Entanglement Mean Field Theory and the Curie-Weiss Law

Aditi Sen(De) and Ujjwal Sen

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India

The mean field theory, in its different hues, form one of the most useful tools for calculating the single-body physical properties of a many-body system. It provides important information, like critical exponents, of the systems that do not yield to an exact analytical treatment. Here we propose an entanglement mean field theory (EMFT) to obtain the behavior of the two-body physical properties of such systems. We apply this theory to predict the phases in paradigmatic strongly correlated systems, viz. the transverse anisotropic XY, the transverse XX, and the Heisenberg models. We find the critical exponents of different physical quantities in the EMFT limit, and in the case of the Heisenberg model, we obtain the Curie-Weiss law for correlations. While the exemplary models have all been chosen to be quantum ones, classical many-body models also render themselves to such a treatment, at the level of correlations.

I. INTRODUCTION

Characterizing many-body systems by understanding their different phases is a key issue in physics. However, it is only in a few cases that exact analytical techniques can be applied [1]. It is therefore crucial to have approximate methods to deal with such systems to predict their different physical properties [2]. A very useful method is to use mean field theory (MFT) [2, 3], which renders the many-body physical system into one with a single particle. MFT allows one to predict the single-body physical properties, like magnetization, susceptibility, of the system. Its importance lies in the fact that these MFT-reduced one-body properties can correctly predict the thermal fluctuation- and quantum fluctuation- driven phase transitions of the system, as well as the critical exponents of the such single-body physical quantities [4, 5].

Useful as it is, there are important limitations of a mean field theory [3], a first being that it is not possible to predict the multi-party physical properties, like entanglement, of the system by using this theory.

In this paper, we propose an "entanglement mean field theory" (EMFT), which transforms an interacting manybody physical system into a two-body one, while still retaining certain footprints of the interactions in the many-body parent, using which it is possible to calculate the two-body properties, like two-point entanglement and two-point correlations, of the system, and predict the critical phenomena in it. Moreover, it is possible to calculate the critical exponents of two-body properties. The theory can be applied for detecting phase transitions driven by thermal fluctuations as well as quantum fluctuations. At the same time, both quantum as well as classical models can be treated by EMFT, where in the classical case, this will be only until the level of correlations. In this paper, we will only consider the applications of EMFT to quantum systems. We will consider three paradigmatic classes of interacting spin models, viz. the transverse anisotropic quantum XY (which includes the transverse quantum Ising), the transverse quantum XX, and the quantum Heisenberg models. The phase transitions of these models, both temperatureinduced phase transitions as well as zero temperature quantum phase transitions, are faithfully signaled by the corresponding EMFT entanglements. The Curie-Weiss law of susceptibility is an important prediction of the mean field-reduced Heisenberg model in the paramagnetic regime [3]. We indicate the corresponding law for EMFT-reduced correlations by using the EMFT-reduced Heisenberg model.

The entanglement mean field theory opens up the possibility of investigating the behavior of entanglement and other two-body properties of many-body systems, particularly for the ones which does not lend themselves to an analytical treatment. At the fundamental level, this forms, potentially, an important link between many-body physics and quantum information science [6].

II. THE ENTANGLEMENT MEAN FIELD THEORY: XY MODEL

Before presenting the entanglement mean field theory, let us briefly remind ourselves the mean field theory. Consider the transverse quantum anisotropic XY model with nearest-neighbor interactions, described by the Hamiltonian

$$H^{XY} = -\frac{J}{2} \sum_{\langle \vec{i}\vec{j} \rangle} \left[(1+\gamma)\sigma_x^{\vec{i}}\sigma_x^{\vec{j}} + (1-\gamma)\sigma_y^{\vec{i}}\sigma_y^{\vec{j}} \right] - h \sum_{\vec{i}} \sigma_z^{\vec{i}},$$
(1)

which represents a system of interacting spin-1/2 particles on a *d*-dimensional cubic lattice. The coupling strength J/2 is positive, the anisotropy $\gamma \in (0, 1]$, and the transverse field strength *h* is also positive. σ_x , σ_y , and σ_z are the Pauli matrices for the spin degree of freedom of a spin-1/2 particle. $\langle \vec{ij} \rangle$ indicates that the corresponding sum runs over nearest neighbor lattice sites only. The mean field theory consists in assuming that a particular spin, say $\vec{i_0}$, is special, and replacing all other spin operators by their mean values. Denoting the mean values of the spin operators σ_x , σ_y , σ_z as m_x , m_y , m_z , respectively, this leads to an MFT Hamiltonian H_{MFT}^{XY}

