Information complementarity in multipartite quantum states and security in cryptography

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We derive complementarity relations for arbitrary quantum states of multiparty systems, of arbitrary number of parties and dimensions, between the purity of a part of the system and several correlation quantities, including entanglement and other quantum correlations as well as classical and total correlations, of that part with the remainder of the system. We subsequently use such a complementarity relation, between purity and quantum mutual information in the tripartite scenario, to provide a bound on the secret key rate for individual attacks on a quantum key distribution protocol.

I. INTRODUCTION

Quantum key distribution (QKD) is a protocol that allows two distant parties to share a secret without meeting and in the presence of a malicious eavesdropper [1]. The security of the protocol is based on the validity of quantum mechanics [2]. Quantum key distribution protocols broadly fall under two main categories, viz. the product-state and the entanglement based ones, the Bennett-Brassard 1984 (BB84) [3] and the Ekert 1991 [4] protocols being prominent examples of the respective categories. It was later realized that the two categories are similar from several perspectives [5], with a notable difference being in the device independent security proof [4, 6–8], which is based on the Ekert protocol with security obtained via violation of Bell inequality [9]. We may note that in the two-qubit scenario, strong violation of Bell inequality is possible only by states close to maximally entangled states, and such states are almost pure [10, 11]. An almost pure state shared between the legitimate users of the key distribution channel implies that these users are informationally detached from the rest of the world, including a possible eavesdropper, indicating the security of the information flowing in the channel between the legitimate users.

In this work, we ask whether we can start a step later to obtain a security proof of quantum cryptography for individual attacks via a variation [12] of the Ekert key distribution protocol. Precisely, we consider states with high purity and that also has correlations which allow generation of correlated bit sequences. We consider a scenario with three parties called Alice, Bob, and Eve, where Alice and Bob represent the legitimate users of the protocol, while Eve represents the potential eavesdropper. We prove complementarity relations between the purity of the Alice-Bob system and the quantum mutual information in the Alice-Bob versus Eve bipartition. We then show that such a complementarity can potentially lead to a bound on the secret key rate in the case of individual attacks. This secret key rate is obtained by concepts independent of Bell inequalities, which, along with providing another perspective of the security of the Ekert

protocol, can also be important to avoid vulnerability of the proof from loopholes in experiments that violate Bell inequalities [8, 13] (cf. [14]). Note that the complementarity relation is true for any quantum state of arbitrary dimensions of the individual parties, and is also true for an arbitrary number of parties, and so can have applications in other quantum information protocols with or without security.

On the way, we also show that a similar complementarity exists in all multiparty quantum systems between the purity of the "legitimate" users, and a large number of quantum characteristics including quantum correlations in the legitimate versus "eavesdroppers" bipartition, irrespective of whether the multipartite quantum state is pure or mixed and irrespective of the dimensions of the subsystems and the number of parties involved. We also numerically investigate the tightness of the obtained inequalities by Haar uniformly generating states of three qubits of different ranks.

The paper is organized as follows. In the following section, we derive the complementarity relations. Their tightness is considered in the Sec. III. In the succeeding section (Sec. IV), we use a complementarity relation to provide bounds on the secret key rate. A concluding section is presented at the end.

II. COMPLEMENTARITY RELATIONS

We now derive the complementarity relations for arbitrary multiparty quantum states. The parties involved are divided into "legitimate" users and "eavesdroppers". The relations show a trade-off between two quantities, one of which is the purity of the state of the legitimate users, while the other is a quantum characteristic in the legitimate users versus eavesdroppers bipartition. To be specific, we begin with the three party case, where there are two legitimate users and a single eavesdropper. We will briefly mention the case of an arbitrary number of parties later.

