Coherence measure of ensembles with nonlocality without entanglement

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Irreversibility between preparation and discrimination processes is manifested in the indistinguishability of orthogonal product states via local operations and classical communication (LOCC). Characterizing quantum properties for sets of states according to their local distinguishing property is one of the avenues to explain the surprising results obtained in the LOCC indistinguishability domain. We introduce a measure based on l_1 norm and relative entropy of coherence whose lower values capture more quantumness in ensembles. In particular, it reaches the maximum for two-way distinguishable product ensembles with minimum rounds. Moreover, to establish the hierarchy between different product ensembles, we report that the trends of success probabilities for orthogonal product ensembles with one-way LOCC via minimum error discrimination scheme are also connected with the coherence-based measures.

I. INTRODUCTION

One of the fundamental tasks in quantum mechanics is to detect quantum states that are given from a known ensemble. Under globally allowed operations, sets of orthogonal quantum states can always be distinguished while for nonorthogonal ensembles, the useful upper bound on the accessible information, quantifying the maximum amount of information extractable from the ensemble is known as the Holevo quantity [1].

On the other hand, if quantum information is encoded into the composite system and subsystems of it are sent to spatially separated observers, a set of globally orthogonal states, in general, cannot be distinguished under a set of allowed operations, local operations assisted by classical communications (LOCC) which is a strict subset of global operations [2]. Initially, it was thought that entanglement which cannot be created by LOCC is responsible for local instinguishability. However, such intuitive understandings turn out to be false on several occasions. One of the surprising results in this direction is the discovery of a set consisting of nine orthogonal product states of two qutrits, which cannot be distinguished perfectly by LOCC - known as 'nonlocality without entanglement' [3]. In a similar spirit, unextendible product qudit basis (UPB) have been discovered [4, 5], which are also LOCC indistinguishable and they provide a systematic way of constructing bound entangled states [4–6]. Further investigations in this direction were carried out which found several complete and incomplete LOCC indistinguishable product ensembles [7–12]. On the other hand, two orthogonal states are shown to be always distinguishable via LOCC irrespective of their entanglement content [13]. Moreover, it was exhibited in two qutrits that decreasing average entanglement from ensembles can increase local indistinguishability, a phenomenon known as "more nonlocality with less entanglement" [14]. All the results strongly indicate that there are other quantum characteristics in ensembles other than average entanglement content which are responsible for LOCC indistinguishability.

Over the years, the studies of local indistinguishability are performed into two distinct directions – on one hand, several counter-intuitive examples of ensembles that are LOCC indistinguishable are reported, while on the other hand, there are few attempts to quantify quantumness in the ensembles which can capture the difficulties in local distinguishing [15–22]. To address the latter direction, the upper bound on locally accessible information like Holevo bound in global case was obtained which is useful to prove local indistinguishability of ensembles with entangled states [15, 16] although it fails to capture the results for product ensembles and more nonlocality with less entanglement. Some of us have resolved this problem by defining quantumness for ensembles from two different perspectives – one is based on the minimal entropy production after dephasing the states in the set of a LOCC distinguishable basis [15] while the other one is based on the generation of entanglement by LOCC indistinguishable sets of product states under some specific transformations on the whole ensemble [22]. In the present work, we characterize quantumness in ensembles using measures of coherence [23, 24]. In modern day-quantum technology, 'coherence' has been shown to be the key ingredient which underlies phenomena such as quantum interference, quantum metrology, multipartite entanglement, quantum communication, thereby establishing it as a resource. In this respect, see the recent work which characterizes the coherence of sets [25]. On the other hand, the quantumness that we want to assess in ensembles is due to the difficulty in distinguishing states via LOCC with the help of coherence. Notice that sets of product states belonging to the LOCC indistinguishable class are globally orthogonal although they are locally nonorthogonal, which indicates an implicit role of coherence under this phenomenon. We uncover how 'the unity of opposites' determines local indistinguishability by explicitly constructing coherence-based measure of 'quantumness' associated with these sets. Broadly these sets can be categorized into four classes – (1) distinguishable by minimum round of LOCC, independent of the parties who start the protocol, referred to as two-way distinguishable, (2) distinguishable by minimum round of LOCC although it requires a specific party to start with, called as one-way distinguishable ensembles, (3) distinguishable by finite but more than minimum rounds of LOCC, which include both one and two ways of distinguishable sets, and (4) sets of states which are indistinguishable via infinite rounds of LOCC. The measure introduced in this paper can distinguish the different classes mentioned above.

When a set of states are not distinguishable by finite or infinite rounds of LOCC, the natural question is to find their distinguishability probabilistically. There are two directions in which imperfect strategies can be provided – unambiguous state discrimination [26–28] in which the results are always correct although there are certain probabilities in which the protocol fails, while another direction is to design the protocol, known as minimum error discrimination in which the error in success probability has to be minimized [29–31]. Both the protocols are developed in the LOCC paradigm. In this work, we also establish a connection between the optimal success probability for product ensembles with two qubits and higher dimensions having restricted classical communication (CC) [32] and the coherence-based measure for ensembles. Moreover, we provide a prescription to maximize optimal success probability in a LOCC protocol for product ensembles.

We organize the paper in the following way. In Sec. II, we describe LOCC distinguishability for sets of states and define two measures of coherence for states used in later sections. In Sec. III, we introduce coherence-based measures for ensembles and show their effectiveness by considering different examples of locally distinguishable and indistinguishable ensembles. The prescriptions for obtaining maximal success probabilities are presented in Sec. IV and the relation between coherence-based measures for sets of states and probabilities are also argued. We conclude in Sec. V.

II. LOCC DISTINGUISHABILITY AND COHERENCE MEASURES

LOCC distinguishability. In $\mathcal{C}^{d_1} \otimes \mathcal{C}^{d_2}$, suppose two parties, Alice (A) and Bob (B), situated in distant locations, share a state which are given with equal probability from an ensemble consisting of orthonormal set of states, $\{|\xi_i\rangle_{i=1,2,..,N=d_1\times d_2}\}$. The task of Alice-Bob duo is to distinguish the shared state via local operations and classical communication. In this work, $|\xi_i\rangle$s are always taken to be product and in the literature, several general but striking results are known even for product ensembles.