$$m_x = \operatorname{tr}(\sigma_x \rho_\beta), \quad m_y = \operatorname{tr}(\sigma_y \rho_\beta), \quad (2)$$

where ρ_{β} is the mean field canonical equilibrium state $\exp(-\beta H_{MFT}^{XY})/\operatorname{tr}(\exp(-\beta H_{MFT}^{XY}))$, for m_x and m_y , substitute them in H_{MFT}^{XY} and ρ_{β} , and we are then ready to find the single-body physical properties of the system in the mean field limit. Here $\beta = \frac{1}{k_B T}$, with T denoting temperature on the absolute scale, and k_B denoting the Boltzmann constant.

The entanglement mean field theory begins by replacing an identity, on a site that is neighboring the interacting spins of a two-spin interaction term, by a square of the Pauli matrix that is involved in the interaction, for all the two-spin interaction terms in the Hamiltonian. [An averaging needs to be done for two-spin interactions involving two Pauli matrices.] Therefore, the term $\sigma_x^{k-1,l}\sigma_x^{k,l}$ in a Hamiltonian on a two-dimensional square lattice can be replaced by $\sigma_x^{k-1,l}\sigma_x^{k,l}\sigma_x^{k+1,l}\sigma_x^{k+1,l}$. The latter can be re-written as AB, with $A = \sigma_x^{k,l} \sigma_x^{k+1,l}$, and $B = \sigma_x^{k-1,l} \sigma_x^{k+1,l}$. We then assume that a certain pair of two neighboring spins are "special". We then replace the non-special two-spin interactions (with nearby spins) in all the interaction terms by their mean values. In a *d*-dimensional cubic lattice, there will be 2ν terms (with $\nu + 1 = 2d$ being the coordination number of the lattice), that will have the special pair, along with two other spins in two other lattice sites. The EMFT-reduced Hamiltonian, for the transverse quantum ferromagnetic XY model on a *d*-dimensional cubic lattice, will therefore be

$$\mathcal{H}_{EMFT}^{XY} = -J\nu \left[(1+\gamma)C_{xx}\sigma_x^{\vec{i}}\sigma_x^{\vec{j}} + (1-\gamma)C_{yy}\sigma_y^{\vec{i}}\sigma_y^{\vec{j}} \right] -h \left[\sigma_z^{\vec{i}} + \sigma_z^{\vec{j}}\right],(3)$$

where we have ignored the terms in the Hamiltonian which will not contribute to the EMFT equations below, and where we have assumed that the neighboring lattice sites \vec{i} and \vec{j} are special. C_{xx} and C_{yy} are respectively the xx and yy correlators, and the self-consistency equations (EMFT equations) are

$$C_{xx} = \operatorname{tr}\left(\sigma_x^{\vec{i}}\sigma_y^{\vec{j}}\varrho_\beta\right), \quad C_{yy} = \operatorname{tr}\left(\sigma_y^{\vec{i}}\sigma_y^{\vec{j}}\varrho_\beta\right), \quad (4)$$

where ρ_{β} is the canonical equilibrium state $\exp(-\beta \mathcal{H}_{EMFT}^{XY})/\operatorname{tr}(\exp(-\beta \mathcal{H}_{EMFT}^{XY}))$. The EMFT equations are to be solved for C_{xx} and C_{yy} , and substituted in \mathcal{H}_{EMFT}^{XY} and ρ_{β} . We can then calculate the two-particle physical properties of the physical system described by H^{XY} in the EMFT limit, and look for possible phase transitions of the system.

A similar formalism works, with slight modifications, for classical spins, higher quantum spins, more complex lattices, etc. Also, both the mean field theory as well as the EMFT has been described for the ferromagnetic cases. The antiferromagnetic case requires some modifications in the mean field theory, and correspondingly some changes in the EMFT. These will not be discussed

A. The Transverse Ising model

in this paper.

