Robustness of Higher Dimensional Nonlocality against dual noise and sequential measurements

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Robustness in the violation of Collins-Linden-Gisin-Masser-Popescu (CGLMP) inequality is investigated from the dual perspective of noise in measurements as well as in states. To quantify it, we introduce a quantity called area of nonlocal region which reveals a dimensional advantage. Specifically, we report that with the increase of dimension, the maximally violating states (MVS) show a greater enhancement in the area of nonlocal region in comparison to the maximally entangled states (MES) and the scaling of the increment, in this case, grows faster than visibility. Moreover, we examine the robustness in the sequential violation of CGLMP inequality using weak measurements, and find that even for higher dimensions, two observers showing a simultaneous violation of the CGLMP inequality as obtained for two-qubit states persists. We notice that the complementarity between information gain and disturbance by measurements is manifested by the decrease of the visibility in the first round and the increase of the same in the second round with dimensions. Furthermore, the amount of white noise that can be added to an MES so that it gives two rounds of the violation, decreases with the dimension, while the same does not appreciably change for the MVS.

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I. INTRODUCTION

The journey from the Einstein-Podolski-Rosen paradox [1] to Bell theorem [2] via Bohmian mechanics [3] is a fascinating story that contributed towards our present outlook of a physical theory. It asserts that a satisfactory description of nature cannot assume both the assumptions of locality and realism simultaneously, which have recently been supported experimentally by loophole-free Bell test [4–6]. Apart from the foundational significance, Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality [7] enables the ground for the device-independent certification of randomness [8], secure key distribution [9–11], entanglement [12] etc.

Going beyond the much-studied simplest Bell scenario involving two settings of measurements for two party with two outcomes, denoted by $(2-2-2)$, new insightful and qualitatively different results have been derived which was otherwise impossible. In particular, violation of local realism is manifested more sharply than $(2-2-2)$-case via Greenberger-Horne-Zeilinger (GHZ) argument [13], which require at least a three-qubit system. With a suitable choice of binary observables, it has been shown that maximal violation of Bell inequality persists for a singlet state of arbitrary spins [14], turning the old quantum wisdom [15] down. Later, dimensional advantage in violation of local realism has been established considering more general choice of observables [16–18], since dichotomic measurements can not exploit the higher dimensional system with full generality. In the bipartite system of arbitrary local dimension, with two choices of non-degenerate measurements, named as $(2-2-d)$-situation, corresponding Bell inequalities have been derived by Collins-Gisin-Linden-Massar-Popescu (CGLMP) [19], which is violated maximally by a nonmaximally entangled state [20] for the specific choices of observables [16–18]. These tight higher dimensional Bell inequalities [21] exhibit enhancement in visibility with the increase of dimension, thereby showing more robustness against noise [16, 19]. It is also important to mention here that higher dimensional bipartite systems also turn out to be useful in several quantum information processing tasks ranging from quantum key distribution, quantum dense coding [22], teleportation [23] to computational complexity [24–37].

In another direction, the conventional Bell scenario has been extended where half of a bipartite system is possessed by a single observer, called Alice, while the other half is possessed by a series of observers, named as Bobs, who can measure sequentially [38]. In this new scheme of Bell test, it has been shown that no more than two observers can violate Bell-CHSH inequality if the series of observers measure independently [38, 39]. Such a sequential scenario has also been tested experimentally [40, 41] and is further extended in several situations which include detecting steerable correlation [42, 43], witnessing entanglement [44, 45], testing Bell inequalities other than CHSH [46], identifying genuine entanglement [47], preparation contextuality [48]. An interesting twist in this situation is that with the slight modification to the independent and unbiased measurement scheme, the unbounded sequence of observers can be found who can certify non-classical correlation with the single observer in another side [38, 49]. Recently, some interesting applications of the sequential scheme like self testing unsharp measurement [50, 51], reusing teleportation channel [52], generating randomness [53] have been proposed, thereby showing its potentiality in quantum technologies. An interesting observation from the above studies is that if one restricts to a particular measurement scheme, i.e., independent and unbiased measurement by the series of observers [39], the number of successful detection of nonclassical correlation depends on the strength of the underlying correlation, detection and measurement processes in an intricate way which is not well understood yet. The number is finite and dictated by the trade-off between the disturbance and information gain by measurements. For example, it is found that at most twelve Bobs can witness entanglement with single Alice [44] while only two can violate CHSH inequality [38, 39]. In the sequential measurement, partial information is extracted which is sufficient for the detection scheme and at the same time, some residual correlation remains for other rounds which gradually diminishes with a longer sequence of Bobs. It also reveals that witnessing entanglement possibly disturbs the state less compared to the situation when the Bell-CHSH test is performed, thereby admitting more robustness of the former scheme against noise. Similarly, measurement-device-independent entanglement witness [54] turns out to be more suitable in the sequential situation than that of the standard entanglement witness [55] as shown through the increased number of Bobs [45].