Consider therefore a three-party quantum system in the state ρ_{ABC}. For several non-classical bipartite cor-

relation measures, $\mathcal{Q}'$, the relation $\mathcal{Q}'_{AB:C} - S_{AB} \leq 0$ holds, where $S_{AB} \equiv S(\rho_{AB})$ is the von Neumann entropy of its argument, and $\rho_{AB} = \mathrm{tr}_C \rho_{ABC}$. [$S_{AC}$, etc. are similarly defined.] And $\mathcal{Q}'_{AB:C}$ is the non-classical correlation $\mathcal{Q}'$ of the state ρ_{ABC} in the $AB : C$ bipartition. The relation holds [15] e.g. for entanglement of formation [16], entanglement cost [17], distillable entanglement [16], relative entropy of entanglement [18], and one-way distilled key rate [19]. Interestingly, $\mathcal{Q}'_{AB:C}$

can also be a measure of classical correlation, as quantified by the measured quantum mutual information, defined as follows. The measured quantum mutual information [20], of a bipartite quantum state ϱ_{XY} is defined as $\mathcal{J}(\varrho_{XY}) = S(\rho_X) - \min \sum p_k S(\varrho_{XY}^k)$, where the minimization is over measurements performed by the party Y that creates the ensemble $\{p_k, \varrho_{XY}^k\}$. Here, $\rho_X = \mathrm{tr}_Y \varrho_{XY}$. That $\mathcal{Q}'_{AB:C} - S_{AB} \leq 0$ is valid for $\mathcal{Q}'$ identified with this classical correlation, follows e.g. from Ref. [21].
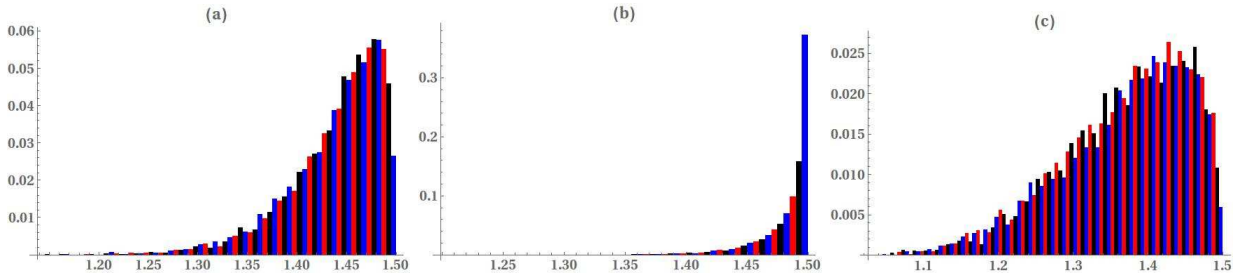

FIG. 1. (Color.) The complementarity for three qubit systems. The different panels exhibit histograms for the sum of two quantities, viz. the normalized purity and a normalized correlation. The correlation is the negativity in panel (a), logarithmic negativity in (b), and quantum mutual information in (c), for rank 1 states. We Haar uniformly generate 10^4 three qubit states for generating the histograms. The histograms for normalized measured quantum mutual information, quantum discord, and quantum work deficit are almost identical to that of the normalized quantum mutual information. The vertical axis represents the relative frequency of occurence of a randomly generated three-qubit state in the corresponding range of the sum of the two quantities on the horizontal axis. All quantities are dimensionless.

This relation $\mathcal{Q}'_{AB:C} - S_{AB} \leq 0$ can be rewritten, using appropriate normalized quantities, as

$$\frac{\log_2 d_{AB} - S_{AB}}{\log_2 d_{AB}} + \frac{\mathcal{Q}'_{AB:C}}{\min\{\log_2 d_{AB}, \log_2 d_C\}}$$
$$\leq 1 + \mathcal{Q}'_{AB:C} \left(\frac{1}{\min\{\log_2 d_{AB}, \log_2 d_C\}} - \frac{1}{\log_2 d_{AB}} \right). \tag{1}$$

While the above relation only needs $\mathcal{Q}'_{AB:C} - S_{AB} \leq 0$, we may note that the choice of the denominators in the terms on the left hand side have been guided by the fact that $0 \leq S(\rho_{AB}) \leq \log_2 d_{AB}$, so that $0 \leq \log_2 d_{AB} - S(\rho_{AB}) \leq \log_2 d_{AB}$, and the oft-true relation $0 \leq \mathcal{Q}'_{AB:C} \leq \min\{\log_2 d_{AB}, \log_2 d_C\}$. We indicate the dimension of the Hilbert space corresponding to a system denoted as X by d_X.