We first illustrate the contrasting features even in the lowest dimension, i.e., in $\mathcal{C}^2 \otimes \mathcal{C}^2$. Let us consider the computational basis, $\mathcal{E}_1 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, which is distinguishable via one round of LOCC, irrespective of the party, i.e., Alice or Bob starting the protocol. Notice that the characteristics of LOCC distinguishability of the ensemble does not change if one replaces $\{|0\rangle, |1\rangle\}$ at Alice or Bob's side by $\{|\eta\rangle, |\eta^\perp\rangle\}$, where $|\eta\rangle = \cos\frac{\theta}{2}|0\rangle + \exp(i\phi)\sin\frac{\theta}{2}|1\rangle$ and its corresponding orthogonal state, with $\{|0\rangle, |1\rangle\}$ being the eigenvectors of Pauli matrix, σ_z. This ensemble is an example of two-way LOCC distinguishable with minimum round (i.e., one round) of CC required. On the other hand, consider another ensemble of orthogonal product basis, given by $\mathcal{E}_2 = \{|00\rangle, |01\rangle, |1+\rangle, |1-\rangle\}$ [7], where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ are the eigenbasis of the Pauli σ_x operator. This ensemble is perfectly one-way distinguishable via LOCC. i.e., first A measures her qubit in the computational basis and sends the result to B who depending on the message, should measure his qubit in $\{|0\rangle, |1\rangle\}$ (if A finds her qubit in $|0\rangle$ state) or $\{|+\rangle, |-\rangle\}$ (if A finds her qubit in $|1\rangle$ state) basis. Notice that $\mathcal{E}_2$ is different than $\mathcal{E}_1$ since the former cannot be distinguished if Bob starts the protocol. Hence it is an example of one-way LOCC distinguishable set with minimum round of CC. If difficulties in local distinguishabillity is a signature of nonclassicality in ensembles, quantumness present in $\mathcal{E}_2$ is expected to be higher than that of $\mathcal{E}_1$ which can also be visualized from the Bennett-Brassard (BB84) quantum key distribution protocol [33].

In this work, our aim is to capture quantumness present in ensembles which we will do with the help of coherence measures.

Measures of coherence. We here use two distance-based coherence measures, namely l_1 norm- and relative entropy-based coherence [23]. The l_1 norm of coherence for the state ρ, $C_{l_1}(\rho)$ [24] is defined as

$$C_{l_1}(\rho) = \min_{\sigma\in S_I} ||\rho - \sigma||_{l_1} = \sum_{i\neq j} |\rho_{ij}|,$$

where minimization is taken over the set of incoherent states, σ. On the other hand, the relative entropy of coherence, denoted by $C_{rel}(\rho)$, which can be connected to thermodynamics, is given by

$$C_{rel}(\rho) = S(\delta[\rho]) - S(\rho),$$

where S(ρ)=-Tr($\rho \log_2 \rho$) is the von Neumann entropy, and δ is the dephasing operation described by

$$\delta[\rho] = \sum_{k=0}^{d-1} |k\rangle\langle k| \rho |k\rangle \langle k| .$$

$C_{rel}(\rho)$ can also be interpreted as the deviation of ρ from thermal equilibrium. Since coherence measures are basis-dependent, we choose computational basis as

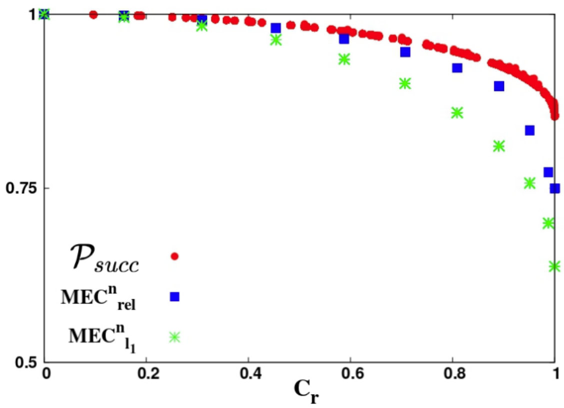

Figure 1. Normalised minimum ensemble coherence (MEC) both obtained via l_1 norm and relative entropy of coherence (ordinate) vs. relative local coherence, C_r, (abscissa). Blue squares and green stars represent $\text{MEC}^n_{\text{rel}}$ and $\text{MEC}^n_{l_1}$ respectively. It is calculated for arbitrary product ensembles in $2 \otimes 2$ with real coefficients, $\mathcal{E}^{4(R)}_{arb}$. Moreover, the trends of success probabilities, $\mathcal{P}_{succ}$ (ordinate) (red circles) are plotted against C_r. The departure from the maximum values obtained for the all three quantities implies that the ensemble is not distinguishable by two-way LOCC with minimum rounds. Moreover, the patterns of $\text{MEC}^n_{\text{rel}}$, $\text{MEC}^n_{l_1}$ and $\mathcal{P}_{succ}$ also match with the increase of C_r in ensembles. Both the axes are dimensionless.

the reference basis and any state that is diagonal in that basis is called the incoherent.

Before introducing coherent-based measure for ensembles, let us first note the following observations. If we measure the l_1 norm of coherence for the equal superposition of states in $\mathcal{E}_1$, the value is found to be 3.0 while for the second ensemble, $\mathcal{E}_2$, it is 1.914. It indicates that the coherence of the ensembles which are two-way LOCC-distinguishable with minimum round is maximal while for the one-way distinguishable set with minimum round, the value of C_{l_1} is non-maximal.

III. COHERENCE-BASED MEASURES FOR ENSEMBLES

In this section, we introduce two coherence-based quantifiers which can characterize quantumness in ensembles. We will also discuss the effectiveness and shortfalls of each measure.

A. Quantifying quantumness in ensembles: Minimum ensemble coherence (MEC)

Let us first define the measure for the full basis consisting of product states although it also works for incomplete basis. For a given ensemble of a bipartite system, $\mathcal{E} = \{|\psi_i\rangle \otimes |\phi_i\rangle\}_{i=1,2,..,N=d_1\times d_2}$ in dimension $\mathcal{C}^{d_1} \otimes \mathcal{C}^{d_2}$ (which, in short, will be mentioned as $d_1 \otimes d_2$), we define a quantity called *total local coherence* as

$$\tau^C_{l_1(\text{rel})} = \min_{\{U_1,U_2\}} \sum_i [C_{l_1(\text{rel})}(U_1|\psi_i\rangle) + C_{l_1(\text{rel})}(U_2|\phi_i\rangle)], \tag{1}$$

where minimization is taken over the set of local unitary operators, $U_i, i = 1,2$. Suppose now that the minimum is attained by $U^* = U_1^* \otimes U_2^*$. Rotating all the states of the original ensemble by U^*, i.e., $\{U^*|\psi_i\rangle \otimes |\phi_i\rangle\}_{i=1,2,..,N=d_1\times d_2}$, we now consider unitarily rotated superposed state of the ensemble represented by

$$|\Psi\rangle^* = U_1^* \otimes U_2^* \frac{1}{\sqrt{N}} \sum_i^N |\psi_i\rangle \otimes |\phi_i\rangle. \tag{2}$$