In the present work, we first investigate the robustness of CGLMP inequality by going beyond the visibility measure of 'nonlocality' [16, 19]. Specifically, in addition to white noise in the state, we consider noisy measurement (which we call as weak/unsharp measurement) on the maximally entangled state (MES) as well as on the maximally CGLMP violating states (MVS). Such a consideration of dual noise leads to a measure of robustness, dubbed as 'area of nonlocal region' (where nonlocality means the violation of CGLMP inequality), which scales with dimension more sharply than the visibility one. Introduction of noise to the measurement enables the possibility of sequential violation of CGLMP inequality. In particular, we find that the violation by two Bobs' persists even with the increase of dimension, as found in the two-qubit case with CHSH inequality. In this respect, the pertinent question is how the robustness of CGLMP reflects in the sequential scenario. It was noticed that in the context of violation of CGLMP inequality, the visibility decreases with the increase of dimension [16, 19]. However, we observe that if we demand the violation of CGLMP inequality in two rounds of a sequential scheme, the required visibility increases with the dimension for maximally entangled states while surprisingly, it remains constant for maximally violating states. It also demonstrates that the sequential scenario can reveal a kind of robustness which is qualitatively different from the visibility and 'area of nonlocal region' obtained for a single round. It is due to the trade-off present in the disturbance by the weak measurements and the information gain via measurements in a sequential scheme.

The paper is organised in the following way. In Sec. II, we briefly discuss the prerequisite of the present work. In Sec. III, robustness of CGLMP is discussed with new measure introducing dual noise. For higher dimensional pure states, CGLMP inequality is used to certify entanglement sequentially in Sec. IV and similar study is carried out for noisy mixed states in Sec. V. We conclude in Sec. VI with a brief discussion.

II. PREREQUISITES: BELL INEQUALITIES IN HIGHER DIMENSIONS AND SEQUENTIAL MEASUREMENT SCHEME

Before presenting our results, let us briefly discuss the CGLMP inequality and sequential scenario of Bell test.

A. CGLMP inequality

Let Alice and Bob be two observers allowed to perform two d outcome measurements. If A_1 and A_2 are measurement settings of Alice while B_1 and B_2 are of Bob, the CGLMP inequality reads as [19]

$$I_d = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor -1} [f(k) - f(-k-1)] \leq 2, \tag{1}$$

where

$$\begin{aligned} f(k) = P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\ + P(A_2 = B_2 + k) + P(B_2 = A_1 + k). \end{aligned} \tag{2}$$

The probabilities of the outcomes of Alice's measurement, A_a and Bob's measurement, B_b, $(a, b = 1, 2)$ in $f(k)$ differ by k mod d and can be written as

$$P(A_1 = B_1 + k) = \sum_{j=0}^{d-1} P(A_a = j, B_b = j + k \ \ mod \ \ d).$$

The strongest violation of CGLMP inequality is obtained for maximally entangled state and a particular class of non-maximally entangled state if the measurement settings on Alice and Bob are chosen to be

$$|k\rangle_{A_a} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp(i \frac{2\pi}{d} j(k + \alpha_a)) |j\rangle_A, \tag{3}$$

and

$$|l\rangle_{B_b} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp(i \frac{2\pi}{d} j(-l + \alpha_a)) |j\rangle_B, \tag{4}$$

with

$$\alpha_1 = 0, \quad \alpha_2 = 1/2, \quad \beta_1 = 1/4 \quad and \quad \beta_2 = -1/4 \tag{5}$$

respectively. Special thing about the above inequality is that its quantum violation increases with dimension d.