The first term, $\mathcal{P}_{AB} \equiv \frac{\log_2 d_{AB} - S_{AB}}{\log_2 d_{AB}}$, on the left hand side of ineq. (1) quantifies the purity of the system in the AB part, i.e. of ρ_{AB}. We have normalized the quantity so that it varies between 0 and 1. The second term, $\mathcal{Q}_{AB:C} \equiv \frac{\mathcal{Q}'_{AB:C}}{\min\{\log_2 d_{AB}, \log_2 d_C\}}$, represents the normalized non-classical correlation of the system in the $AB : C$ bipartition. Again, it has been normalized, and if we as-

sume that $0 \leq \mathcal{Q}'_{AB:C} \leq \min\{\log_2 d_{AB}, \log_2 d_C\}$ is true, we once more have $0 \leq \mathcal{Q}_{AB:C} \leq 1$. Thus the trivial upper bound of the quantity $\mathcal{P}_{AB} + \mathcal{Q}_{AB:C}$ is 2.

Eq. (1) can further be shuffled into

$$\mathcal{P}_{AB} + \mathcal{Q}_{AB:C} \leq 1, \text{ when } d_{AB} \leq d_C, \tag{2}$$

while it reads

$$\mathcal{P}_{AB} + \mathcal{Q}_{AB:C} \leq 2 - \frac{\log_2 d_C}{\log_2 d_{AB}}, \text{ when } d_{AB} > d_C \tag{3}$$

$$\text{and } \mathcal{Q}'_{AB:C} \leq \log_2 d_C.$$

For systems in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, the complementarity relation reads

$$\mathcal{P}_{AB} + \mathcal{Q}_{AB:C} \leq \frac{3}{2}. \tag{4}$$

Note that the right-hand-side is independent of dimension in this case. In particular, for three-qubit quantum states, the bound is three-halves, and is saturated by the Greenberger-Horne-Zeilinger state $(|000\rangle + |111\rangle)/\sqrt{2}$ [22]. As mentioned before, the complementarity holds for several entanglement measures. It is natural to ask whether the same holds for other quantum correlation

measures including entanglement measures like negativity [23] and logarithmic negativity [23], and information theoretic quantum correlations like quantum discord [20] and quantum work-deficit [24]. If quantum discord is defined by considering the measurement in its definition to be in the first party, then its value for an arbitrary quantum state in $AB : C$ can be shown to be bounded above by S_{AB} [25], so that the complementarity relation (2) is valid for the measure. Henceforth, we consider that the quantum discord is defined by performing the measure-

ment in the second party. Numerical simulations by Haar uniform generation of three-qubit states having different ranks (ranging from 1 till 4) show that several correlation measures indeed obey the above complementarity (ineq. (4)). See Figs. 1 and 2 for depictions of the cases of ranks 1 and 2. For higher ranks, the complementarities become less and less tight. We will discuss in the succeeding section about the tightness of these relations. It is also clear that similar complementarities will hold also when purity is quantified by utilizing Rényi [26] and Tsallis [27] entropies.
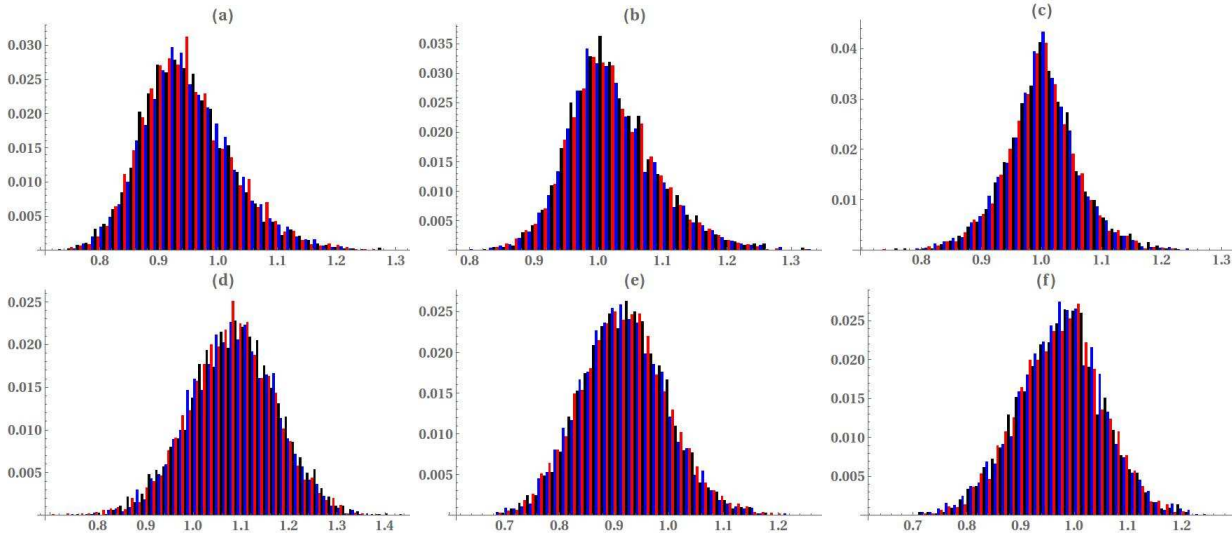

