell basis [21] at  $AB$ , which creates a state at  $CD$ , that is local unitarily equivalent to the singlet. Therefore, we have  $\mathcal{C}_a(|W_2\rangle) = 1$ . To find the unassisted capacity, we will turn to Proposition C2, to find that  $\mathcal{C}_{ua} \leq 2/3$ . Suppose now that both Alice and Bob measure in the  $\{|0\rangle, |1\rangle\}$  basis. The  $|00\rangle_{AB}$  and  $|11\rangle_{AB}$  outcomes at  $AB$  produce product states at  $CD$ . However, both the  $|01\rangle_{AB}$  and the  $|10\rangle_{AB}$  outcomes at  $AB$  produce the state  $(|01\rangle + |10\rangle) / \sqrt{2}$  at  $CD$ , each with probability  $1/3$ . Consequently, we have  $\mathcal{C}_{ua} \geq 2/3$ , so that  $\mathcal{C}_{ua}(|W_2\rangle) = 2/3$ . The GGM for the  $W_2$  state is  $1/3$ . [ $\mathcal{E}_G(|W_2\rangle) = 5/8 = 0.625$ .]

### F. Case V: Two singlets

In this case, which we denote by  $|SS\rangle$ , any two observers share a singlet, and the other two share another singlet. This is a non-symmetric case. The unassisted transfer probability for the state  $|SS\rangle_1 = |\psi^-\rangle_{AB} \otimes |\psi^-\rangle_{CD}$  is unity. For the other options, viz.  $|SS\rangle_2 = |\psi^-\rangle_{AC} \otimes |\psi^-\rangle_{BD}$  or  $|SS\rangle_3 = |\psi^-\rangle_{AD} \otimes |\psi^-\rangle_{BC}$ , it is zero. But the unassisted capacity will still be given by  $\mathcal{C}_{ua}(|SS\rangle) = 1$ . Using the method of entanglement swapping [20], we have that the assisted capacity is unity for all the three options:  $\mathcal{C}_a(|SS\rangle) = 1$ . To see this, note that the assisted transfer probability is unity by construc-

tion for the state  $|SS\rangle_1$ , while entanglement swapping can be employed to produce unit probabilities for the states  $|SS\rangle_2$  and  $|SS\rangle_3$ . By definition, the GGM of  $|SS\rangle$  is zero, while the GM of this state can be calculated to be  $3/4$ .

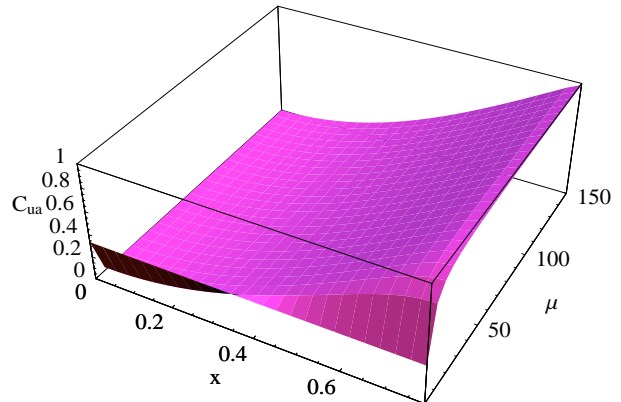


FIG. 1: (Color online.) From resonating valence bond states to ferromagnets. The unassisted capacity (in ebits) is plotted on the vertical axis against a base consisting of the measurement parameter  $x$  (for arbitrary projective measurements in the basis  $\{\cos x|0\rangle + \exp(i\varphi)\sin x|1\rangle, \exp(-i\varphi)\sin x|0\rangle - \cos x|1\rangle\}$  at two parties), and the state parameter  $\mu$ , for the family of four-qubit quantum states  $|\phi_{RF}^\mu\rangle$ . (The base variables are dimensionless.) The capacity for this strategy can be readily read off from the figure for any  $\mu$ . E.g. for the resonating valence bond state, situated at  $\mu = 0$ , the capacity is  $1/3$ , while for the ferromagnetic state at  $\mu \rightarrow \infty$ , the corresponding quantity is unity.

