Activation of Nonmonogamous Multipartite Quantum States

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The monogamy relation for quantum correlations is not satisfied by all measures for all multiparty quantum states. We prove that an arbitrary quantum state which is nonmonogamous for negativity will become monogamous if a finite number of copies of the same state is provided. We refer this as activation of nonmonogamous states. We also show that multiple copies of a state satisfy monogamy for negativity if it does so for a single copy. The results are true for all quantum states of an arbitrary number of parties. Moreover, we find that two different three-qubit pure states which individually violate monogamy relation for negativity, taken together can satisfy the three-party monogamy relation. This holds for almost all three-qubit pure states while it is true for all three-qubit pure states when a four-party monogamy relation is used to check for their activation. We finally connect monogamy of negativity with genuine multipartite entanglement.

I. INTRODUCTION

Quantum mechanical postulates of linearity and unitarity put restrictions on tasks like cloning, broadcasting and deleting [1–5] which can be executed efficiently in a classical world. However, these no-go theorems turn out to be crucial in several quantum information protocols, including secure quantum communication [6] and quantum teleportation [7].

Constraints in a many-body scenario include the existence of bound entangled states [8] and unconvertible state pairs by local quantum operations and classical communication (LOCC) [9]. Bound entangled states are ones that require entanglement for their creation implying a nonvanishing entanglement cost [8], although after preparation, it is impossible to distil [10] its entanglement content. On the other hand, it was shown by using the mazorization criterion that there are pairs of bipartite pure quantum states that cannot be transformed into each other by LOCC [9]. Existence of these states seems to limit the protocols that can be performed by using them. However, this is not true in general. It was found that although the bound entangled states are not useful for quantum teleportation [11], these states can be useful in increasing the teleportation fidelity of distillable states [12] (see also [11]). It was also shown that a pair of bipartite quantum states can be distillable even though the individual states are respectively bound entangled with positive partial transpose and conjectured bound entangled with negative partial transpose [13]. In a different spirit, a locally unconvertible bipartite pure quantum state pair may become convertible with the help of another entangled state, a process known as catalysis [14].

In the multiparty domain, there appears the restriction of monogamy so that entanglement can not be shared arbitrarily between different partners. Monogamy of entanglement puts limitations on sharing of entanglement among different subsystems of a multipartite state which is not the case for the distribution of classical correlation [15–17]. Monogamy of entanglement has practical ap-

plications in a variety of areas including quantum cryptography [16]. In this sense, monogamous states can be thought of as a useful resource in quantum communication between several parties. However, there exist multiparty entangled states which are nonmonogamous under the known measures of quantum correlation like negativity [18, 19], entanglement of formation [17, 20], quantum discord [21, 22], and quantum work-deficit [23, 24].

At this point, a natural question can be about the conditions for which multiparty states become monogamous with respect to some quantum correlation measure. Specifically, in this paper, we address the following question:

Given a bipartite quantum correlation measure $\mathcal{Q}$, if a multiparty quantum state is nonmonogamous under $\mathcal{Q}$, is it possible to obtain monogamy with respect to $\mathcal{Q}$ when finite copies of the same state are available or with addition of another state which, considered alone, is also nonmonogamous with respect to $\mathcal{Q}$?

We find that the answer is in the affirmative for a nonadditive quantum correlation measure and we term the phenomenon as activation of nonmonogamous states. In particular, we prove that in the case of negativity, if a multiparty state is monogamous, then its multicopy versions, i.e., several copies of the state also remains so. We then show that a nonmonogamous state with respect to negativity becomes monogamous, when several copies of the same state are provided. The result is true for both pure and mixed states for arbitrary number of parties. From extensive numerical searches for three-qubit pure states, we observe that to get rid of nonmonogamous nature of a state, the states from the W-class require much higher number of copies than the states from the Greenberger-Horne-Zeilinger (GHZ)-class [25–28]. We also prove that if two different states individually satisfy the monogamy inequality for negativity, they continue to obey so even jointly. Moreover, we observe that almost all pairs of three-qubit pure states can jointly obey three-party monogamy with respect to negativity, even when they individually are nonmonogamous. We also introduce a different kind of activation protocol of an N-party state, where two copies of N-party state is con-
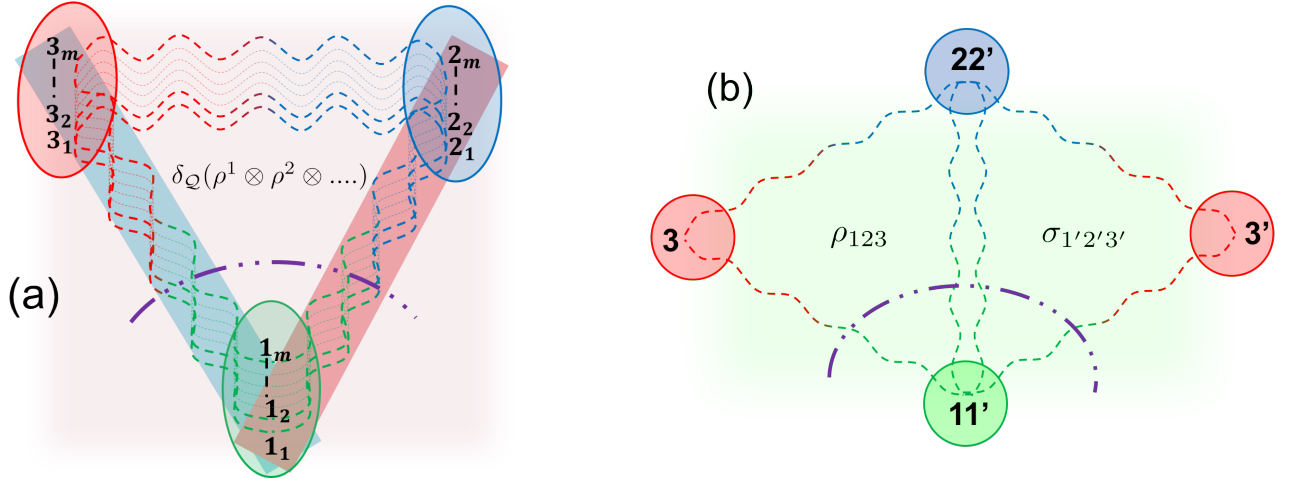
FIG. 1. (Color online.) The two scenarios of activation. (a) The multicopy case. Schematic diagram of $\delta_{\mathcal{Q}}(\rho^1_{1_1 2_1 3_1} \otimes \rho^2_{1_2 2_2 3_2} \otimes \ldots)$, where three-party monogamy score is considered. For activation, we consider two scenarios – (i) when all ρ^i's are the same and (ii) when they are different. In both the cases, activation occurs, when $\delta_{\mathcal{Q}}(\rho^1_{1_1 2_1 3_1} \otimes \rho^2_{1_2 2_2 3_2} \otimes \ldots) \geq 0$ although $\delta_{\mathcal{Q}}(\rho^i) < 0$. (b) A schematic representation of four-party monogamy score of $\delta_{\mathcal{Q}}(\rho_{123} \otimes \sigma_{1'2'3'})$, with 3 and 3' being located on separate locations. We look for positivity of $\delta_{\mathcal{Q}}(\rho_{123} \otimes \sigma_{1'2'3'})$ with $\delta_{\mathcal{Q}}(\rho_{123}) < 0$ and $\delta_{\mathcal{Q}}(\sigma_{1'2'3'}) < 0$