FIG. 2. (Color.) The complementarity relation, again for three qubit systems, but for rank 2 states. The different panels exhibit histograms for the sum of two quantities, viz. the normalized purity and a normalized correlation. The correlation is the negativity in panel (a), logarithmic negativity in (b), quantum mutual information in (c), measured quantum mutual information in (d), quantum discord in (e), and quantum work deficit in (f). All other considerations are the same as in Fig. 1.

The quantum mutual information [28, 29] of a bipartite quantum state ϱ_{XY}, identified with the total correlation in the state, is defined as $I'_{X:Y} = S_X + S_Y - S_{XY}$. Using the Araki-Lieb triangle inequality [30], $|S_X - S_Y| \le S_{XY}$, it follows that

$$I'_{X:Y} \le 2\min\{S_X, S_Y\} \le 2\min\{\log_2 d_X, \log_2 d_Y\}. \quad (5)$$

The Araki-Lieb inequality therefore helps us to obtain the relations in (2) and (3) with the normalized correlation $\mathcal{Q}$ replaced by the normalized quantum mutual information, with the additional property that the parallel of the condition $\mathcal{Q}'_{AB:C} \le \log_2 d_C$ being automatically satisfied here. More precisely, we have

$$\mathcal{P}_{AB} + \mathcal{I}_{AB:C} \le 1 \text{ when } d_{AB} \le d_C, \quad (6)$$

$$\mathcal{P}_{AB} + \mathcal{I}_{AB:C} \le 2 - \frac{\log_2 d_C}{\log_2 d_{AB}} \text{ when } d_{AB} > d_C, \quad (7)$$

where $\mathcal{I}_{AB:C} = \frac{I'_{AB:C}}{2\min\{\log_2 d_{AB}, \log_2 d_C\}}$.

For quantum states of three qudits, we again have the dimension-independent complementarity bound,

$$\mathcal{P}_{AB} + \mathcal{I}_{AB:C} \le \frac{3}{2}, \quad (8)$$

and again for three qubits, the GHZ state saturates the bound.

The (classical) mutual information [28] between two observables $\mathcal{X}$ and $\mathcal{Y}$ of the systems X and Y is given by

$$\tilde{I}'_{\mathcal{X}:\mathcal{Y}} = H(\mathcal{X}) + H(\mathcal{Y}) - H(\mathcal{X}, \mathcal{Y}), \quad (9)$$

where $H(\mathcal{X}) = -\sum_i p_i^{\mathcal{X}} \log_2 p_i^{\mathcal{X}}$ is the Shannon entropy of the observable $\mathcal{X}$, with the observable $\mathcal{X}$ having been measured in the state ϱ_{XY} and the outcomes x_i obtained with the Born probabilities $p_i^{\mathcal{X}} = \text{tr}_X(\text{tr}_Y(\varrho_{XY})\mathcal{X})$.