A quantifier, dubbed it as *'minimum ensemble coherence'* (MEC), based on l_1 norm or relative entropy of coherence for ensembles, capturing nonclassicality in terms of LOCC indistinguishability, can be introduced as

$$\text{MEC}_{l_1(\text{rel})} = C_{l_1(\text{rel})}(|\Psi\rangle^*). \tag{3}$$

It turns out that MEC_{l_1} reaches its maximum value, 3.0 for $\mathcal{E}_1$ and it is 1.914 for $\mathcal{E}_2$. The normalized MEC can then be computed as

$$\text{MEC}^n_{l_1(\text{rel})} = \frac{\text{MEC}_{l_1(\text{rel})}}{\max(\text{MEC}_{l_1(\text{rel})})}, \tag{4}$$

so that its maximum value is always unity irrespective of the dimension. Notice that the maximum value is $d_1 d_2 - 1$ for the l_1-norm measure while it is $\log_2 d_1 d_2$ in case of relative entropy of coherence. We demand and establish that like entanglement witness for quantum states,
nonmaximal value of $\text{MEC}_{l_1(\text{rel})}$, *i.e., deviation from unity of* $\text{MEC}^n_{l_1(\text{rel})}$ *is capable to detect quantumness present in the ensembles which are either one-way LOCC distinguishable with minimum round or with rounds more than the minimum or two-way LOCC distinguishable but requires more rounds than the minimum round possible.*

Remark 1. The same coherence measure should be used to find τ_C and MEC. Although we present all the results by using l_1 norm of coherence or relative entropy of coherence, the qualitatively similar results can be obtained by using other coherence measures, satisfying the basic postulates of coherence [23].

Remark 2. Note that if any side belongs to the computational basis, unitaries applied on that side to minimize coherence can be shown to be the Identity operator.

1. *Trends of MEC for arbitrary product basis in two qubits*

Let us first consider an arbitrary full product basis in $2 \otimes 2$, given by $\mathcal{E}^{4(R)}_{arb} = \{|0\eta_1\rangle, |0\eta_1^\perp\rangle, |1\eta_2\rangle, |1\eta_2^\perp\rangle\}$ [34]

where $|\eta_1\rangle$ and $|\eta_2\rangle$ are arbitrary nonorthogonal qubits without complex coefficients, i.e.,

$$|\eta_1\rangle = \cos\frac{\theta_1}{2}|0\rangle + \sin\frac{\theta_1}{2}|1\rangle\text{, and}$$
$$|\eta_2\rangle = \cos\frac{\theta_2}{2}|0\rangle + \sin\frac{\theta_2}{2}|1\rangle.$$

In the superscript, '(R)' is used to indicate the coefficients of states to be real. The imaginary coefficients will also be considered separately.

To distinguish the set of states from the ensemble which are two-way LOCC distinguishable with minimum round, we introduce a quantity, referred to as *relative local coherence*, C_r, corresponding to the ensemble $\{|0\eta_1\rangle, |0\eta_1^\perp\rangle, |1\eta_2\rangle, |1\eta_2^\perp\rangle\}$, given by

$$C_r = (C_{l_1}(|\eta_2\rangle\langle\eta_2|) + C_{l_1}(|\eta_2^\perp\rangle\langle\eta_2^\perp|))/2.0$$

where $C_{l_1}(\rho)$ is the l_1 norm coherence of the state ρ and coherence is measured in $\{|\eta_1\rangle, |\eta_1^\perp\rangle\}$ basis. If the value of C_r vanishes, the set $\{|\eta_1\rangle, |\eta_1^\perp\rangle\}$ and $\{|\eta_2\rangle, |\eta_2^\perp\rangle\}$ coincide, and hence the corresponding ensemble is both way LOCC-distinguishable with minimum round. Fig. 1 shows the plot of the normalised MEC value in terms of l_1 norm and relative entropy of coherence for the ensemble $\mathcal{E}_{arb}^{4(R)}$ with respect to C_r. The decreasing trends of $\text{MEC}^n_{l_1(\text{rel})}$ with the increase of relative local coherence, C_r actually establishes that it can quantify the quantumness in the ensembles according to their complexity of LOCC distinguishability. In fact, we will show in the next section that the pattern of MEC with relative coherence is connected to the success probabilities of restricted LOCC protocol [32]. Moreover, it is clear from the figure that the measure for ensembles is independent of the choices of the coherence measures.

2. *MEC beyond two qubits*

Let us now analyze whether the ensemble quantifier also works in higher dimension or not. Towards answering it, let us first consider a class of orthogonal product ensemble in $2\otimes 3$, given by $\mathcal{E}_{arb}^{6(R)} = \{|0\eta_1\rangle, |0\eta_1^\perp\rangle, |0\eta_1^{\perp\perp}\rangle, |1\eta_2\rangle, |1\eta_2^\perp\rangle, |1\eta_2^{\perp\perp}\rangle\}$, where the components of the nonorthogonal qutrit states $|\eta_1\rangle$ and $|\eta_2\rangle$ are real, i.e.,

$$|\eta_1\rangle = \sin\theta_1\cos\phi_1|0\rangle + \sin\theta_1\sin\phi_1|1\rangle + \cos\theta_1|2\rangle,$$
and
$$|\eta_2\rangle = \sin\theta_2\cos\phi_2|0\rangle + \sin\theta_2\sin\phi_2|1\rangle + \cos\theta_2|2\rangle.$$

Here $\{|0\rangle, |1\rangle, |2\rangle\}$ form the computational basis for qutrit system. In this situation, if coherence is measured in the $\{|\eta_1\rangle, |\eta_1^\perp\rangle, |\eta_1^{\perp\perp}\rangle\}$ basis, the relative local coherence, in this case, turns out to be

$$C_r = \frac{1}{3}(C_{l_1}(|\eta_2\rangle\langle\eta_2|) + C_{l_1}(|\eta_2^\perp\rangle\langle\eta_2^\perp|) + C_{l_1}(|\eta_2^{\perp\perp}\rangle\langle\eta_2^{\perp\perp}|)).$$

Again C_r vanishes when $\{|\eta_1\rangle, |\eta_1^\perp\rangle, |\eta_1^{\perp\perp}\rangle\}$ and $\{|\eta_2\rangle, |\eta_2^\perp\rangle, |\eta_2^{\perp\perp}\rangle\}$ coincide, and they are two-way LOCC distinguishable with minimum rounds. Otherwise, it is nonvanishing. Notice that unlike in $2\otimes 2$, numerical optimizations involved in τ_C for obtaining $\text{MEC}^n_{l_1}$ with the variation of state parameters become hard with the increase of dimensions due to the increase of parameters in U_is, thereby making numerical analysis difficult for general ensembles in higher dimensions.