In this paper, the issue of robustness in the CGLMP inequality is addressed in two ways – one is by studying the trade-off in noise given in states and in measurements while the other one is by considering the sequential measurements which we will now briefly describe.

B. Sequential measurement scenario

The sequential measurement scenario considers an entangled state of two d-dimensional systems shared in such a way that the half of the system is in possession with the observer (say, Alice) and the other half is with several observers (say, n Bobs, referred as Bob_1, Bob_2, Bob_3, ..., Bob_n). The task of Bob_1 is to pass the system to Bob_2 after performing an unsharp measurement on his part. Similarly, Bob_2 passes the system to Bob_3 after the measurement and so on. In other words, several Bobs measure their part sequentially, and hence the name of sequential measurement scheme. Note that the measurement of each Bob is independent and all the measurement settings of each Bob are equally probable.

In order to know the number of Bob sharing nonlocality of a shared entangled state (say, ρ) between Alice and n Bobs, we have to assume that measurement of Alice and Bob_n is sharp (i.e., they perform projection measurements on their parts) while $1, \ldots, n-1$ Bobs perform unsharp measurements represented by positive-operator valued measurements (POVMs). If measurement settings at Alice are denoted by $\{|k\rangle_A\langle k|\}$ and the measurement settings of Bob_m is represented by

$$E^l_{B_m} = \lambda_m |l\rangle_B\langle l| + \frac{1-\lambda_m}{d}\mathbb{I}_d, \tag{6}$$

where $k, l = 0, 1, 2, d-1$, $m = 1, 2, 3, 4.....n-1$, λ_m ($0 < \lambda_m \leq 1$) is the sharpness parameter of Bob_m and $\mathbb{I}_d$ is the d-dimensional identity matrix. The state after the measurements of $(m-1)$th Bob and without any measurement at Alice's end transforms as

$$\rho_m = \frac{1}{d}\sum_{l=0}^{d-1}(\mathbb{I}_d \otimes \sqrt{E^l_{B_{m-1}}})\rho_{m-1}(\mathbb{I}_d \otimes \sqrt{E^l_{B_{m-1}}}), \tag{7}$$

where ρ_{m-1} is the state before the unsharp measurement performed by Bob_{m-1}. We will use the post measured state ρ_m and POVM in Eqs. (6) and (7) respectively, when we certify nonlocality via CGLMP inequality in this scenario.

III. ROBUSTNESS IN CGLMP VIOLATION: AREA OF NONLOCAL REGION

The study of violation of Bell-type inequalities is a major endeavor in studies of nonlocality. Another important aspect is the investigation of robustness in the obtained violation. Typical studies of robustness consist of addition of noise to the state and tracking the response of violation due to the amount of noise added to the state. However, for the violation of Bell-type inequalities, measurements play as crucial a role as states. Therefore, robustness analysis should also be carried out when noise is added to the measurements as well.

We perform a general robustness analysis when both the state as well as the measurements are simultaneously noisy. In particular, we explore the role of dimension of the bipartite state whose nonlocal characteristics in terms of violation of CGLMP inequality in Eq. (1) are under investigation. Before that, we briefly discuss the scenario when white noise is mixed with the state, given by

$$\rho = p|\psi\rangle\langle\psi| + \frac{1-p}{d^2}\mathbb{I}_{d^2}, \tag{8}$$

where $|\psi\rangle$ is a bipartite pure state with each party of dimension d, and $\mathbb{I}_{d^2}$ is the $d \otimes d$ maximally mixed state (white noise). It was observed [19] that when $|\psi\rangle$ is a maximally entangled state in $d \otimes d$, given by $|\psi^d_{MES}\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$, the robustness to noise which can be called as visibity, measured as p, increases with the increase in d. This is in the sense that the maximal white noise that can be added to $|\psi^d_{MES}\rangle$ such that the resultant mixed state ρ violates the CGLMP inequality which increases with dimension d. For a given d, this maximal amount of white noise is denoted by $1 - p_{\min}$. In other words, for $|\psi^d_{MES}\rangle$, as $p_{\min}$ decreases, $1 - p_{\min}$ increases with d. Such dimensional advantage of robustness is enhanced when instead of a MES, $|\psi\rangle$ is chosen to be the non-maximally entangled state which violates CGLMP inequality maximally [20]. We call such maximally violating states as MVS. The exact form of MVS upto $d = 10$ can be found in Ref. [56]. The MVS offer a greater robustness with d in comparison to MES.