$H(\mathcal{Y})$ is similarly defined, and $H(\mathcal{X}, \mathcal{Y})$ is the joint entropy of the observable $\mathcal{X} \otimes \mathcal{Y}$ when measured in the state ρ_{XY}. Now the classical mutual information is bounded above by the quantum one [31], and so the relations for the quantum mutual information derived above are also true for the classical variety. We will use the notation $\tilde{I}_{\mathcal{X}:\mathcal{Y}} = \frac{\tilde{I}'_{\mathcal{X}:\mathcal{Y}}}{2\min\{\log_2 d_X, \log_2 d_Y\}}$.

Since quantum mutual information is non-increasing under discarding of parties [29], and since classical mutual information is a lower bound for the quantum one [31], the quantum mutual information in the $AB : C$ bipartition, in the complementarity, can be replaced by the minimum of the quantum or classical mutual information in $A : C$ and $B : C$. Similarly, the quantum correlation in the $AB : C$ partition can be replaced by the minimum of the quantum correlation in $A : C$ and $B : C$, by assuming that the quantum correlation is non-increasing under discarding of parties.

As mentioned earlier, the complementarity relations are also true for N-party systems in the following sense. We envisage a quantum communication protocol of $N - r$ "legitimate" parties, and a further r "eavesdroppers" who are trying to obtain some information from the legitimate users. The entire system of N parties share a quantum state ρ. It is now possible to obtain complementarities in this N-party system between purity of the state of the $N - r$ legitimate users and correlation in the legitimate users versus eavesdroppers bipartition.

It is clear that the complementarity relations considered will also be true for $\mathcal{P} + \min\{\mathcal{Q}_{A:C}, \mathcal{Q}_{B:C}\}$, provided $\mathcal{Q}$ is non-increasing under discarding of subsystems. Numerical analysis of this quantity for 10^4 Haar uniformly generated states of three qubits, separately for rank-1, 2, and 3 states, reveals that it is indeed true for all the correlation measures considered. See Fig. 3.
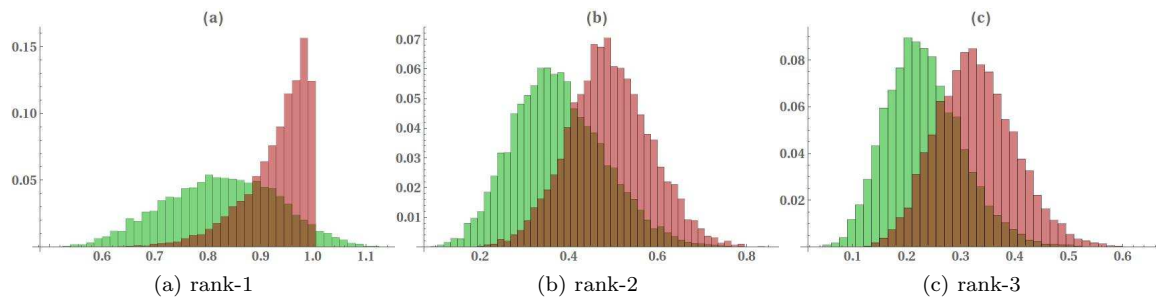

(a) rank-1   (b) rank-2   (c) rank-3

FIG. 3. (Color.) The quantity $\mathcal{P} + \min\{\mathcal{Q}_{A:C}, \mathcal{Q}_{B:C}\}$. The different panels exhibit overlapping histograms, of three qubit states of different ranks, for this quantity. The green (red) bars correspond to the cases when $\mathcal{Q}$ represents the normalized negativity (normalized quantum mutual information). The panels (a), (b), and (c) are respectively for ranks 1, 2, and 3. For each rank, we Haar uniformly generate 10^4 three qubit states of that rank. The vertical axis represents the relative frequency of occurence of a randomly generated three-qubit state of the considered rank in the corresponding range of the sum of the two quantities on the horizontal axis. All quantities are dimensionless.