Observation 1. In $2\otimes 2$ and $2\otimes 3$ dimensions, if a given ensemble $\{|\psi_i\rangle\otimes|\phi_i\rangle\}_{i=1,2,\cdots,N}$ ($N = 4$ and 6 for $2\otimes 2$ and $2\otimes 3$ dimensions respectively), equipped with the condition that all the components of the states of the ensemble are real, is both way LOCC-distinguishable with minimum rounds, we find that the resulting superposed local unitarily rotated states, $|\chi\rangle = U^*\frac{1}{\sqrt{N}}\sum_{i=1}^N|\psi_i\rangle\otimes|\phi_i\rangle$ are maximally coherent ones [23, 35].

Remark 3. The measure does not give satisfactory results, when Alice or Bob's side possess arbitrary qubits, i.e., when

$$|\eta_1\rangle = \cos\tfrac{\theta_1}{2}|0\rangle + e^{i\phi_1}\sin\tfrac{\theta_1}{2}|1\rangle\text{, and}$$
$$|\eta_2\rangle = \cos\tfrac{\theta_2}{2}|0\rangle + e^{i\phi_2}\sin\tfrac{\theta_2}{2}|1\rangle. \tag{5}$$

E.g., consider $|\eta_1\rangle = (\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}})^T$ and $|\eta_2\rangle = (\cos(\frac{\pi}{8}), i\sin(\frac{\pi}{8}))^T$, the corresponding ensemble, $\mathcal{E}_2$ is one-way LOCC-distinguishable. Since the absolute value of the inner product between $|\eta_1\rangle$ and $|\eta_2\rangle$ is 0.92388, the value of C_r is nonvanishing. However, one can find that $\text{MEC}_{l_1(\text{rel})}$ is maximum irrespective of the coherence measures. There are other examples of ensembles having nonvanishing $\phi_i, i = 1,2$ for which the similar maximal values of MEC can be found. In the next subsection, we will show that one can define another coherence-based measure with the help of MEC for ensembles which are effective in characterizing any ensembles.

Moving beyond qubits, we know that for orthogonal product ensembles, several interesting results are reported both for the full product basis as well as for the incomplete basis [3, 4]. In $3\otimes 3$, prominent ones are the basis known as nonlocality without entanglement (NLWE) [3] and unextendible product basis (UPB) [4]. The common property of these ensembles is that they are indistinguishable by LOCC (even if one involves infinite rounds of classical communication), thereby leading to an irreversibility between preparation and distinguishing

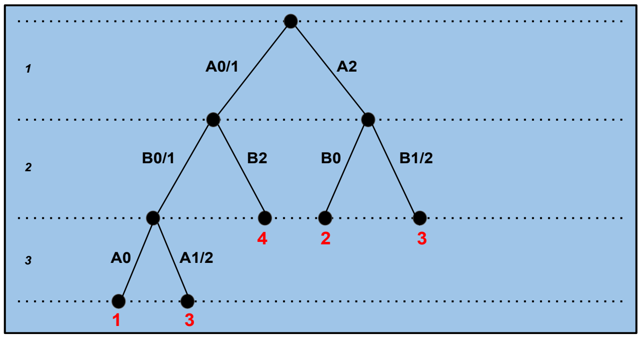

Figure 2. Schematic diagram of the three stages of LOCC rounds to distinguish the Tiles UPB without the fifth states (stopper), denoted by $\mathcal{E}'_{tiles}$. A and B indicate the party performing the measurements. The numbers mentioned next A or B specifies the outcome of the measurements, i.e., $A0/1$ means that A obtains the outcome $|0\rangle$ or $|1\rangle$ and so on. As mentioned in the text, the ensemble is two-way LOCC distinguishable and here we show the protocol when A starts.

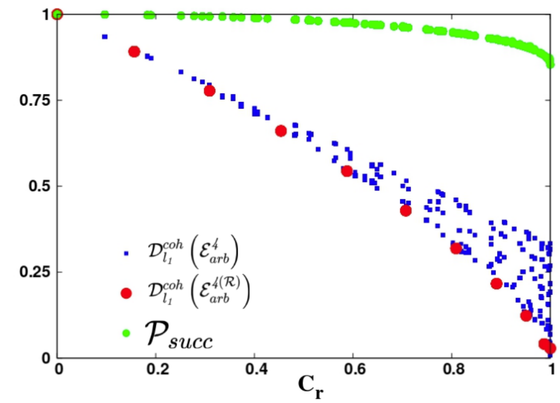

Figure 3. Coherence deficit (CD), $\mathcal{D}^{coh}_{l_1}$ (ordinate) and $\mathcal{P}_{succ}$ (ordinate) with respect to C_r (abscissa) for arbitrary product ensembles in $2 \otimes 2$. Squares (blue), big circles (red) and small circles (green) correspond to CD values for general ensembles with complex coefficients, CD values for arbitrary ensembles with real coefficients and success probabilities respectively. Again the maximum value is obtained for the ensembles which is two-way distinguishable by LOCC with minimum rounds and otherwise, they are non optimal. Both the axes are dimensionless.

processes. We find that for the NLWE basis, $\mathcal{E}_{nlwe} = \{|1\rangle|1\rangle, |0\rangle|0+1\rangle, |0\rangle|0-1\rangle, |2\rangle|1+2\rangle, |2\rangle|1-2\rangle, |1+2\rangle|0\rangle, |1-2\rangle|0\rangle, |0+1\rangle|2\rangle, |0-1\rangle|2\rangle\}$ where $|i+j\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |j\rangle)$, $\text{MEC}^n_{l_1}(\mathcal{E}_{nlwe}) = 0.491$ while for the "Tiles UPB", $\text{MEC}^n_{l_1}(\mathcal{E}_{tiles}) = 0.772$ with $\mathcal{E}_{tiles} = \{|0\rangle|0-1\rangle, |2\rangle|1-2\rangle, |1-2\rangle|0\rangle, |0-1\rangle|2\rangle, \frac{1}{3}(|0\rangle+|1\rangle+|2\rangle)\otimes(|0\rangle+|1\rangle+|2\rangle)\}$. All of them are less than unity, thereby detecting their quantumness via MEC successfully.

To check the effectiveness of MEC, let us omit one of the states of the ensemble and make them distinguishable. For example, if one omits the fourth state from NLWE basis which we call it as $\mathcal{E}'_{nlwe}$, the new ensemble is distinguishable with four rounds of LOCC protocol provided B starts the protocol as shown in Ref. [3] and we compute $\text{MEC}^n_{l_1}(\mathcal{E}'_{nlwe}) = 0.567$ which is higher than the value obtained for the full product basis. On the other hand, removing fifth state from the Tiles UPB, the ensemble can be distinguished with three rounds of LOCC although it is independent of the parties who start the protocol as shown schematically in Fig. 2 and $\text{MEC}^n_{l_1}$ gets increased and is 0.875.