sidered in an $N + 1$-party situation, and the monogamy of the state in the $N + 1$-party situation is considered. See Fig. 1(b) for an illustration. In this situation, we numerically confirm that all three-qubit nonmonogamous states can be activated for negativity. Finally, we establish a relation between "monogamy scores" for negativity of single or two copies of three-qubit pure states and the amount of a genuine multipartite entanglement.

The paper is organized as follows. In Sec. II, we introduce the concept of activation of nonmonogamous states after briefly defining the monogamy score for a quantum correlation measure. Negativity and its closed form for multiple copies of arbitrary states are discussed in the next section (Sec. III). The main results of this paper are presented in Sec. IV. Specifically, we derive the effects on monogamy score for negativity with multiple copies in Sec. IV A, and nonadditivity of monogamy score for negativity is shown in Sec. IV B. In Sec. V, we make a connection between negativity monogamy score and a genuine multipartite entanglement. We present a conclusion in Sec. VI .

II. MONOGAMY SCORE OF QUANTUM CORRELATION MEASURES

Suppose $\rho_{12\ldots N}$ is an arbitrary N-party quantum state, and let $\mathcal{Q}$ be a bipartite entanglement measure. The "monogamy score" [15–17, 29] of $\mathcal{Q}$ for the state $\rho_{12\ldots N}$, with the party 1 as the "nodal" observer, is defined as

$$\delta_{\mathcal{Q}} \equiv \delta_{\mathcal{Q}}(\rho_{12\ldots N}) = \mathcal{Q}_{1:rest} - \sum_{i=2}^{N} \mathcal{Q}_{1:i}, \quad (1)$$

where $\mathcal{Q}_{1:rest} \equiv \mathcal{Q}(\rho_{1:rest})$ and $\mathcal{Q}_{1:i} \equiv \mathcal{Q}(\rho_{1:i})$. One can consider another party as the nodal observer as well. $\delta_{\mathcal{Q}}(\rho_{12\ldots N}) \geq 0$ implies that the state, $\rho_{12\ldots N}$, is monogamous with respect to $\mathcal{Q}$, and otherwise it is nonmonogamous. And when it is non-negative for all states, then the quantum correlation measure $\mathcal{Q}$ is said to be monogamous. If there exists a state for which $\mathcal{Q}$ fails to satisfy the monogamy condition, then $\mathcal{Q}$ is called nonmonogamous. The squares of concurrence [15], negativity [19], and quantum discord [30] have been shown to satisfy monogamy, although without the power, they violate the relation [22, 31]. Recently, it was shown that in general, given any quantum correlation measure, $\mathcal{Q}$, for which $\delta_{\mathcal{Q}} < 0$ for some state, it is always possible to find a monotonically increasing function of $\mathcal{Q}$ satisfying monogamy for that state [24]. In this paper, instead of taking functions of $\mathcal{Q}$, we choose a different path to obtain monogamy.