III. TIGHTNESS OF THE COMPLEMENTARITY RELATIONS

We now investigate how tight the complementarity relations are for the different measures. To examine the tightness of the relation, we compute the average perpendicular distance of the points from the straight line representing the corresponding complementarity relation. The average is performed over the uniform Haar distribution for every considered rank, and over the corresponding quantum state space (for that rank). In Table I, we list the average perpendicular distances for the different measures for states of different ranks. We see that rank-1 (pure) quantum states satisfy the complementarity relations quite tightly, while it becomes comparatively

weaker with the increase in the rank. See also Figs. 1 and 2 in this regard. Note that increasing the rank of the density matrix, typically, increases its mixedness, driving it towards the (normalized) identity matrix, for which the sum of the purity and any of the correlation measures considered vanishes. It is therefore expected that the average perpendicular distance would increase with increasing rank, as also observed in Table I.

	rank-1	rank-2	rank-3	rank-4
$\mathcal{N}$	0.043	0.390	0.590	0.706
$E_{\mathcal{N}}$	0.013	0.338	0.531	0.651
$\mathcal{I}$	0.093	0.354	0.509	0.612
$\mathcal{J}$	0.093	0.296	0.481	0.605
$\mathcal{D}$	0.093	0.412	0.536	0.619
$\triangle$	0.093	0.372	0.504	0.594

TABLE I. Average perpendicular distance of the point $(\mathcal{P}, \tilde{\mathcal{Q}})$ from the line $\mathcal{P} + \tilde{\mathcal{Q}} = \frac{3}{2}$, where $\tilde{\mathcal{Q}}$ represents the normalized versions of negativity ($\mathcal{N}$), or logarithmic negativity ($E_{\mathcal{N}}$), or quantum mutual information ($\mathcal{I}$), or measured quantum mutual information $\mathcal{J}$, or quantum discord ($\mathcal{D}$), or quantum work-deficit ($\triangle$). The different columns represent values for different ranks of the three qubit states, while the different rows are for different measures. For every rank, 10^4 states are generated Haar uniformly.

IV. APPLICATION IN QUANTUM CRYPTOGRAPHY

Let us now discuss whether the derived complementarity relations can have implications in quantum information protocols.

The QKD setup.– We consider the QKD protocol proposed in Ref. [12] (see also [7]) that is a modification of the Ekert 1991 protocol [4]. Suppose that Alice and Bob share a two-party state, and Alice chooses between the measurement settings A_0, A_1, and A_2, while Bob chooses between B_1 and B_2 on their respective portions of the shared state. Each measurement is assumed to have two outcomes. The measurement results of A_0 and B_1 are used to obtain the raw key, and the corresponding bit error rate is given by $e = \text{prob}(a_0 \neq b_1)$, where a_0 and b_1 are the measurement results of A_0 and B_1. We will now use the complementarity between purity and quantum mutual information, as obtained above in Sec. II, to establish the security of the protocol. Indeed, we will obtain a bound on the secret key rate by using the complementarity.

A potential eavesdropper, Eve, denoted by E, tries to gather information about the key of Alice and Bob. To this end, Eve plants ancillas near the channels carrying the states of Alice and Bob, and consequently, the Alice-Bob-Eve trio share the state ρ_{ABE}.

Key rates.– The optimal key rate for individual attacks and obtained via one-way communication is provided by the Csiszár-Körner criterion [32], given by

$$r_{CK} = \tilde{I}'_{A_0:B_1} - \min\{\tilde{I}'_{A_0:\tilde{E}}, \tilde{I}'_{B_1:\tilde{E}}\}, \qquad (10)$$

where we have used the notation $\tilde{E}$ for the measurement setting at Eve, and $\tilde{I}'_{A_0:B_1} = 1 - h(e)$, with $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$ being the binary entropy for $0 \leq p \leq 1$.