### G. Case VI: RVB to Ferromagnets

The resonating-valence-bond state [22], apart from its significance in many-body physics, has potential applications in quantum information [23]. In our case, it can be expressed as  $|RVB\rangle_{ABCD} = (\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AB} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{DC} + \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AC} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{DB}) / \sqrt{3}$ , where  $A$  and  $D$  are in sublattice 1, while  $B$  and  $C$  are in sublattice 2, of the bipartite lattice formed by  $A, B, C$ , and  $D$ , with the singlets being always directed from sublattice 1 to sublattice 2. The state can be rewritten as  $|\psi_{RF}^\mu\rangle_{ABCD} = (|0101\rangle + |1010\rangle + |0011\rangle + |1100\rangle - \mu|1001\rangle - \mu|0110\rangle)_{ABCD} / \sqrt{4 + 2\mu^2}$ , for  $\mu = 2$ . Interestingly, the state  $|\psi_{RF}^\mu\rangle$  is the ferromagnetic ground state (GHZ state of Case I) for  $\mu \rightarrow \infty$ , and we will consider the state for the whole range  $[2, \infty)$ . Certainly we have  $\mathcal{C}_a \leq 1$ , for all  $\mu$ , but this bound can be attained by the following protocol. Suppose that  $A$  and  $B$  are allowed to share the additional singlet state resource. This implies that entangled measurements are

allowed in the  $AB$  sector of the state  $|\psi_{RF}^\mu\rangle$ . A measurement in the Bell basis [21] at  $AB$  and subsequent local unitary transformations at  $C$  and  $D$ , attains the bound for all  $\mu$ . By explicit calculation, we find that  $p_{\max\min}^d(|\psi_{RF}^\mu\rangle) = (\mu^2 - 2\mu + 3)/(\mu^2 + 2)$ , so that by Proposition C2,  $\mathcal{C}_{ua}(|\psi_{RF}^\mu\rangle) \leq (\mu^2 - 2\mu + 3)/(\mu^2 + 2)$ . Measurements in the  $\{|+\rangle, |-\rangle\}$  basis by two parties, and local operations by the remaining parties, produces a lower bound:  $\mathcal{C}_{ua}(|\psi_{RF}^\mu\rangle) \geq (\mu^2 - 2\mu + 2)/(\mu^2 + 2)$ . For the RVB state, this reduces to  $1/3 \leq \mathcal{C}_{ua}(|RVB\rangle) \leq 1/2$ . We have optimized over all projective measurements at two parties and all LOCC at the other two, for all  $\mu$ , and the results are summarized in Fig. 1. For the RVB state, the lower bound is the optimal one for the considered strategy. The generalized geometric measure of  $|\psi_{RF}^\mu\rangle$  is  $(\mu + 1)^2/(4 + 2\mu^2)$  (by Proposition E1), so that for the RVB state,  $\mathcal{E} = 1/4$ . [ $\mathcal{E}_G(|RVB\rangle) = 2/3$ .]

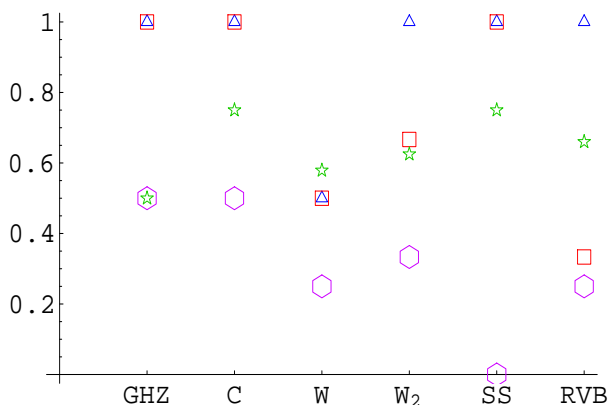


FIG. 2: (Color online.) The capacities and the measures. The assisted capacities (blue triangles), unassisted capacities (red boxes), generalized geometric measures (pink hexagons), and geometric measures (green stars) for a selection of four-party quantum states that are important from a quantum networks perspective. While the capacities are measured in ebits, the measures are dimensionless.

### H. Bipartite versus multipartite revisited

Consider now the generalized GHZ (Case I) and RVB states ( $|\psi_{RF}^\mu\rangle$  for  $\mu = 2$  in Case VI). Choosing  $\beta$  in the range  $1/4 < |\beta|^2 < 1/2$ , we find that while the assisted capacity increases from generalized GHZ to RVB, the unassisted capacity actually decreases. This therefore leaves us with no option to reconcile with the picture

in the bipartite domain by using *any* multipartite entanglement measure. The multi-access channel capacities therefore presents a much richer picture than its bipartite variety. A similar situation arises if we compare the generalized GHZ states for  $\beta$  in the range  $1/3 < |\beta|^2 < 1/2$  with the  $|W_2\rangle$  state (Case IV).

It is plausible that the unassisted capacity for the RVB state is  $1/3$  (see Case VI). In that case, again such an irreconcilable situation arises for the W (Case III) and RVB pair.

The richness of the multiparty picture is further enforced by the other examples considered. In particular, the generalized GHZ and generalized W states reveal a situation where both the assisted and unassisted capacities are equal (for certain choices of the parameters), while the GGM can still be different. A synopsis of the whole picture is presented in Fig. 2.