A. Activation vs. additivity of quantum correlation

Let us consider an N-party state $\rho_{12\ldots N}$ which does not satisfy the monogamy relation for some $\mathcal{Q}$, i.e., $\delta_{\mathcal{Q}}(\rho_{12\ldots N}) < 0$. Considering m copies of the same state, the monogamy score is

$$\delta_{\mathcal{Q}}^{(m)} \equiv \delta_{\mathcal{Q}}(\rho_{12\ldots N}^{\otimes m}) = \mathcal{Q}_{1_1 1_2 \ldots 1_m :rest} - \sum_{i=2}^{N} \mathcal{Q}_{1_1 1_2 \ldots 1_m : i_1 i_2 \ldots i_m},$$

$$(2)$$

where the notation $i_r (r = 1, 2, \ldots m)$ denotes the i-th party of the r-th copy of the state. For a given $\mathcal{Q}$, we want to know the status of $\delta_{\mathcal{Q}}(\rho_{12\ldots N}^{\otimes m})$. If we find that

$\delta_{\mathcal{Q}}(\rho_{12\ldots N}^{\otimes m}) \geq 0$, with $\delta_{\mathcal{Q}}(\rho_{12\ldots N}) < 0$, we say that "activation" of the nonmonogamous state, $\rho_{12\ldots N}$, has occurred with respect to $\mathcal{Q}$. See Fig. 1(a) for a schematic representation of the activation.

In another case, we also want to know the status of $\delta_{\mathcal{Q}}(\rho_{12\ldots N} \otimes \sigma_{1'2'\ldots N'})$ where $\rho_{12\ldots N}$ and $\sigma_{1'2'\ldots N'}$ are both N-party states. Specifically, we want to see whether for a given $\mathcal{Q}$ and $\rho_{12\ldots N}$ with $\delta_{\mathcal{Q}}(\rho_{12\ldots N}) < 0$, it is possible to find a $\sigma_{1'2'\ldots N'}$ having $\delta_{\mathcal{Q}}(\sigma_{1'2'\ldots N'}) < 0$ such that $\delta_{\mathcal{Q}}(\rho_{12\ldots N} \otimes \sigma_{1'2'\ldots N'}) \equiv \mathcal{Q}_{11':rest} - \sum_{i,i'=2}^{N,N'} \mathcal{Q}_{11':ii'} \geq 0$. We again call this situation as activation.

In the two preceding cases, we have considered N-party monogamy scores for N-party quantum states, even when more than a single copy of the state is available. Relaxing this restriction opens up a large number of possibilities, of which we choose only one. We consider an $(N+1)$-party monogamy score, $\delta_{\mathcal{Q}}(\rho_{12\ldots N} \otimes \sigma_{1'2'\ldots N'}) \equiv \mathcal{Q}_{11':rest} - \sum_{i,i'=2}^{N-1,N'-1} \mathcal{Q}_{11':ii'} - \mathcal{Q}_{11':N} - \mathcal{Q}_{11':N'}$, and find whether $\delta_{\mathcal{Q}}(\rho_{12\ldots N} \otimes \sigma_{1'2'\ldots N'})$, can be positive when both $\delta_{\mathcal{Q}}(\rho_{12\ldots N}) < 0$ and $\delta_{\mathcal{Q}}(\sigma_{1'2'\ldots N'}) < 0$. This situation can also be called activation of nonmonogamous states. See Fig. 1(b) for a schematic description.

In this respect, it is interesting to note that any quantum correlation measure, $\mathcal{Q}$ is said to be additive, if $\mathcal{Q}(\rho \otimes \sigma) = \mathcal{Q}(\rho) + \mathcal{Q}(\sigma)$. For some measure $\mathcal{Q}$, if we are able to show that activation is possible for all states, it implies that $\delta_{\mathcal{Q}}$ is non-additive.

One should note that both the above questions remain invalid if the entanglement measures are additive like squashed entanglement [32], relative entropy of entanglement for two qubits [33], and logarithmic negativity [18].

III. NEGATIVITY

Having defined the activation of monogamy for arbitrary $\mathcal{Q}$, in the rest of the paper, we consider a specific entanglement measure, namely negativity. We begin by giving a the definition of negativity [18]. Thereafter, we discuss the closed form of negativity for multiple copies of a given state, and also when a pair of two different states is considered .

The negativity of a bipartite quantum state ρ_{12} defined on the composite Hilbert space of $\mathcal{H}_1^{d_1} \otimes \mathcal{H}_2^{d_2}$, is based on the partial transposition criterion [34]. If a bipartite quantum state is separable, then the partial transposed state, $\rho_{12}^{T_1}$, with transposition being taken with respect to the first party, is positive semi-definite. The negativity of ρ_{12} is defined as

$$\mathcal{N}_{12} \equiv \mathcal{N}(\rho_{12}) = \frac{||\rho_{12}^{T_1}||_1 - 1}{2}, \tag{3}$$

where $||\rho||_1$ is the trace norm [35], defined as $||\rho||_1 = \text{tr}(\sqrt{\rho^\dagger \rho})$. The above equation reduces to

$$\mathcal{N}(\rho_{12}) = \sum_j |\lambda_j^n|, \tag{4}$$

where λ_j^n are the negative eigenvalues of $\rho_{12}^{T_1}$. Since partial transposition does not change the trace of the matrix, we have $\text{tr}(\rho_{12}^{T_1}) = \sum_j \lambda_j^n + \sum_k \lambda_k^p = 1$, where λ_k^p are non-negative eigenvalues. The total number of eigenvalues of $\rho_{12}^{T_1}$ is $d_1 d_2$, and we denote them as $\{\lambda_i\}$. The λ_i's can be either λ_j^n or λ_k^p.