Writing the complementarity between purity and quantum mutual information of ρ_{ABE} as

$$\mathcal{P}_{AB} + \mathcal{I}_{AB:E} \leq b, \qquad (11)$$

we have

$$\frac{I'_{AB:E}}{2\min\{\log_2 d_{AB}, \log_2 d_E\}} \leq b - \mathcal{P}_{AB}, \qquad (12)$$

following ineqs. (6) and (7). Now, quantum mutual information is non-decreasing under discarding of subsystems [29], so that $I'_{A:E} \leq I'_{AB:E}$ and $I'_{B:E} \leq I'_{AB:E}$. Therefore,

$$\min\{I'_{A:E}, I'_{B:E}\} \leq 2\min\{\log_2 d_{AB}, \log_2 d_E\}(b - \mathcal{P}_{AB}). \qquad (13)$$

Furthermore, classical mutual information is upper bounded by the corresponding quantum mutual information [31], i.e. $\tilde{I}'_{X:Y} \leq I'_{X:Y}$, resulting in

$$\min\{\tilde{I}'_{A_0:\tilde{E}}, \tilde{I}'_{B_1:\tilde{E}}\} \leq 2\min\{\log_2 d_{AB}, \log_2 d_E\}(b - \mathcal{P}_{AB}). \qquad (14)$$

Therefore, we have

$$r_{CK} \geq 1 - h(e) - 2\min\{\log_2 d_{AB}, \log_2 d_E\}(b - \mathcal{P}_{AB}). \qquad (15)$$

It is reasonable to allow the eavesdropper to be of a larger dimension, so that we choose $d_{AB} \leq d_E$, whence

$$r_{CK} \geq 1 - h(e) - 2S(\rho_{AB}). \qquad (16)$$

To illustrate the rate obtained, consider that the legitimate users of the key distribution channel share the Werner state [33], given by $\rho_W = p|\phi^+\rangle\langle\phi^+| + (1-p)\frac{1}{2}I_2 \otimes \frac{1}{2}I_2$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $0 \leq p \leq 1$, and I_2 denotes the identity operator on the qubit Hilbert space. In this case, $e = \frac{1}{2}(1 - p)$, and the entropy of the shared state between Alice and Bob is given by $H(\{\frac{1}{2}e, \frac{1}{2}e, \frac{1}{2}e, 1 - \frac{3}{2}e\})$, where $H(\{p_i\}) = -\sum_i p_i \log_2 p_i$ is the Shannon entropy of the probability distribution $\{p_i\}$. Consequently, the maximal bit error rate that is allowed before the protocol becomes insecure is about 3.6%. The Shor-Preskill security proof provides a rate of 11% [2].

Let us note here that if Alice and Bob share a pure state, then the quantum mutual information between the Alice-Bob system and the eavesdropper vanishes. It is therefore apparent that for large values of p, in the shared Werner state, a stronger complementarity similar to $\mathcal{P}_{AB} + \mathcal{I}_{AB:E} \lesssim 1$ is active, and this is independent of whether $d_{AB} \leq d_E$ is valid. Using this stronger complementarity would provide a security proof without the assumption $d_{AB} \leq d_E$. We also note that there are recent results on device independent protocols for estimating entropies [34], which if possible to be used in the present scenario, would provide the potential for a device independent security proof of quantum cryptography [7, 8] via information complementarity relations, instead of Bell inequality violations. We mention here that the Devetak-Winter result [19] also provides an entropic bound on the secret key rate for collective attacks. We however believe that the bound obtained here is via methods that provide an independent perspective on the Ekert protocol and its security, albeit for individual attacks.

V. CONCLUSION

To sum up, we have derived complementarity relations in multiparty quantum systems connecting purity of a part of the system with correlations – classical, quantum, and total – between that part and the rest of the system. We found that they have the potential to provide bounds on the secret key rates in quantum cryptography.

A relation for three-party quantum states that has a similar topology to the ones derived here is in Ref. [35], where local measurement-induced changes in two-party entanglement is related to the measurement-induced changes in entanglement of those two parties with the third party, in a three party system. See also Ref. [21].

Since the derived complementarity relations are rather unalike to the previously existing ones, they are potentially useful in getting perspectives, hitherto not known, on protocols and phenomena involving many parties, including e.g. the black hole information paradox [36].

ACKNOWLEDGMENTS

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