B. Coherence Deficit

Based on the previously introduced quantities, one can intuitively understand that the minimal difference between the total local coherence and minimum ensemble coherence can characterize the inherent quantum features responsible in local indistinguishability among product states. The difference can be referred to as *'coherence deficit'* (CD), represented as

$$\mathcal{D}^{coh}_{l_1(\text{rel})} = |\tau^C_{l_1(\text{rel})} - \text{MEC}_{l_1(\text{rel})}|. \quad (6)$$

Again the ensemble which are two-way locally distinguishable with minimum rounds gives the maximum while the deviation of the maximal value detects the difficulty in distinguishing product ensembles via LOCC.

Arbitrary ensembles in $2 \otimes 2$. Considering the ensemble, $\mathcal{E}^4_{arb}$ with complex coefficients (i.e., with $\phi_i \neq 0, i = 1, 2$ in Eq. (5)), we observe that $\mathcal{D}^{coh}_{l_1}$ decreases with the variation of C_r and the maximum value is attained for states having $C_r = 0$ (see Fig. 3). Clearly, the ensemble for which it reaches unity (after dividing it with the maximum value 3.0) is two-way distinguishable with minimum round, i.e., the ensemble, $\mathcal{E}_1$, even when $\{|0\rangle, |1\rangle\}$ is replaced by $\{|\eta\rangle, |\eta^{\perp}\rangle\}$.

Non-uniqueness. We notice that several ensembles corresponding to different values of DC may yield same C_r value as shown in Fig. 3, although we observe that nonmaximal CD never corresponds to vanishing C_r.

Higher dimensional ensembles. The similar decreasing pattern for $\mathcal{D}^{coh}_{l_1}(\mathcal{E}^6_{arb})$ with the increase of relative local coherence can also be found in $2 \otimes 3$ and the nonoptimality of $\mathcal{D}^{coh}_{l_1}$ guarantees the difficulties in distinguishing sets of orthogonal product states. Moreover, we find that $\mathcal{D}^{coh}_{l_1}(\mathcal{E}_{nlwe}) = 4.076$ and $\mathcal{D}^{coh}_{l_1}(\mathcal{E}_{tiles}) = 1.823$. Similar calculation reveals that for Pyramid UPB, given by $\{|v_0\rangle|v_0\rangle, |v_1\rangle|v_2\rangle, |v_2\rangle|v_4\rangle, |v_3\rangle|v_1\rangle, |v_4\rangle|v_3\rangle\}$ where $v_i = N(\cos\frac{2\pi i}{5}, \sin\frac{2\pi i}{5}, h)$ with $N = \frac{2}{\sqrt{5+\sqrt{5}}}$ and

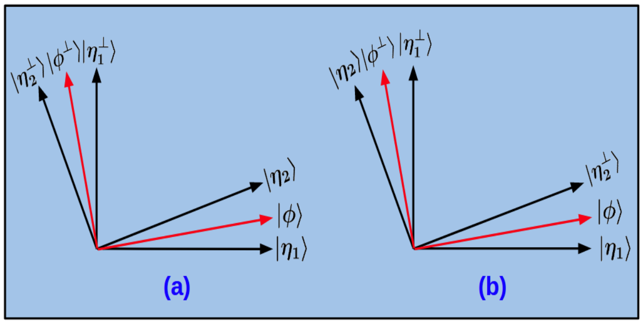

Figure 4. Schematic diagram of finding the configuration of the ensemble and searching for the set of rank-1 projectors to be performed by B for arbitrary product ensembles in $2 \otimes 2$. For a given ensemble in $2 \otimes 2$ dimension, there are two possible configurations which are shown in (a) and (b). Relevant rank-1 projective measurements to reduce the ensemble to any of the four possible sets, $S_i, i = 1 \ldots 4$, given in Eq. (7), are also shown for each configurations. For case (a), the outcome of the projectors P and $P^\perp$ ensure that the ensemble reduces to the set S_1 and S_4 respectively, while for case (b), the outcome of the same conclude the reduction to S_2 and S_3 respectively.

$h = \dfrac{\sqrt{1+\sqrt{5}}}{2}$, MEC_{l_1} and $\mathcal{D}_{l_1}^{coh}$ are respectively 7.055 and 1.197. Therefore, we again find that the CD values deviate from the maximum value for locally indistinguishable sets of states.

IV. PRESCRIPTION FOR PROBABILISTIC LOCC DISCRIMINATION AND ITS CONNECTION WITH COHERENCE-BASED MEASURES

The aim of this section is two-fold. First, we provide a prescription of discriminating arbitrary orthogonal product ensembles which are one-way locally distinguishable, following the similar techniques of BB84 protocol and the protocol introduced in Ref. [32]. Secondly, we show that the optimal success probabilities follow the similar trend like the quantumness measures based on coherence, MEC and CD, for ensembles with the variation of relative local coherence.

A. Success probabilities of arbitrary two qubit product ensembles and its relation with coherence

Let us illustrate the method of probabilistic discrimination for $\mathcal{E}^4_{arb}$ which can be distinguished by LOCC provided Alice starts the protocol. For discriminating probabilistically, we assume that instead of Alice, Bob starts the protocol, for which the deterministic discrimination of the ensemble is not possible, and hence Bob's aim is to maximize the success probability of LOCC discrimination. In general, if one restricts the LOCC protocol in such a way that the party who does not share the computational basis, has to start the protocol, the deterministic discrimination is not possible and hence the probability of discriminating ensembles has to be optimized. In particular, B first measures his qubit, shares the outcome with A via CC, A then measures her qubit and decides the state which was given to them from the ensemble. A has to measure her qubit in $\{|0\rangle, |1\rangle\}$ basis and can distinguish the state if the measurement of B reduces the ensemble to any one of the four possible sets, given by

$$\begin{aligned} S_1 &= \{|0\eta_1\rangle, |1\eta_2\rangle\},\ S_2 = \{|0\eta_1\rangle, |1\eta_2^\perp\rangle\}, \\ S_3 &= \{|0\eta_1^\perp\rangle, |1\eta_2\rangle\},\ \text{and } S_4 = \{|0\eta_1^\perp\rangle, |1\eta_2^\perp\rangle\}. \end{aligned} \tag{7}$$

The procedure described below can then be followed to distinguish $\mathcal{E}^4_{arb}$ with maximum success probability.

1. *Finding the configuration.* For a given ensemble, $\mathcal{E}^4_{arb}$ for which $|\eta_1\rangle$ and $|\eta_2\rangle$ are known, B can find the absolute value of the inner products $\langle\eta_1|\eta_2\rangle$ and $\langle\eta_1|\eta_2^\perp\rangle$ for the given ensemble in order to determine the configuration. If he finds that $|\langle\eta_1|\eta_2|\rangle$ is higher than $|\langle\eta_1|\eta_2^\perp\rangle|$, the possible configuration of the ensemble is shown in Fig. 4(a), otherwise the configuration to be chosen is depicted in Fig. 4(b).