A. Closed form of negativity with multiple copies and a pair of states

Let us first calculate the negativity of m copies of ρ_{12}, where $m \in \mathbb{N}$. Without loss of generality, we assume that in all the copies, partial transposition is taken in the first part. Note that $\left(\rho_{12}^{\otimes m}\right)^{T_{1_1 1_2 \ldots 1_m}} = \left(\rho_{12}^{T_1}\right)^{\otimes m}$. Hence, the eigenvalues of $\left(\rho_{12}^{\otimes m}\right)^{T_{1_1 1_2 \ldots 1_m}}$ are $\{\Pi_{j=1}^m \lambda_{i_j}\}$, Here $\rho_{12}^{\otimes m}$ is the shorthand for $\rho_{1_1 2_1} \otimes \rho_{1_2 2_2} \otimes \ldots \otimes \rho_{1_m 2_m}$. Therefore, we get

$$\begin{aligned}
\mathcal{N}(\rho_{12}^{\otimes m}) &= \sum_{r=odd} \sum_{\substack{j_1 \ldots j_r \\ k_{r+1} \ldots k_m}} \binom{m}{r} |\lambda_{j_1}^n| \ldots |\lambda_{j_r}^n| \lambda_{k_{r+1}}^p \ldots \lambda_{k_m}^p \\
&= \sum_{r=odd} \binom{m}{r} \mathcal{N}_{12}^r (1 + \mathcal{N}_{12})^{m-r} \\
&= \frac{1}{2}\left[(1 + 2\mathcal{N}_{12})^m - 1\right].
\end{aligned} \tag{5}$$

Similar calculation also leads to

$$\begin{aligned}
\mathcal{N}(\rho_{12} \otimes \sigma_{1'2'}) &= \sum_{j,k}(|\lambda_{j_\rho}^n| \lambda_{k_\sigma}^p + |\lambda_{j_\sigma}^n| \lambda_{k_\rho}^p) \\
&= \mathcal{N}^\rho(1 + \mathcal{N}^\sigma) + \mathcal{N}^\sigma(1 + \mathcal{N}^\rho),
\end{aligned} \tag{6}$$

where λ_{j_ρ} and λ_{j_σ} denote the eigenvalues of ρ and σ respectively, and the superscripts p and n respectively indicate when the eigenvalue is non-negative and negative. Here $\mathcal{N}^\rho$ represents $\mathcal{N}(\rho_{12})$. Similarly, $\mathcal{N}^\sigma \equiv \mathcal{N}(\sigma_{12})$.

IV. ACTIVATING NONMONOGAMOUS STATES WITH RESPECT TO NEGATIVITY

In this section, we prove the main results. The first subsection is devoted to the situation where many copies of a state is considered, and a single copy of which violates monogamy for negativity. Next, we discuss the status of monogamy score of negativity for a pair of states which are separately nonmonogamous.

A. Effect of multiple copies on monogamy

Before answering the main question, we first ask the following question: If negativity monogamy score is non-negative for a single copy of a state, will it remain so with multiple copies of the same state? The answer is "yes" and it holds in arbitrary dimensions and for arbitrary

number of parties. The proofs are mainly given for three-party states, which can be easily generalized to arbitrary number of parties.

Theorem 1. If negativity of a tripartite state, ρ_{123}, satisfy monogamy i.e. if $\delta_{\mathcal{N}}(\rho_{123}) \geq 0$, then the m copies of the same state also remains monogamous.

Proof : By using Eq. (5), the monogamy score of negativity for the m copies of ρ_{123} can be written as

$$
\begin{aligned}
\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) &= \mathcal{N}(\rho_{1:23}^{\otimes m}) - \mathcal{N}(\rho_{12}^{\otimes m}) - \mathcal{N}(\rho_{13}^{\otimes m}) \\
&= \frac{1}{2}\big[(1 + 2\,\mathcal{N}_{1:23})^m - (1 + 2\,\mathcal{N}_{12})^m \\
&\qquad\qquad - (1 + 2\,\mathcal{N}_{13})^m + 1\big]. \quad (7)
\end{aligned}
$$

We now use the method of induction to prove $\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) \geq 0$ for all m, provided $\delta_{\mathcal{N}}(\rho_{123}) \geq 0$.

For $m = 2$, Eq. (7) reduces to

$$
\begin{aligned}
\delta_{\mathcal{N}}(\rho_{123}^{\otimes 2}) &= 2\big[\mathcal{N}_{1:23}(1 + \mathcal{N}_{1:23}) - \mathcal{N}_{12}(1 + \mathcal{N}_{12}) \\
&\qquad\qquad - \mathcal{N}_{13}(1 + \mathcal{N}_{13})\big] \\
&\geq 2\big[(\mathcal{N}_{12} + \mathcal{N}_{13})(1 + \mathcal{N}_{12} + \mathcal{N}_{13}) \\
&\qquad\quad - \mathcal{N}_{12}(1 + \mathcal{N}_{12}) - \mathcal{N}_{13}(1 + \mathcal{N}_{13})\big] \\
&= 4\,\mathcal{N}_{12}\mathcal{N}_{13} \geq 0, \quad\quad (8)
\end{aligned}
$$

where the inequality follows from $\delta_{\mathcal{N}}(\rho_{123}) \geq 0$. Thus the theorem is true for $m = 2$. Let us now assume that $\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) \geq 0$ for some $m = n$, i.e., $\delta_{\mathcal{N}}(\rho_{123}^{\otimes n}) \geq 0$. This implies that,

$$
(1 + 2\mathcal{N}_{1:23})^n + 1 \geq (1 + 2\mathcal{N}_{12})^n + (1 + 2\mathcal{N}_{13})^n \quad (9)
$$