On the other hand, the effect operators for the noisy measurements are described by POVMs given in Eq. (6). Considering $|\psi\rangle$ to be MES and MVS, the amount by which measurements can be made noisy is denoted by $1 - \lambda_{\min}$, which also increases with increasing d. We observe an exactly similar dimensional dependence with noise in the measurements denoted by λ, as obtained in the case of noisy states since adding white noise to the state or measurements is equivalent. Mathematically, for any pure state $|\psi\rangle$,

$$I_d(p = 1, \lambda = x) = I_d(p = x, \lambda = 1). \tag{9}$$

Things become more interesting and involved when both the state and measurements suffer from noise simultaneously, which we will discuss in the next subsection.

A. Complimentarity of Robustness

We are now going to study the robustness obtained from the violation of CGLMP inequality by considering

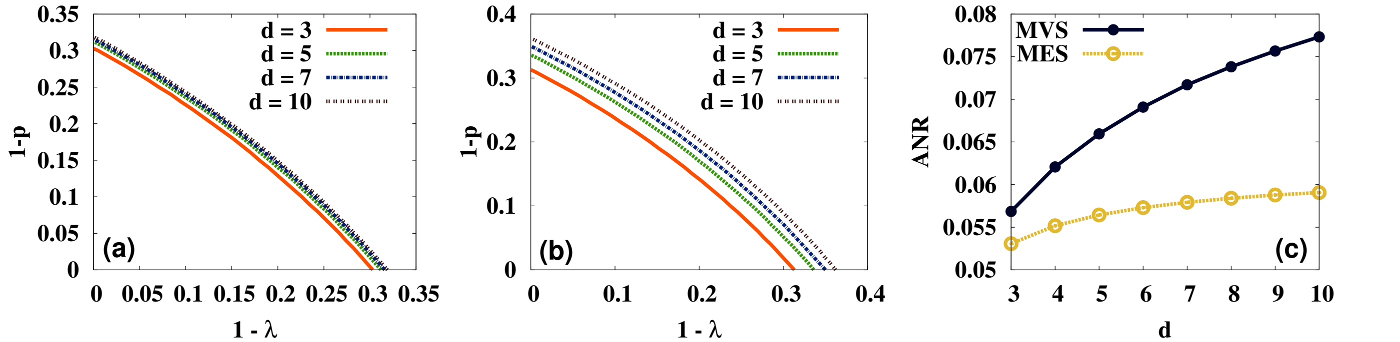

FIG. 1. (a) For a fixed d, each point in the curve just crosses the local realist value of 2 i.e., when the value of I_d in Eq. (1) is just above 2 by taking maximally entangled state (MES) in the $(1-\lambda, 1-p)$-plane. Different d values are considered. (b) Similar plot when the shared state is maximally CGLMP violating state (MVS). (c) ANR (ordinate) defined in Eq. (10) vs. d (abscissa) for MES and MVS.

both the state and the measurements noisy. We again start with a d-dimensional maximally entangled state, $|\psi^d_{MES}\rangle$ as well as MVS, $|\psi_{MVS}\rangle$. In this general framework, $p_{\min}$ is a function of the noise in the measurements, which we denote as $p_{\min}(\lambda)$, and naturally $\lambda_{\min}$ in turn becomes a function of the noise added to the state, which is referred as $\lambda_{\min}(p)$. For convenience, we drop the min and functional labels, thereby indicating $1-p_{\min}(\lambda)$ and $1-\lambda_{\min}(p)$ as $1-p$ and $1-\lambda$ respectively. We investigate the dual version of robustness by tracking the locus of all the points in the $(1-\lambda, 1-p)$-plane that just crosses the local realist value of 2 by considering MES and MVS, see Figs. 1 (a) and (b). Note that all noise configurations that fall below the curve leads to the violation of the CGLMP inequality. Motivated by this observation, we introduce a generalized robustness as the area under this curve. Mathematically, the area of nonlocal region (ANR) in the noise plane can be defined as

$$\text{ANR} = \int_0^{1-p_{\min}(\lambda=1)} \big(1-\lambda_{min}(1-p)\big)\, dp. \quad (10)$$