2. *Fixing the set of rank-1 projectors for party B.* After determining the configuration, B has to find a set of rank-1 projectors which can reduce the ensemble to any one of the four possible sets, $S_i, i = 1, \ldots, 4$, as mentioned before with maximum success probability.
E.g., if a given ensemble belongs to the configuration in Fig. 4(a), B can construct a set of rank-1 projectors, $\{P = |\phi\rangle\langle\phi|, P^\perp = |\phi^\perp\rangle\langle\phi^\perp|\}$, such that the outcome of P ensures that the qubit of B is either in the state $|\eta_1\rangle$ or in the state $|\eta_2\rangle$ which leads to the reduced set, S_1. On the other hand, the outcome of $P^\perp$ indicates that the reduced set is S_4. In this case, the set of optimal projectors is chosen in such a way that the quantity $|\langle\phi|\eta_1\rangle|^2 + |\langle\phi|\eta_2\rangle|^2$ or $|\langle\phi^\perp|\eta_1^\perp\rangle|^2 + |\langle\phi^\perp|\eta_2^\perp\rangle|^2$) is maximum over all projectors. In this case, after optimizing, $|\phi\rangle$ can be found to be intermediate state between $|\eta_1\rangle$ and $|\eta_2\rangle$ and the optimal probability of successful discrimination is given by $\mathcal{P}_{succ} = |\langle\phi|\eta_1\rangle|^2$.

3. *Measurements of A.* After determining the correct reduced set, A has to measure her qubit in the computation basis, $\{|0\rangle, |1\rangle\}$, to decide the state given from the ensemble.

The success probabilities with the variation of relative local coherence are shown for arbitrary two-qubit product ensembles $\mathcal{E}^4_{arb}$ with real and complex coefficients in Figs. 1 and 3. In both the scenarios, we find that $\mathcal{P}_{succ}$ reaches its maximum value, unity when C_r vanishes. It decreases with the increase of C_r which is in good agreements with the patterns of minimum ensemble coherence and coherence deficit. Therefore, it demonstrates that the coherence-based measures introduced here to characterize quantumness in ensembles can not only serve as a detector but it has potential to quantify quantum features in ensembles.

B. Relating success probabilities in higher dimensional ensembles with coherence

Let us now check whether the connections between $\mathcal{P}_{succ}$ and coherence remain valid in higher dimensions. Before doing that, let us present briefly the modifications required in each step to obtain success probabilities, discussed for $\mathcal{E}^4_{arb}$. Let us first start the discussion with product ensemble, $\mathcal{E}^6_{arb}$ in $2\otimes 3$. In this scenario with restricted LOCC from B to A, the ensemble can be distinguished with optimum probability if the measurement of B reduces the ensemble to any one of the nine possible sets,

$$\begin{aligned}
&S_1=\{|0\eta_1\rangle,|1\eta_2\rangle\}, S_2=\{|0\eta_1^{\perp}\rangle,|1\eta_2\rangle\},\\
&S_3=\{|0\eta_1^{\perp\perp}\rangle,|1\eta_2\rangle\}, S_4=\{|0\eta_1\rangle,|1\eta_2^{\perp}\rangle\},\\
&S_5=\{|0\eta_1^{\perp}\rangle,|1\eta_2^{\perp}\rangle\}, S_6=\{|0\eta_1^{\perp\perp}\rangle,|1\eta_2^{\perp}\rangle\},\\
&S_7=\{|0\eta_1\rangle,|1\eta_2^{\perp\perp}\rangle\}, S_8=\{|0\eta_1^{\perp}\rangle,|1\eta_2^{\perp\perp}\rangle\}, \text{ and}\\
&S_9=\{|0\eta_1^{\perp\perp}\rangle,|1\eta_2^{\perp\perp}\rangle\}. \qquad (8)
\end{aligned}$$

The prescription consisting of finding the configuration, setting of rank-1 projective measurements by B and finding the probabilities gets modified with the increase of dimensions in the following way:

1. Since the states in the ensemble are known to both the parties, they can find the configuration easily by just finding the absolute value of the inner products, given by $\langle\eta_1|\eta_2\rangle$, $\langle\eta_1|\eta_2^{\perp}\rangle$, $\langle\eta_1|\eta_2^{\perp\perp}\rangle$, $\langle\eta_1^{\perp}|\eta_2\rangle$, $\langle\eta_1^{\perp}|\eta_2^{\perp}\rangle$, $\langle\eta_1^{\perp}|\eta_2^{\perp\perp}\rangle$, $\langle\eta_1^{\perp\perp}|\eta_2\rangle$, $\langle\eta_1^{\perp\perp}|\eta_2^{\perp}\rangle$, $\langle\eta_1^{\perp\perp}|\eta_2^{\perp\perp}\rangle$.

   E.g., $|\langle\eta_1|\eta_2\rangle||$, $\langle\eta_1^{\perp}|\eta_2^{\perp}\rangle|$ and $|\langle\eta_1^{\perp\perp}|\eta_2^{\perp\perp}\rangle|$ are relatively higher than the rest of the absolute values of the inner products, the possible configuration of the ensemble is given in Fig. 5(a).

2. After fixing the configuration, B requires to find a set of optimal rank-1 projective measurement that can reduce the ensemble to any one of the sets stated earlier with maximum success probability. If the given ensemble has the configuration in Fig. 5(a), the measurement of B reduces to the ensemble to any one of the sets S_1, S_5 and S_9 depending on the outcome of the measurement. In this situation, a set of rank-1 projectors for B reads as $\{P_1=|\phi\rangle\langle\phi|, P_2=|\phi^{\perp}\rangle\langle\phi^{\perp}|, P_3=|\phi^{\perp\perp}\rangle\langle\phi^{\perp\perp}|\}$, where the outcome of P_1 ensures that state at B's end is either $|\eta_1\rangle$ or $|\eta_2\rangle$ and the corresponding reduced set is S_1. Similarly, for the outcome of P_2, the sets are either S_5 or S_9. Hence to confirm the actual reduced set corresponding to the projector P_2, B has to find the absolute value of the inner products $\langle\phi^{\perp}|\eta_1^{\perp}\rangle$, $\langle\phi^{\perp}|\eta_2^{\perp}\rangle$, $\langle\phi^{\perp}|\eta_1^{\perp\perp}\rangle$ and $\langle\phi^{\perp}|\eta_2^{\perp\perp}\rangle$. For this configuration, given in Fig. 5(a), either the first two inner products are relatively higher than the rest or the opposite with the last two occurs. If B finds the former, the outcome of P_2 infers the reduced set to be S_5 and evidently the outcome of P_3 reduces to the set S_9, and vice versa (see Fig. 6). Let us now address the question how the set of optimal projectors can be found. We now illustrate a procedure that B can follow to systematically find the optimal projector if the ensemble has the configuration Fig. 5(a). Of course, the similar procedure after modification can be followed for other configurations.