Then for $m = n + 1$, we have

$$
\begin{aligned}
(1 + 2\mathcal{N}_{123})^{n+1} &= (1 + 2\mathcal{N}_{123})^n(1 + 2\mathcal{N}_{123}) \\
&\geq \big\{(1 + 2\mathcal{N}_{12})^n + (1 + 2\mathcal{N}_{13})^n - 1\big\} \times \\
&\qquad\qquad (1 + 2\mathcal{N}_{12} + 2\mathcal{N}_{13}) \\
&= (1 + 2\mathcal{N}_{12})^{n+1} + (1 + 2\mathcal{N}_{13})^{n+1} - 1 \\
&\quad + 2\mathcal{N}_{12}\big\{\underbrace{(1 + 2\mathcal{N}_{13})^n - 1}_{\geq 0}\big\} + 2\mathcal{N}_{13}\big\{\underbrace{(1 + 2\mathcal{N}_{12})^n - 1}_{\geq 0}\big\} \\
&\geq (1 + 2\mathcal{N}_{12})^{n+1} + (1 + 2\mathcal{N}_{13})^{n+1} - 1. \quad (10)
\end{aligned}
$$

Hence the proof [36]. ∎

Corollary. For an N-party state, $\rho_{12\ldots N}$, $\delta_{\mathcal{N}}(\rho_{12\ldots N}) \geq 0$ implies $\delta_{\mathcal{N}}(\rho_{12\ldots N}^{\otimes m}) \geq 0$ for some $m \in \mathbb{N}$.

Proof : The proof is similar to the proof of Theorem 1. Here, we notice that

$$
\delta_{\mathcal{N}}(\rho_{12\ldots N}^{\otimes 2}) \geq 4 \sum_{i=2, j>i}^{N-1} \mathcal{N}_{1i}\mathcal{N}_{1j}, \quad\quad (11)
$$

which is again a non-negative quantity. Then by using induction, we can have the proof. ∎

We now consider activation of nonmonogamous states for negativity. Again, we give the proof for three-qubit mixed states, which can be easily extended to an arbitrary number of parties.

Theorem 2: For a three-qubit mixed state, ρ_{123}, if the negativity monogamy score is negative, i.e., if $\delta_{\mathcal{N}}(\rho_{123}) < 0$, then $\rho_{123}^{\otimes m}$ becomes monogamous for some positive integer m.

Proof : We again note that Eq. (7) can be expressed with the help of Eq. (5) as

$$
\begin{aligned}
2\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) &= 2\big[\mathcal{N}(\rho_{1:23}^{\otimes m}) - \mathcal{N}(\rho_{12}^{\otimes m}) - \mathcal{N}(\rho_{13}^{\otimes m})\big] \\
&= \sum_{k=1}^{m} \binom{m}{k} 2^k \big(\mathcal{N}_{1:23}^k - \mathcal{N}_{12}^k - \mathcal{N}_{13}^k\big). \quad (12)
\end{aligned}
$$

Since negativity is a convex function [18], and the squared negativity is monogamous for three- and more qubit pure [19, 31], and mixed [37] quantum states, we can rewrite Eq. (12) as

$$
\begin{aligned}
2\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) &= \sum_{k=2}^{m} \binom{m}{k} 2^k \big(\mathcal{N}_{1:23}^k - \mathcal{N}_{12}^k - \mathcal{N}_{13}^k\big) \\
&\quad + 2m\big(\mathcal{N}_{123} - \mathcal{N}_{12} - \mathcal{N}_{13}\big). \quad (13)
\end{aligned}
$$

Now the last term of the above equation is negative, since we assume that $\delta_{\mathcal{N}}(\rho_{123}) < 0$ for ρ_{123} [31], and it is linear with m. On the other hand, the first term is polynomial of degree $k \geq 2$. It is therefore always possible to choose some positive integer $m \geq 2$ such that the first term of $\delta_{\mathcal{N}}(\rho_{123}^{\otimes m})$ is bigger than the last one. Hence the proof. ∎

Note 1. The above proof uses the fact that negativity is convex and its squared monogamy score is non-negative for pure as well as mixed three-qubit states. Since squared negativity satisfies monogamy for all states with arbitrary number of qubits, Theorem 2 also holds for arbitrary multiqubit states.

Note 2. The above theorem also proves that negativity monogamy score is not additive.

To visualize the effect of multiple copies on negativity monogamy score, we Haar uniformly generate 10^7 three-qubit pure states from both the GHZ and the W-classes. depicts The histogram in Fig. 2 depicts the probability for arbitrary nonmonogamous pure states becoming activated with the increase of m. The analysis also reveals that there is a marked difference between the GHZ-class and the W-class states with respect to their activation. In particular a W-class state which is nonmonogamous with respect to negativity, generally requires a higher value of m for satisfying the monogamy relation as compared to a GHZ-class state. For example, we find that there are only few random states from the GHZ-class, which require more than 10 copies to obtain monogamy for negativity. However, this is not the case for the W-class states as shown in the inset of Fig. 2.