We then compute ANR values for both MES and MVS, and make a comparative analysis of their respective scalings with d, see Figs. 1 (a), (b), and (c). The ANR values for the MES and MVS are listed in Table I. Our findings are listed below:

1. The ANR values for MVS are strictly greater than those obtained for MES. Furthermore, the gap of ANR values for the MVS and the MES grows with d as clearly discernible from Table I and Fig. 1 (a) - (c).

2. The ANR scales much faster with d for the MVS in comparison to the MES, see Fig. 1 (c).

3. The gap in the growth between MVS and MES in case of ANR grows much faster than that of the visibility.

d	ANR (MES)	ANR (MVS)	Diff.
3	0.05307	0.05685	7.14%
4	0.05517	0.06207	12.51%
5	0.05644	0.06595	16.85%
6	0.0573	0.06909	23.81%
7	0.05792	0.07171	27.45%
8	0.0584	0.07382	26.40%
9	0.05878	0.07567	28.73%
10	0.05906	0.07733	30.93%

TABLE I. The ANR values for maximally entangled (MES) as well as maximally violating states (MVS), and their percentage differences (labelled as Diff.) from $d=3$ to $d=10$ are given in different columns. The difference grows with d since the ARN for MVS scales much faster with increase in d than that of MES.

Typically, noise in the system has an adverse effect on system in the form of lowering the visibility. As shown in this section, the bane can turn out to be boon in disguise if we look at the situation from a different point of view. In the context of sequential measurements, the "white noise" in the measurement actually constitutes a POVM strategy which allows multiple Bobs to share nonlocality, thereby manifesting the robustness from a different perspective, as will be showing in the succeeding section.

IV. SHARING OF NONLOCALITY IN HIGHER DIMENSION

In the sharing scenario considered in this section, we deal with the maximally entangled and maximally violating states shared by Alice and Bob$_1$ in arbitrary dimension. We will start our discussion from $d=3$ and a detailed analysis is presented for MES in $d=3$ to $d=5$.

We then repeat the investigation for the maximally violating states.

After substituting $d = 3$ in Eq. (1), the CGLMP inequality reads as

$$\begin{aligned} I_3 &= P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\ &+ P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ &+ P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \leq 2. \end{aligned} \tag{11}$$

If the shared state is the two-qutrit MES, given by

$$|\psi^3_{MES}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{12}$$

by performing POVM at Bob$_1$'s side, and by considering the measurement settings for CGLMP test given in Eqs. (3), (4) and (5) for Alice and Bob$_1$, the quantum expression of CGLMP inequality, I_3 (Eq.(11)) for Alice-Bob$_1$-pair reduces to

$$I^1_3 = \frac{4}{9}(3 + 2\sqrt{3})\lambda_1, \tag{13}$$

where the superscript, "1" represents the number of rounds in the sequential scenario. Hence, the nonlocality can be demonstrated by showing the violation of CGLMP inequality between Alice and Bob$_1$ if $\lambda_1 > 2/(\frac{4}{9}(3 + 2\sqrt{3})) = 0.69615$ while the optimal quantum value for Alice and Bob$_1$ is 2.87293 obtained at $\lambda_1 = 1$. In a similar fashion, we can find the quantum expressions for Alice-Bob$_2$ and Alice-Bob$_3$-pairs are respectively

$$\begin{aligned} I^2_3 = \frac{4\lambda_2}{81}\Big[&-2\left(\sqrt{3}+3\right)\lambda_1 + 12\sqrt{1-\lambda_1}\sqrt{2\lambda_1+1} \\ &+ 4\sqrt{2\lambda_1+1}\sqrt{3-3\lambda_1} + 14, \sqrt{3} + 15\Big], \end{aligned} \tag{14}$$