   (a) First, B finds the following set of the elements:
   $P=\{|\langle\eta_1|\phi\rangle|$, $|\langle\eta_2|\phi\rangle|$, $|\langle\phi^{\perp}|\eta_1^{\perp}\rangle|$, $|\langle\phi^{\perp}|\eta_2^{\perp}\rangle|$, $|\langle\phi^{\perp}|\eta_1^{\perp\perp}\rangle|$,$|\langle\phi^{\perp}|\eta_2^{\perp\perp}\rangle|$, $|\langle\phi^{\perp\perp}|\eta_1^{\perp}\rangle|$, $|\langle\phi^{\perp\perp}|\eta_2^{\perp}\rangle|$, $|\langle\phi^{\perp\perp}|\eta_1^{\perp\perp}\rangle|$, $|\langle\phi^{\perp\perp}|\eta_2^{\perp\perp}\rangle|\}$
   with $\{|\phi\rangle,|\phi^{\perp}\rangle,|\phi^{\perp\perp}\rangle\}$ being given by

   $|\phi\rangle = \sin\theta\cos\phi|0\rangle+\sin\theta\sin\phi e^{i\phi_1}|1\rangle+\cos\theta e^{i\phi_2}|2\rangle$, $|\phi^{\perp}\rangle = -\sin\phi|0\rangle+e^{i\phi_1}\cos\phi|1\rangle$, $|\phi^{\perp\perp}\rangle=\cos\theta\cos\phi|0\rangle+\cos\theta\sin\phi e^{i\phi_1}|1\rangle-\sin\theta e^{i\phi_2}|2\rangle$.

   (b) Secondly, for a fixed ensemble, the parameters in the state $|\phi\rangle$ (i.e., $\theta,\phi,\phi_1,\phi_2$) can be varied and arranging the set P in descending order for the specified values of the parameters $(\theta,\phi,\phi_1,\phi_2)$, the set, P_{des} is formed where each pair from the first three pairs of the elements corresponds to the absolute value of the inner products between any one of the three states $\{|\phi\rangle,|\phi^{\perp}\rangle,|\phi^{\perp\perp}\rangle\}$ and its two adjacent states (e.g., if we choose $|\phi\rangle$ such that its adjacent states are $|\eta_1\rangle$ and $|\eta_2\rangle$, the pair of the elements $|\langle\phi|\eta_1\rangle|$ and $|\langle\phi|\eta_2\rangle|$ will be one in the first three pairs). The sum of the first six elements from the set P_{des} is maximized over all possible values of the parameters which lead to the optimal projectors. E.g., if the first six elements of the set P_{des} are $|\langle\eta_1|\phi\rangle|$, $|\langle\eta_2|\phi\rangle|$, $|\langle\phi^{\perp}|\eta_1^{\perp}\rangle|$, $|\langle\phi^{\perp}|\eta_2^{\perp}\rangle|$, $|\langle\phi^{\perp\perp}|\eta_1^{\perp\perp}\rangle|$ and $|\langle\phi^{\perp\perp}|\eta_2^{\perp\perp}\rangle|$ respectively (i.e., the outcome of P_1, P_2 and P_3 corresponds to the reduced set S_1, S_5 and S_9 respectively), the probability of

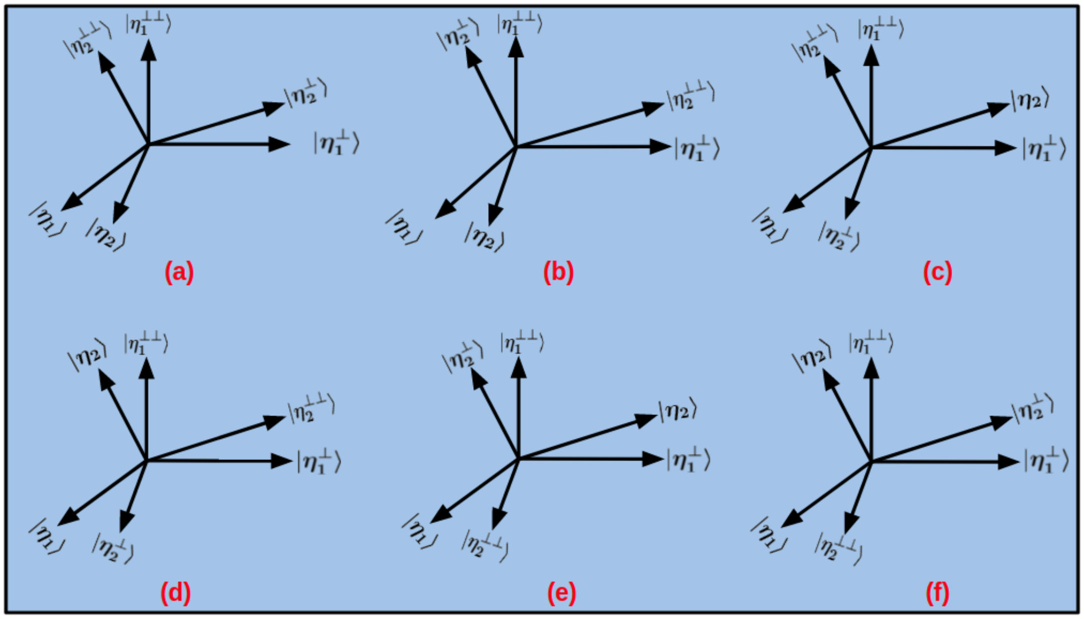

Figure 5. Schematic diagram of the configuration for the arbitrary product ensemble, $\mathcal{E}^6_{arb}$ in $2 \otimes 3$ dimension. In this case, there are six possible configurations as depicted from (a) to (f).

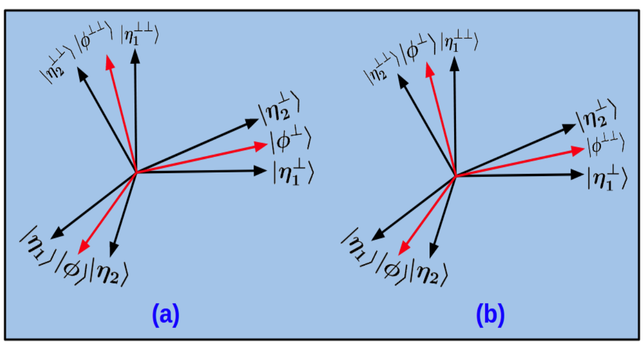

Figure 6. Schematic diagram of finding the set of rank-1 projectors for party B in $2 \otimes 3$ dimension provided the ensemble is found to be in the configuration given in Fig. 5(a). In (a), the outcome of the projectors P_1, P_2 and P_3 ensure that the ensemble reduces to the set S_1, S_5 and S_9 in Eq. (8) respectively. In (b), the outcome of the same reduces to the set S_1, S_9 and S_5 corresponding to the projects P_1, P_2 and P_3 respectively.