In Fig. 3, we compare the distribution of negativity monogamy scores of $|\psi_{123}\rangle$ and $|\psi_{123}\rangle^{\otimes 2}$, when we Haar uniformly generate three-qubit pure states. We observe that almost all states from the GHZ-class become monogamous when two copies are considered, while this is not the case for the states from the W-class. The difference in the behavior of the negativity monogamy scores of $|\psi_{123}\rangle$,

and $|\psi_{123}\rangle^{\otimes 2}$ is analysed in the succeeding section from the perspective of genuine multiparty entanglement.
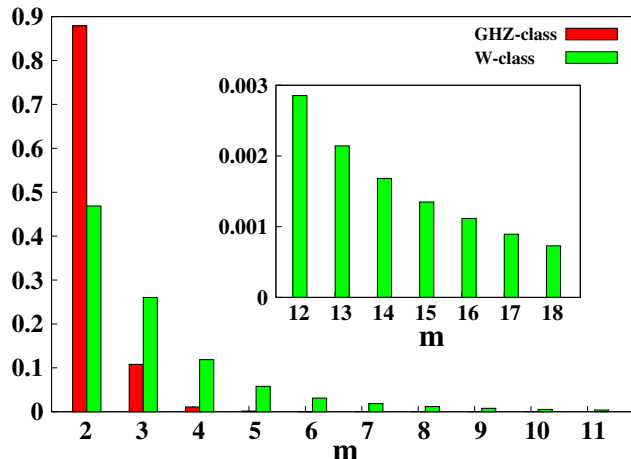

FIG. 2. (Color online.) Histogram of probability for obtaining $\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) \geq 0$, against the minimum number of copies m. We Haar uniformly generate 10^7 states from both the GHZ- and the W-classes. We observe that approximately 88% states from the GHZ-class, which do not satisfy monogamy when a single copy is provided, become monogamous when two copies are considered. For the W-class, the same is 47%. (Inset) The same plot for only W-class states but for higher m. All the axes are dimensionless.
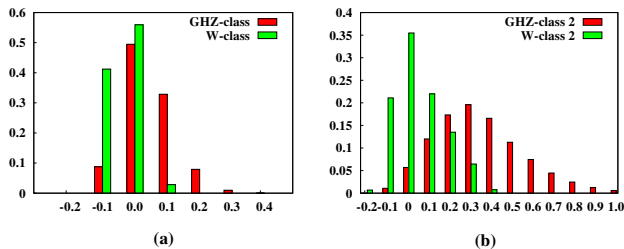

FIG. 3. (Color online.) (a) Distribution of negativity monogamy score of three-qubit pure states, $|\psi\rangle$. On the horizontal axis, the value of the tick (x_i) indicates that the corresponding probability columns are for $x_i \leq \delta_{\mathcal{N}}(\psi) < x_{i+1}$, where x_{i+1} denotes the value of the next tick. A higher number of states from the GHZ class satisfy monogamy compared to the W-class states, confirming the results in Fig. 2. (b) Similar distribution is depicted for $|\psi\rangle^{\otimes 2}$. Other notations are the same. We see that almost all states from the GHZ-class becomes monogamous when two copies are considered. The abscissa is in ebits while the ordinate is dimensionless.

B. Nonadditivity of monogamy score for negativity

In this section, we choose two tripartite states, ρ_{123} and $\sigma_{1'2'3'}$ such that

$$\mathcal{N}(\rho_{1:23}) < \mathcal{N}(\rho_{12}) + \mathcal{N}(\rho_{13}),$$
$$\mathcal{N}(\sigma_{1':2'3'}) < \mathcal{N}(\sigma_{1'2'}) + \mathcal{N}(\sigma_{1'3'}). \quad (14)$$

We are interested in the sign of

$$\delta_{\mathcal{N}}(\rho_{123} \otimes \sigma_{1'2'3'})$$
$$= \mathcal{N}(\rho_{1:23} \otimes \sigma_{1':2'3'}) - \mathcal{N}(\rho_{12} \otimes \sigma_{1'2'}) - \mathcal{N}(\rho_{13} \otimes \sigma_{1'3'}). \quad (15)$$

In particular, corresponding to each ρ_{123}, our task is to find out $\sigma_{1'2'3'}$ so that both relations in Eqs. (14) are satisfied, where $\delta_{\mathcal{N}}(\rho_{123} \otimes \sigma_{1'2'3'}) \geq 0$. By using Eq. (6), we obtain that the above relations hold, only when the expression given by

$$2\mathcal{N}_{1:23}^{\rho}\mathcal{N}_{1':2'3'}^{\sigma} - \left(2\mathcal{N}_{12}^{\rho}\mathcal{N}_{1'2'}^{\sigma} + 2\mathcal{N}_{13}^{\rho}\mathcal{N}_{1'3'}^{\sigma}\right)$$
$$- \left(|\delta_{\mathcal{N}}(\rho_{1:23})| + |\delta_{\mathcal{N}}(\sigma_{1':2'3'})|\right) \geq 0. (16)$$

In this case, we numerically simulate 10^7 three-qubit pure states from the GHZ-class. We find that it is always possible to find a pure state $\sigma = |\phi\rangle\langle\phi|$ from the GHZ-class which satisfies both the relations in (14), and $\delta_{\mathcal{N}}(\rho_{123} \otimes \sigma_{1'2'3'}) \geq 0$. Among Haar uniformly generated GHZ-class states, we find that there are only 8.8% states, for which $\delta_{\mathcal{N}} < 0$.

On the other hand, a large number of states from the W-class show nonmonogamous nature for $\mathcal{N}$. In particular, within 10^7 randomly generated states of the W-class, 43.3% states are nonmonogamous. Numerical simulations show that in this case, there are less than 1% states which do not satisfy monogamy relation for negativity even with the help of σ, chosen from the W-class as well as from the GHZ-class.