and

$$\begin{aligned} I^3_3 = \frac{4\lambda_3}{729}\Big[&4(\sqrt{3}+6)\left(2\sqrt{1-\lambda_2}\sqrt{2\lambda_2+1} - \lambda_2\right) \\ &\times \sqrt{1-\lambda_1}\sqrt{2\lambda_1+1} - 2\lambda_1(7\sqrt{3} + 15 - \left(\sqrt{3}+6\right)\lambda_2 \\ &+ 2(\sqrt{3}+6)\sqrt{1-\lambda_2}\sqrt{2\lambda_2+1}) - 2\left(7\sqrt{3}+15\right)\lambda_2 \\ &+ 4\left(7\sqrt{3}+15\right)(\sqrt{1-\lambda_1}\sqrt{2\lambda_1+1} \\ &+ \sqrt{1-\lambda_2}\sqrt{2\lambda_2+1} + 75 + 98\sqrt{3})\Big]. \end{aligned} \tag{15}$$

Considering the situation of minimum violation of I^1_3 by Alice and Bob$_1$, quantum expression of I^2_3 reduces to be $2.40856\lambda_2$. In this case, the violation of CGLMP inequality for Alice and Bob$_2$ is possible if $\lambda_2 > 0.830372$ while the optimal quantum value is 2.40856 with $\lambda_2 = 1$. Substituting the conditions for λ_1 and λ_2, we get that two Bobs surely violate CGLMP inequality. Let us now check whether the third Bob, Bob$_3$ can also violate CGLMP inequality or not. In this case, the optimal quantum value of I^3_3 turns out to be $1.83798 < 2$ by taking minimum violation condition for Bob$_2$ and Bob$_3$. Since optimal quantum value of I^3_3 is strictly less than 2, we can claim that only two Bobs, Bob$_1$ and Bob$_2$, can exhibit nonlocality with Alice by using CGLMP inequality for $d = 3$. Notice here that only two Bobs can violate CHSH inequality with Alice if they initially share a two-qubit maximally entangled state [38].

Let us now move to $d = 4$ and $d = 5$. In this case, I_d in Eq. (1) reduces to

$$\begin{aligned} I_4 &= P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\ &+ P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ &+ P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \\ &+ \frac{1}{3}\big(P(A_1 = B_1 + 1) + P(B_1 = A_2 + 2) \\ &+ P(A_2 = B_2 + 1) + P(B_2 = A_1 + 1) \\ &- [P(A_1 = B_1 - 2)P(B_1 = A_2 - 1) \\ &+ P(A_2 = B_2 - 2) + P(B_2 = A_1 - 2)]\big) \leq 2. \end{aligned} \tag{16}$$

By following similar prescription, for maximally entangled state, $|\psi^4_{MES}\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$, we find that Bob$_1$ starts sharing nonlocality with Alice through the violation of CGLMP when $\lambda_1 > 0.690551$ and $\max I^1_4 = 2.89624$ for $\lambda_1 = 1$. Again, if we restrict the situation such that Alice-Bob$_1$ duo just shows violation, Alice and Bob$_2$ violates CGLMP when $\lambda_2 > 0.834603$ and in the second round, the maximal quantum value is reduced which is 2.39635 ($\lambda_2 = 1$). By taking minimum violation condition of sharpness parameter for Bob$_2$ and Bob$_3$, the optimal quantum value of I^3_3, given in Table II, again tuns out to be less than 2. For $d = 5$,

$$\begin{aligned} I_5 &= P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\ &+ P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ &+ P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \\ &+ \frac{1}{2}\big(P(A_1 = B_1 + 1) + P(B_1 = A_2 + 2) \\ &+ P(A_2 = B_2 + 1) + P(B_2 = A_1 + 1) \\ &- [P(A_1 = B_1 - 2)P(B_1 = A_2 - 1) \\ &+ P(A_2 = B_2 - 2) + P(B_2 = A_1 - 2)]\big) \leq 2 \end{aligned} \tag{17}$$

can be used to obtain violations of CGLMP in a sequential situation with $|\psi^5_{MES}\rangle = \frac{1}{\sqrt{5}}(|00\rangle + |11\rangle + |22\rangle + |33\rangle + |44\rangle)$. The optimal quantum violation of CGLMP inequality for Bob$_1$, Bob$_2$ and Bob$_3$ are given in Table. II upto $d = 10$.