## V. CONCLUSIONS

Capacities of quantum channels corresponding to shared bipartite pure quantum states presents a relatively simple image, viz. the capacities are monotonically increasing with similar behavior for entanglement of the states. Capacities of multi-access channels, however, offers a much richer picture. Two such quantum capacities are defined and considered for paradigmatic multiparty quantum states, and compared against a measure of genuine multiparty entanglement. The quantum capacities are defined from the perspective of quantum computational networks.

The measure of genuine multiparty entanglement, which we call the generalized geometric measure, is defined, its properties are explored. In particular, we find that it is possible to render it into a computable form for any multiparty quantum state of any dimension and of any number of parties.

The investigation also points to the fact that at least for some multiparty situations, the additional singlet state does *not* help to increase the capacity with respect to that in the unassisted case. This is in contrast to the bipartite case, where the singlet state is almost always the most important resource.

### Acknowledgments

We acknowledge partial support from the Spanish MEC (TOQATA (FIS2008-00784)).

- 
- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).  
 [2] See e.g. M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (CUP, Cambridge,

- 2000); *Lectures on Quantum Information*, eds. D. Bruß and G. Leuchs (Wiley, Weinheim, 2006).  
 [3] See e.g. M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen(De), and U. Sen, Adv. Phys. **56**, 243

- (2007); L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Rev. Mod. Phys.* **80**, 517 (2008).
- [4] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, *Phys. Rev. A* **54**, 3824 (1997); V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997); M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **84**, 2014 (2000).
- [5] See e.g. P.W. Shor, *The quantum channel capacity and coherent information*, Lecture at MSRI Workshop on quantum computation, 2002; P.W. Shor, *Mathematical Programming* **97**, 311 (2003).
- [6] C.H. Bennett and S.J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).
- [7] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
- [8] T. Hiroshima, quant-ph/0009048; M. Ziman and V. Bužek, *Phys. Rev. A* **67**, 042321 (2003); D. Bruß, G.M. D’Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen, *Phys. Rev. Lett.* **93**, 210501 (2004).
- [9] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [10] A. Shimony, *Ann. N.Y. Acad. Sci.* **755**, 675 (1995); H. Barnum and N. Linden, *J. Phys. A* **34**, 6787 (2001); T.-C. Wei and P.M. Goldbart, *Phys. Rev. A* **68**, 042307 (2003).
- [11] G. Vidal and R.F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
- [12] E. Knill, R. Laflamme, and G.J. Milburn, *Nature* **409**, 46 (2001).
- [13] R. Raussendorf and H.J. Briegel, *Phys. Rev. Lett.* **86**, 5188 (2001); R. Raussendorf, D.E. Browne, and H.J. Briegel, *Phys. Rev. A* **68**, 022312 (2003).
- [14] M.A. Nielsen, *Phys. Rev. Lett.* **93**, 040503 (2004); D.E. Browne and T. Rudolph, *Phys. Rev. Lett.* **95**, 010501 (2005).
- [15] M.A. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999); G. Vidal, *Phys. Rev. Lett.* **83**, 1046 (1999); G. Vidal, *J. Mod. Opt.* **47**, 355 (2000).
- [16] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, ed. M. Kafatos (Kluwer, Dordrecht, (1989)).
- [17] Y. Yeo and W.K. Chua, *Phys. Rev. Lett.* **96**, 060502 (2006).
- [18] A. Zeilinger, M.A. Horne, and D. M. Greenberger, in *Squeezed States and Quantum Uncertainty*, eds. D. Han, Y.S. Kim, and W.W. Zachary (NASA Conference Publication 3135, NASA, College Park, 1992); W. Dür, G. Vidal, and J.I. Cirac, *Phys. Rev. A* **62**, 062314 (2000).
- [19] A. Sen(De), U. Sen, M. Wieśniak, D. Kaszlikowski, and M. Żukowski, *Phys. Rev. A* **68**, 062306 (2003).
- [20] M. Żukowski, A. Zeilinger, M.A. Horne, and A.K. Ekert, *Phys. Rev. Lett.* **71**, 4287 (1993); M. Żukowski, A. Zeilinger, and H. Weinfurter, *Ann. N.Y. Acad. Sci.* **755**, 91 (1995); S. Bose, V. Vedral, and P.L. Knight, *Phys. Rev. A* **57**, 822 (1998).
- [21] The Bell basis consists of  $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$  and  $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ .
- [22] P.W. Anderson, *Science* **235**, 1196 (1987); S. Liang, B. Doucot, and P.W. Anderson, *Phys. Rev. Lett.* **61**, 365 (1988).
- [23] A.Y. Kitaev, *Ann. Phys.* **303**, 2 (2003).