With these observations, we can conclude that almost all three-qubit pure states satisfy negativity monogamy relation with the help of another nonmonogamous state from the same class. It also confirms the non-additivity of negativity monogamy score.

Let us now consider another activation scenario, given in Fig. 1(b). We again choose two tripartite states which satisfy the relations in (14). We now find the condition when the four-party monogamy score becomes nonnegative, i.e., when

$$\delta_{\mathcal{N}}(\rho_{123} \otimes \sigma_{1'2'3'})$$
$$= \mathcal{N}(\rho_{1:23} \otimes \sigma_{1':2'3'}) - \mathcal{N}(\rho_{12})$$
$$- \mathcal{N}(\rho_{13} \otimes \sigma_{1'3'}) - \mathcal{N}(\sigma_{1'2'}) \geq 0. \quad (17)$$

The above condition reduces to

$$2\mathcal{N}_{1:23}^{\rho}\mathcal{N}_{1':2'3'}^{\sigma} - (2\mathcal{N}_{13}^{\rho}\mathcal{N}_{1'3'}^{\sigma} + |\delta_{\mathcal{N}}(\rho_{1:23})| + |\delta_{\mathcal{N}}(\sigma_{1':2'3'})|) \geq 0. \quad (18)$$

This is similar to the one obtained for three-party monogamy score, given in (16). However, comparing
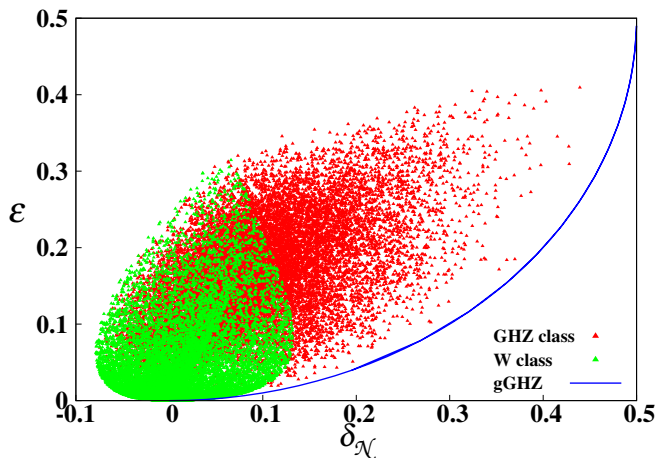
FIG. 4. (Color online.) Negativity monogamy score vs. GGM. We Haar uniformly generate 10^7 arbitrary three-qubit pure states from the GHZ (red dots) and the W-classes (green dots). $\delta_{\mathcal{N}}$ is plotted along the abscissa while the $\mathcal{E}$ is plotted along the ordinate. The blue line corresponds to the generalized GHZ state. The horizontal axis is in ebits, where the vertical axis is dimensionless.

both the relations, one should note that (16) contains one extra term which is absent in (18). Therefore, all the states from the GHZ-class which satisfy three-party negativity monogamy relation, continue to obey four-party monogamy condition. It will be interesting to find whether the 1% states from the W-class which cannot be activated by using three-party negativity monogamy score, can satisfy the four-party monogamy relation. Numerically, we find that it is indeed the case.

V. RELATION OF NEGATIVITY MONOGAMY SCORE WITH GENUINE MULTIPARTY ENTANGLEMENT MEASURES

We now ask whether activation of a state is related to it's multipartite entanglement content. First, we establish a relation between $\delta_{\mathcal{N}}(|\psi\rangle)$ and genuine multiparty entanglement measure, with the later being quantified by the generalized geometric measure (GGM) [38] for three-qubit pure states. We then try to build a connection $\delta_{\mathcal{N}}(|\psi\rangle^{\otimes 2})$ and the GGM of $|\psi\rangle$ of arbitrary $|\psi\rangle$. A multiparty pure quantum state is called genuinely multiparty entangled if it is entangled in all bipartitions. The generalized geometric measure of an arbitrary state is the minimum distance of the given state from the nongenuinely mutiparty entangled states, and can be simplified for a three-party quantum state in arbitrary dimensions as

$$\mathcal{E}(|\psi_{123}\rangle) = 1 - \max\{\lambda_1^{\max}, \lambda_2^{\max}, \lambda_3^{\max}\}, \quad (19)$$

where $\lambda_i^{\max}(i = 1, 2, 3)$ denotes the maximum eigenvalue of the single-site reduced density matrices, $\rho_i(i = 1, 2, 3)$,

of $|\psi_{123}\rangle$. In this respect, we consider the generalized GHZ state [26], given by $|gGHZ\rangle = \sqrt{\alpha}|000\rangle + \sqrt{1-\alpha}e^{i\phi}|111\rangle$ which plays an important role to obtain the connection. In particular, we have the following Proposition:
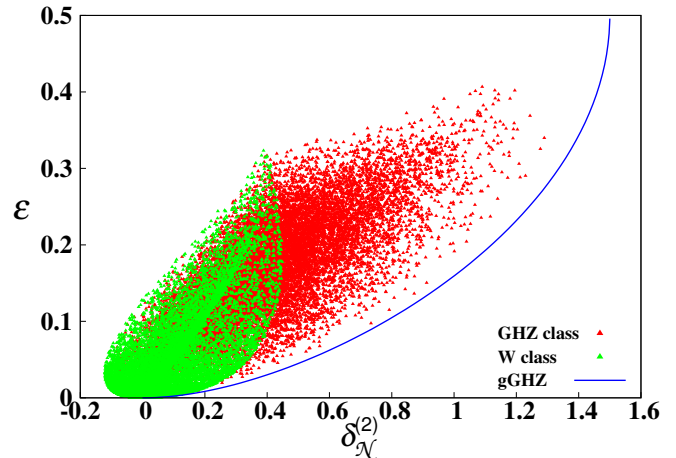

FIG. 5. (Color online.) Negativity monogamy score of two copies of randomly generated states against GGM. All notations are same as in Fig. 4.