From Table. II, we can see that as dimension increases, there is an increment of optimal quantum value for Bob$_1$ although the same decreases with the increase of dimension for Bob$_2$ and Bob$_3$. It also indicates that the trade-off between the information gain by the measurement and the disturbance created by the measurement plays a crucial role in this enterprise.

Optimal quantum value of CGLMP inequality			
Dimension	Bob_1	Bob_2	Bob_3
3	2.8729	2.4086	1.8380
4	2.8962	2.3963	1.7994
5	2.9105	2.3819	1.7650
6	2.9202	2.3699	1.7382
7	2.9272	2.3570	1.7122
8	2.9324	2.3458	1.6910
9	2.9365	2.3360	1.6722
10	2.9398	2.3274	1.6568

TABLE II. Optimal quantum value of Bob_1, Bob_2 and Bob_3 are obtained using CGLMP inequality for maximally entangled state for $d = 3$ to $d = 10$ dimensions.

Optimal quantum value of CGLMP inequality			
Dimension	Bob_1	Bob_2	Bob_3
3	2.9150	2.4402	1.8578
4	2.9729	2.4526	1.8307
5	3.0158	2.4564	1.8015
6	3.0495	2.4522	1.7702
7	3.0771	2.4418	1.7342
8	3.1012	2.4324	1.7041
9	3.1215	2.4231	1.6768
10	3.1393	2.4142	1.6517

TABLE III. I_d^i, $(i = 1, 2, 3)$, for Bob_1, Bob_2 and Bob_3 are listed for maximally violating state (MVS) as the initial state from $d = 3$ to $d = 10$.

Since the CGLMP inequality gives the maximum violation for non-maximally entangled state, let us examine if the initial shared state in a sequential scenario is MVS whether the situation improves or not. The observation is that although the first round of violation is more and increases faster with d in comparison to the MES, the measurements disturb the state to such an extent that violation for more than two Bobs still remains an impossibility. Also note that the third round value of the CGLPM expression decreases on increasing the dimension, so the possibility of getting simultaneous violation for three rounds is unlikely even if d is increased beyond 10. See Table. III for details. Comparing Tables II and III, we observe that the gap between the I_d^3 values obtained for MVS and MES decreases with the increase of dimension. It possibly indicates that the unsharp measurements disturb the MVS more drastically than the MES in higher dimensions.

V. ROBUSTNESS IN SEQUENTIAL EXHIBITION OF NONLOCALITY

In Sec. III A, we analyzed how much noise we could add to the state as well as measurements so that it continues to violate the CGLMP inequality. However, the option of using sequential measurements to obtain violations for multiple Bobs with a single Alice opens up a possibility to examine robustness from a new point of view. In this context, we define robustness as the maximal amount of noise that can be added to a state such that the CGLMP inequality can be violated for multiple rounds, which we claim to be two, since from the previous section, we observed that both for the MES and MVS, the maximum number of Bobs that can violate the CGLPMP inequality with Alice remains two.

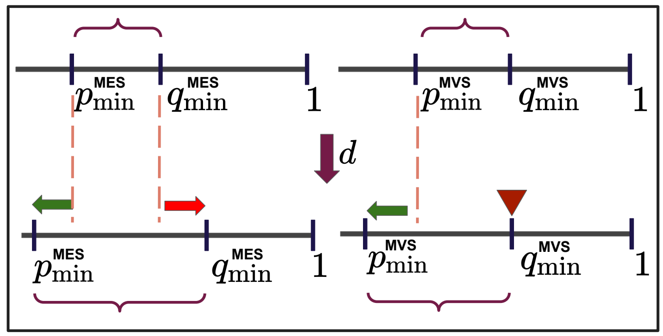

FIG. 2. Schematic depiction of the dynamics of $q_{\min}$ and $p_{\min}$ for both MES and MVS with d. $p_{\min}$ denotes the visibility of the state while $q_{\min}$ is the minimum value of the visibility above which CGLMP inequality in the second round starts violating. The superscripts represent the states considered. The green and red arrows respectively indicate the advantages and disadvantages of robustness with dimensions.