For simplicity, let us consider the transverse quantum Ising model ($\gamma = 1$). This is the simplest model which exhibits a quantum phase transition (at zero temperature) [5, 8], that has been experimentally observed [12]. for which the EMFT equation reads

$$C_{xx} = \frac{2}{\mathcal{Z}_{EMFT}^{Ising}} \Big[\sinh(2\beta\nu C_{xx}J) + \frac{\nu C_{xx}J}{\Gamma} \sinh(2\beta\Gamma) \Big],(5)$$

where the corresponding EMFT Ising partition function is given by $\mathcal{Z}_{EMFT}^{Ising} = 2 \cosh(2\beta\nu C_{xx}J) + 2 \cosh(2\beta\Gamma)$, with $\Gamma = \sqrt{\nu^2 C_{xx}^2 J^2 + h^2}$.

We are now ready to present the EMFT phase diagram, where we use the xx correlator, C_{xx} , as the order parameter of the many-body system. Let us first look at the T = 0 picture, where transitions are driven by quantum fluctuations only. The zero temperature EMFT C_{xx} is given by

$$C_{xx} = \sqrt{1 - \frac{h^2}{\nu^2 J^2}},$$
 (6)

for $h \leq h_c \equiv \nu J$. It is vanishing otherwise. Let us now fix our attention on the other extreme: the behavior of the system with respect to temperature for zero field, in which case the EMFT equation reduces to

$$C_{xx} = \tanh\left(2\beta\nu C_{xx}J\right),\tag{7}$$

whereby a temperature-driven phase transition is obtained at $T = T_c \equiv 2\nu J/k_B$. The complete phase diagram can be seen in Fig. 1, which contains these extreme cases as special instances.

It is possible to calculate the critical exponents in the EMFT limit. The critical exponent for the EMFT xx correlator can be calculated as follows, which we find for both the temperature-driven phase transition on the h = 0 axis, and for the quantum phase transition on the T = 0 axis. In the zero temperature scenario, the critical exponent is $\frac{1}{2}$, as can be found by using Eq. (6). In the zero field case, we can perform an expansion of the equation,

$$2\beta\nu C_{xx}J = \tanh^{-1}C_{xx},\tag{8}$$

around $T = T_c$, and the critical exponent is again $\frac{1}{2}$. In a similar fashion, one can obtain the critical exponent for the EMFT energy gap to be $\frac{1}{2}$. Note that the same exponent is obtained for the gap in the MFT limit [9].

The entanglement mean field theory can not only predict the correlations of the system, but one can also study



FIG. 1: The xx correlator, C_{xx} , of the transverse Ising model in the EMFT limit. The temperature-driven phase transition at $T = T_c$ and the quantum phase transition at $h = h_c$ are clearly visible on the respective axes. All the three axes represent dimensionless quantities.

the behavior of its entanglement. Apart from its fundamental importance, entanglement is known to be the basic ingredient in quantum information tasks [10]. We will quantify the entanglement of a two-party quantum state ρ , by its logarithmic negativity [11], defined as

$$E(\varrho) = \log_2 \parallel \varrho^{T_1} \parallel_1, \tag{9}$$

where $\|\cdot\|$ denotes the trace norm of its argument, and ρ^{T_1} denotes the partial transpose of ρ with respect to one of the two parties forming the state ρ . The behavior of entanglement in the EMFT canonical state with respect to temperature and applied field, in the transverse Ising model, is seen in Fig. 2.

The other members of the class of Hamiltonians given in Eq. (1) have a similar behavior with respect to their two-body physical properties in the EMFT limit. We can compare this with the fact that they fall in the same universality class [5]. A different universality class is considered in the succeeding section.

III. THE EMFT-REDUCED XX MODEL

The transverse field quantum XX model on a *d*dimensional cubic lattice is described by the Hamiltonian

$$H^{XX} = -\frac{J}{2} \sum_{\langle \vec{i}\vec{j} \rangle} \left[\sigma_x^{\vec{i}} \sigma_x^{\vec{j}} + \sigma_y^{\vec{i}} \sigma_y^{\vec{j}} \right] - h \sum_{\vec{i}} \sigma_z^{\vec{i}}, \qquad (10)$$

where J and h are positive. The physical importance of the Hamiltonian includes that it can be obtained from the Bose Hubbard Hamiltonian for hard-core boson limit, by suitably associating the bosonic creation and annihilation



FIG. 2: EMFT entanglement in the transverse Ising model, plotted against temperature and field strength. The base axes represent dimensionless system parameters, while the vertical axis represent entanglement, measured in ebits.

operators with the Pauli matrices [5]. The corresponding EMFT-reduced Hamiltonian is

$$\mathcal{H}_{EMFT}^{XX} = -J\nu C \left[\sigma_x^{\vec{i}} \sigma_x^{\vec{j}} + \sigma_y^{\vec{i}} \sigma_y^{\vec{j}} \right] - h \left[\sigma_z^{\vec{i}} + \sigma_z^{\vec{j}} \right], (11)$$

where C is the xx (which is same as the yy) correlator of the system, and where we have supposed that the neighboring lattice sites \vec{i} and \vec{j} are special.