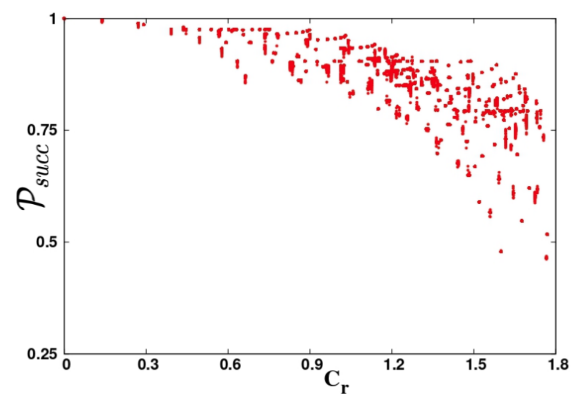

Figure 7. Success probability with one-way LOCC, $\mathcal{P}_{succ}$ (vertical axis) vs. C_r (horizontal axis) for arbitrary product ensembles in $2 \otimes 3$ with real coefficients. Both the axes are dimensionless.

the successful discrimination is the square of the sixth element of the set P_{des} which turns out to be the worst case scenario, i.e., $|\langle \phi^{\perp\perp}|\eta_2^{\perp\perp}\rangle|^2$.

After finding the success probability of discriminating arbitrary product ensembles $\mathcal{E}^6_{arb}$, we observe that the envelopes of $\mathcal{P}_{succ}$ decreases with the increase of relative local coherence (see Fig. 7). Again when C_r vanishes, i.e., the sets of basis vectors $\{|\eta_1\rangle, |\eta_1^{\perp}\rangle, |\eta_1^{\perp\perp}\rangle\}$ and $\{|\eta_2\rangle, |\eta_2^{\perp}\rangle, |\eta_2^{\perp\perp}\rangle\}$ coincide, B can measure and distinguish them deterministically, thereby obtaining $\mathcal{P}_{succ}$ to be unity.

We now briefly sketch the prescription discussed above for the product ensembles in $2 \otimes 3$ to $2 \otimes d$. In this scenario, $\mathcal{E}^{2d}_{arb} = \{|0\eta_1^{(0)}\rangle, |0\eta_1^{(1)}\rangle, \cdots, |0\eta_1^{(d-1)}\rangle, |1\eta_2^{(0)}\rangle, \cdots, |1\eta_2^{(d-1)}\rangle\}$, where $\{|\eta_i^{(0)}\rangle, |\eta^{(1)_i}\rangle, \cdots, |\eta_i^{(d-1)}\rangle\}$, $i = 1, 2$ are d-dimensional mutually orthogonal arbitrary qudit states. If we consider the restricted LOCC from B to A, as described earlier, the ensemble can be locally distinguishable with maximum success probability if the measurement of B reduces the ensemble to any of the d^2 number of possible sets, given by

$$
\begin{aligned}
S_1 = & \quad \{|0\eta_1^{(0)}\rangle, |1\eta_2^{(0)}\rangle\}, S_2 = \{|0\eta_1^{(0)}\rangle, |1\eta_2^{(1)}\rangle\}, \cdots, S_d = \{|0\eta_1^{(0)}\rangle, |1\eta_2^{(d-1)}\rangle\}, \\
S_{d+1} = & \quad \{|0\eta_1^{(1)}\rangle, |1\eta_2^{(0)}\rangle\}, S_{d+2} = \{|0\eta_1^{(1)}\rangle, |1\eta_2^{(1)}\rangle\}, \cdots; S_{2d} = \{|0\eta_1^{(1)}\rangle, |1\eta_2^{(d-1)}\rangle, \\
& \qquad \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
S_{(d^2-d+1)} = & \{|0\eta_1^{(d-1)}\rangle, |1\eta_2^{(0)}\rangle\}, S_{(d^2-d+2)} = \{|0\eta_1^{(d-1)}\rangle, |1\eta_2^{(1)}\rangle\}, \cdots, S_{d^2} = \{|0\eta_1^{(d-1)}\rangle, |1\eta_2^{(d-1)}\rangle\}.
\end{aligned}
$$

In this case, there are $d!$ number of possible configurations possible and hence the first task is to find the configuration of the given ensemble. After fixing the configuration, B can find the optimal projective measurements, discussed above which leads to the final measurement at A's end in the computational basis, giving the optimal $\mathcal{P}_{succ}$.

V. CONCLUSION

In the entanglement resource theory, the free states are the separable ones while the free operations are the local operations and classical communication (LOCC) by which free states can be created. It is natural to predict that the difficulty in discrimination of set of states via LOCC is related to the average entanglement content of the ensembles. However, it was found that such an intuition in LOCC distinguishability does not hold in general. Specifically, it was surprisingly, reported that there are product ensembles, full as well as incomplete basis, which cannot be discriminated by LOCC.

Characterizing properties which are responsible for showing LOCC indistinguishability of product as well as entangled ensembles, is one of the central questions in this field. There are few attempts in this direction which include locally accessible information which fail to characterize product ensembles, entanglement production by using global operations. In this work, we quantify quantumness of ensembles by using the concept of coherence. Specifically, we showed that the coherence of the superposed states in ensembles after suitably rotating them via unitary operations (which are obtained by optimizing coherence locally) can have potential to distinguish product ensembles which are two-way locally distinguishable with minimum rounds of classical communication from the rest. Moreover, we showed that the patterns of coherence-based measures match with the optimal success probability by which the states can be distinguished by LOCC probabilistically.

Among product ensembles, there are several hierarchies present according to their LOCC discrimination protocol. The coherence-based quantifier can capture some features present in the ensembles and so it will be an interesting direction to search for the possible measures of ensembles consisting of both product as well as entangled states which can provide more fine-grained structure.

ACKNOWLEDGEMENT

We acknowledge the support from Interdisciplinary Cyber Physical Systems (ICPS) program of the Department of Science and Technology (DST), India, Grant No.: DST/ICPS/QuST/Theme- 1/2019/23 and SM acknowledges the Ministry of Science and Technology, Taiwan (Grant No. MOST 110- 2124-M-002-012). We acknowledge the use of QIClib – a modern C++ library for general purpose quantum information processing and quantum computing (https://titaschanda.github.io/QIClib) and cluster computing facility at Harish-Chandra Research Institute.

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