Let us consider the pure state, $|\psi\rangle$ admixed with white noise, given in Eq. (8), having the visibility, q, as an initial state in the sequential scenario. We now demand that if two Bobs has to show violation of local realism with Alice, both I_d^1 and I_d^2 have to be greater than 2. We define q_{min} to be the minimum value of q above which both $I_d^1 > 2$ and $I_d^2 > 2$. We now compute how the $q_{\min}$ scales with d and compare it with the scaling obtained for $p_{\min}$ as discussed in Sec. III A for both MES and MVS.

Recall that in CGLMP test, we observed an enhanced amount of robustness (as defined in terms of persistence of the violation on addition of white noise) on increasing d as indicated by lowered values of $p_{\min}$. The maximal amount of white noise that the state can absorb such that the violation persists is simply given by $1 - p_{\min}$. For both MES and MVS, $p_{\min}$ decreases with d [16, 19, 20]. Furthermore, note that, we expectedly find $p_{\min} < q_{\min} < 1$.

When robustness is analyzed in the context of sustaining dual round violation via the use of sequential measurements, we observe a qualitatively different

Dimension	$q_{\min}^{\text{MVS}}$	$q_{\min}^{\text{MES}}$
3	0.8773	0.8845
4	0.8748	0.8872
5	0.8737	0.8900
6	0.8736	0.8933
7	0.8738	0.8963
8	0.8741	0.8987
9	0.8748	0.9012
10	0.8752	0.9034

TABLE IV. The $q_{\min}$ values for maximally entangled states (MES) and maximally violating states (MVS) for $d = 3$ to 10 are reported when we demand violation of CGLMP inequality by two Bobs sequentially with Alice.

trend. For MES, $q_{\min}$ actually increases with d. This implies that robustness actually decreases with d when MES are employed and we demand CGLMP violations by two Bobs. However, for MVS, $q_{\min}$ values do not change significantly on increasing d. See Table. IV for details of the $q_{\min}$ values for both MES and MVS. However, in both the cases, the gap between $q_{\min}$ and $p_{\min}$ increases with d. For a pictorial representation of the situation, see Fig. 2.

The above results explain in part why inspite of an increase in the first round violation with d, one does not get higher number of Bobs which sequentially violates CGLMP inequality i.e., $I_d^k > 2$ with $k > 2$ for higher dimensional systems. Although the amount of maximal first round violation grows, the disturbance induced by the measurements are high enough to actually bring down the violation in the second round with d which ultimately leads to the third round becoming non-violating.

VI. DISCUSSION

To achieve quantum supremacy, manipulating and analysing higher dimensional quantum system is essential, since, in several quantum information processing tasks, higher dimensional quantum systems turn out to be more beneficial than the qubit pairs. CGLMP inequality is a family of tight Bell inequalities for bipartite systems of arbitrary dimension, which is known to exhibit more robustness against noise with increasing dimension. Therefore, it is interesting to investigate how CGLMP inequality responds if noise is present not only in the state but also in measurement.

We introduced a new measure of robustness which we referred as 'area of nonlocal region' under consideration of dual noises, both in states and measurements. In particular, this area indicates the region in noise parameter space where violation of CGLMP can be observed. We found that this region grows more rapidly with the increase of dimension with respect to the increase in visibility associated solely with states or measurements.

The introduction of noise in measurement facilitates to invoke of a sequential violation of CGLMP inequality as it retains some residual correlation after obtaining violation in the first round. Interestingly, we found that the violation of CGLMP inequality by two sequential observers on one side and another observer on the other end persists with dimension. Moreover, minimum visibility required to achieve double violation in the sequential case increases with the increase of dimension, thereby exhibiting opposite behavior as compared to the violation obtained for the shared state without unsharp measurement. It indicates that robustness in the sequential measurement scenario is qualitatively distinct than that of the typical Bell test since it involves the disturbance on the state introduced via the measurement. It is interesting to probe further how the double violation obtained in CGLMP inequality enables applications in the context of information processing tasks involving higher dimensional quantum systems.

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