The EMFT equation in this case reads

$$C = 2\sinh\left(2\beta\nu CJ\right) / \mathcal{Z}_{EMFT}^{XX},\tag{12}$$

where the EMFT XX partition function is given by $\mathcal{Z}_{EMFT}^{XX} = 2 \cosh(2\beta\nu CJ) + 2 \cosh(2\beta h)$. Solving for C from the EMFT equation, we can subsequently find other physical quantities of the system. In particular, the entanglement of the system, as quantified by its logarithmic negativity, is given in Fig. 3.

It is also possible to write an EMFT equation for the ground state of the EMFT-reduced XX Hamiltonian, solving which we find

$$C = 1, \text{ for } \frac{h}{J\nu} < 1$$

$$C = 0, \text{ for } \frac{h}{J\nu} > 1.$$
(13)

It is interesting to compare this result with the fact that the physical system represented by the Hamiltonian H^{XX} , in the one-dimensional case, undergoes a Mott insulator to superfluid transition at h/J = 1 [5, 13].

IV. EMFT AND THE HEISENBERG MODEL: A CORRELATION CURIE-WEISS LAW

Investigations in magnetism in solids very often starts off by using the Heisenberg Hamiltonian [14], which, for



FIG. 3: EMFT entanglement in the transverse XX model. The plot shows the behavior of entanglement of the canonical state of the EMFT-reduced transverse XX model with respect to temperature and applied field. The base axes represent dimensionless system parameters, while the vertical axis represent entanglement, measured in ebits.

a *d*-dimensional cubic lattice, is given by

$$H^{HD} = -J \sum_{\langle \vec{i}\vec{j} \rangle} \left[\sigma_x^{\vec{i}} \sigma_x^{\vec{j}} + \sigma_y^{\vec{i}} \sigma_y^{\vec{j}} + \sigma_z^{\vec{i}} \sigma_z^{\vec{j}} \right] -h\mu \sum_{\vec{i}} \left[\sigma_x^{\vec{i}} + \sigma_y^{\vec{i}} + \sigma_z^{\vec{i}} \right], \qquad (14)$$

where J, h, and μ are positive. An important conclusion of the MFT treatment of this model is the Curie-Weiss Law, which predicts the behavior of magnetization of the physical system in its paramagnetic phase. We will see that it is possible to extract a similar law for the correlations in the system by solving the corresponding EMFT Hamiltonian.

The EMFT-reduced Heisenberg Hamiltonian is given by

$$\mathcal{H}_{EMFT}^{HD} = -2J\nu C \left[\sigma_x^{\vec{i}} \sigma_x^{\vec{j}} + \sigma_y^{\vec{i}} \sigma_y^{\vec{j}} + \sigma_z^{\vec{i}} \sigma_z^{\vec{j}} \right] -h\mu \left[\sigma_x^{\vec{i}} + \sigma_x^{\vec{j}} + \sigma_y^{\vec{i}} + \sigma_y^{\vec{j}} + \sigma_z^{\vec{i}} + \sigma_z^{\vec{j}} \right)], \quad (15)$$

where C is the zz (which is the same as the xx and yy) correlation of the system, and where we have supposed that the lattice sites \vec{i} and \vec{j} are special, for constructing the EMFT Hamiltonian. The EMFT equation in this case is

$$C = \frac{2}{3\mathcal{Z}_{EMFT}^{HD}} \left[\sinh\Gamma_C - \exp(-\Gamma_C) + \exp\Gamma_C \cosh\Gamma_h\right]$$
(16)