Proposition. The GGM of an arbitrary three qubit pure state is bounded below by the GGM of the generalized three qubit GHZ state, whenever $\delta_{\mathcal{N}}(|gGHZ\rangle) = \delta_{\mathcal{N}}(|\psi\rangle)$ and when the nodal observer gives a maximum eigenvalue involved in GGM.

Proof: The negativity monogamy scores with the first party as the nodal observer, of the generalized GHZ state and an arbitrary three-qubit pure state, $|\psi\rangle$, are respectively given by

$$\delta_{\mathcal{N}}(|gGHZ\rangle) = \sqrt{\alpha(1-\alpha)}, \quad (20)$$

$$\delta_{\mathcal{N}}(|\psi\rangle) = \sqrt{\lambda_1(1-\lambda_1)} - \mathcal{N}_{12} - \mathcal{N}_{13} \leq \sqrt{\lambda_1(1-\lambda_1)}, \quad (21)$$

where $\lambda_1 \geq 1/2$ is the largest eigenvalue of the local density matrix ρ_1, of $|\psi\rangle$. Using (20) and (21), we get $\alpha \geq \lambda_1$. Now the GGM of $|\psi\rangle$ is $1 - \max\{\lambda_1, \lambda_2, \lambda_3\}$, where λ_i s are the maximal eigenvalues of the local density matrices of $|\psi\rangle$, while $\mathcal{E}(|gGHZ\rangle) = 1 - \alpha$. If $\lambda_1 \geq \lambda_2, \lambda_3 \geq 1/2$, we have $\mathcal{E}(|gGHZ\rangle) \leq \mathcal{E}(|\psi\rangle)$, and hence the proof. ∎

Numerical simulations indicate that the lower bound is true irrespective of whether the maximum eigenvalue is obtained from the nodal observer or not. The numerical analysis is performed by generating 10^7 random three-qubit pure states, Haar uniformly, from both GHZ and W-classes (see Fig. 4).

Similar analysis can also be carried out between $\delta_{\mathcal{N}}(|\psi\rangle^{\otimes 2})$ and $\mathcal{E}(|\psi\rangle)$. In this situation, by using Eq. (5), it can be proven in a similar fashion that whenever $\delta_{\mathcal{N}}(|gGHZ\rangle^{\otimes 2}) = \delta_{\mathcal{N}}(|\psi\rangle^{\otimes 2})$, we have $\mathcal{E}(|\psi\rangle) \geq \mathcal{E}(|gGHZ\rangle)$, for arbitrary pure three qubit pure states

$|\psi\rangle$, assuming again that the maximum eigenvalue is obtained from the nodal observer. If we once more lift the restriction of maximum eigenvalue in GGM appearing from the nodal observer, we see that the $|gGHZ\rangle$ state still gives the lower boundary, as seen from Fig. 5. Moreover, we notice that the states from the W-class which possess low value of GGM and high negative value of $\delta_{\mathcal{N}}(|\psi\rangle)$ require more than two copies to satisfy the monogamy relation for negativity. Pure states with high genuine multipartite entanglement which violate monogamy for negativity, are seemingly easier to be activated than the states with low genuine multipartite entanglement.

VI. CONCLUSION

Summarizing, we have introduced the concept of activation of nonmonogamous multiparty states for a non-additive quantum correlation measure. Our aim was to obtain monogamous states from nonmonogamous ones by using many copies of the same state. We proved that an arbitrary state, whether pure or mixed, which does not satisfy the monogamy relation for negativity, will satisfy monogamy when multiple copies of the same state is provided. This is true irrespective of the number of parties and their dimension. On the other hand, we showed that negativity monogamy score of multiple copies of a state remains non-negative if it is so for a single copy. We carried out extensive numerical searches among three-qubit pure states and found that nonmonogamous W-class states, generally require a higher number of copies to satisfy monogamy compared to the states from the GHZ-class. We also analysed the status of monogamy for pairs of states that are individually nonmonogamous. We found that almost all pairs of states become monogamous, even when they individually are not. Finally, we provided a relation between the negativity monogamy score and a genuine multipartite entanglement measure for three-qubit pure states.

ACKNOWLEDGMENTS

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$$2\delta_{\mathcal{N}}(\rho_{123}^{\otimes m}) = 2\big[\mathcal{N}(\rho_{123}^{\otimes m}) - \mathcal{N}(\rho_{12}^{\otimes m}) - \mathcal{N}(\rho_{13}^{\otimes m})\big]$$
$$\geq \big[(1 + 2\,\mathcal{N}_{12}) + 2\,\mathcal{N}_{13}\big]^m - (1 + 2\,\mathcal{N}_{12})^m$$
$$-(1 + 2\,\mathcal{N}_{13})^m + 1$$
$$= \sum_{k=1}^{m-1} \binom{m}{k} \big(2\mathcal{N}_{12}\big)^k \big[(1 + 2\,\mathcal{N}_{13})^{m-k} - 1\big].$$
Clearly, the right hand side of the last line is non-negative for $m \geq 1$ which also proves Theorem 1.

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