where the EMFT Heisenberg partition function is given by $\mathcal{Z}_{EMFT}^{HD} = 2 \cosh \Gamma_C + 2 \exp \Gamma_C \cosh \Gamma_h$, with $\Gamma_C = 2\beta\nu CJ$ and $\Gamma_h = 2\sqrt{3}\beta h\mu$. The partial derivative of magnetization with respect to the applied field is defined as the susceptibility of the system, and the usual Curie-Weiss Law is given for that quantity, for vanishing applied field. We define the correlation susceptibility of the system as the partial derivative of the correlation C with respect to the field strength h. This quantity, as solved from the EMFT equation (Eq. (16)), for h = 0, is given in Fig. 4. Note that the correlation susceptibility clearly signals the onset of the paramagnetic phase of the system at $T = T_c^{FP} \equiv \nu J/k_B$. The data obtained for $\frac{\nu J}{\mu} \frac{\partial C}{\partial h}\Big|_{h=0}$ can be fitted to the curve $\left(\frac{\nu J}{k_B}\right)^{\lambda} \alpha/(T - T_c^{FP})^{\lambda}$, for $T > 1.2T_c^{FP}$, and it gives the optimal values of the curve parameters as $\alpha = 0.06895$

optimal values of the curve parameters as $\alpha = 0.06895$ and $\lambda = 1.08162$. The corresponding mean square error is 1.2×10^{-4} .



FIG. 4: Curie-Weiss Law of EMFT-reduced correlation in the Heisenberg model. The partial derivative of the zz correlation (which is same as the xx and yy ones) with respect to the field h is plotted, as red triangles, against temperature, for h = 0. More precisely, $\frac{\nu J}{\mu} \frac{\partial C}{\partial h}\Big|_{h=0}$ is plotted on the vertical axis as a function of $\frac{k_B T}{\nu J}$ on the horizontal axis. Both axes represent dimensionless quantities. The blue squares represent a fit of the plotted data with the function $\left(\frac{\nu J}{k_B}\right)^{\lambda} \alpha/(T - T_c^{FP})^{\lambda}$, for $T > 1.2T_c^{FP}$. See text for further details.

V. CONCLUSIONS

The mean field theory is a useful tool to obtain important information about single-body physical quantities of many-body systems, especially the ones which are not tractable analytically. We have presented an entanglement mean field theory that can be used to obtain information about two-body physical quantities of manybody systems. The theory predicts the phase diagram of the physical system, and the critical exponents of their two-body quantities, as we have shown for several important classes of many-body systems. In particular, we have derived a Curie-Weiss Law for correlations in the Heisenberg spin model. We acknowledge partial support from the Spanish MEC (TOQATA (FIS2008-00784)).

- See e.g. D.C. Mattis, *The Many-Body Problem* (World Scientific, Singapore, 1993).
- [2] See e.g. G.D. Mahan, Many-Particle Physics (Kluwer/Plenum, New York, 2000); X.-G. Wen, Quantum Field Theory of Many-Body Systems (OUP, Oxford, 2004).
- [3] J.M. Ziman, Principles of the Theory of Solids (Cambridge University Press, 1972); N.W. Ashcroft and N.D. Mermin, Solid State Physics (Holt, Rinehert and Winston, New York, 1976).
- [4] See e.g. R.K. Pathria, *Statistical Mechanics* (Butterworth-Heinemann, Oxford, 1996).
- [5] S. Sachdev, Quantum Phase Transitions (CUP, Cambridge, 1999).
- [6] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen(De), and U. Sen, Adv. Phys. 56, 243 (2007); L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
- [7] R. Brout, K.A. Müller, and H. Thomas, Solid State

Comm. 4, 507 (1966); R.B. Stinchcombe, J. Phys. C 6, 2459 (1973).

- [8] B.K. Chakrabarti, A. Dutta, and P. Sen, Quantum Ising Phases and Transitions in Transverse Ising Model (Springer, Berlin, 1996).
- [9] P. Pfeuty and R.J. Elliot, J. Phys. C: Solid State Phys. 4, 2370 (1971).
- [10] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
- [11] G. Vidal and R.F. Werner, Phys. Rev. A 65, 032314 (2002).
- [12] D. Bitko, T.F. Rosenbaum, and G. Appeli, Phys. Rev. Lett. 77, 940 (1996).
- [13] M.P.A Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B 40, 546 (1989).
- [14] See e.g. K. Yosida, Theory of Magnetism (Springer, Berlin, 1996); A. Aharoni, Introduction to the Theory of Magnetism (OUP, Oxford, 2000).