Sequential Reattempt of Telecloning

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The task of a telecloning protocol is to send an arbitrary qubit possessed by a sender to multiple receivers. Instead of performing Bell measurement at the sender's node, if one applies unsharp measurement, we show that the shared state can be recycled for further telecloning protocol. Specifically, in case of a single sender and two receivers, the maximal attempting number, which is defined as the maximum number of rounds used by the channel to obtain quantum advantage in the fidelity, turns out to be *three* both for optimal and nonoptimal shared states for telecloning while the maximal number reduces to *two* in case of three receivers. Although the original telecloning with quantum advantage being possible for arbitrary numbers of receivers, we report that the recycling of resources is not possible in telecloning involving a single sender and more than three receivers, thereby demonstrating a no-go theorem. We also connect the maximal achievable fidelities in each round with the bipartite entanglement content of the reduced state between the sender and one of the receivers as well as with the monogamy score of entanglement.

I. INTRODUCTION

Efficient information transmission among distant parties is one of the thriving avenues in the field of communication. Although the existing communication protocols serve our most of purposes, it has been realized that the performance of these classical protocols can be improved qualitatively by using quantum mechanical laws [1–6]. Specifically, successful transmissions of classical and quantum information between a single sender and a single receiver via quantum channels have been proposed and experimentally verified through various protocols like teleportation [2, 7], dense coding [1, 8, 9], quantum key distribution [3, 10–14]. The possible next step is to generalize these protocols in a multiparty scenario, thereby building a communication network or quantum internet which is one of the centre of attentions in recent years [15]. In this direction, prominent works include the measurement-based method for transmitting information over a long distance, known as quantum repeaters based on entanglement swapping [16, 17], combination of cloning and teleportation to share an arbitrary quantum state between the senders and the receivers, called the telecloning scheme [18, 19], distributed quantum computing in a network [20], quantum dense coding network involving multiple senders and a single or two receivers [21]. Notice that in all these situations, shared multipartite entangled states are shown to be the key ingredient for successful realizations [22].

Despite giving advantages during various information processing tasks over their classical counterparts, quantum mechanical laws also enforce stringent conditions on some available resources, recognized as no-go theorems [23–27]. Among them, no-cloning theorem restrain us from copying an arbitrary quantum state perfectly [23, 28, 29] although an approximate universal cloning machine exists by which a quantum state can be copied with a certain fidelity [19, 30, 31]. Taking optimal cloned state and performing teleportaion, information of a quantum state can be transferred from a sender to multiple receivers with optimal fidelity [18, 19, 32, 33] – the protocol is known as telecloning which will be the main focus of this work. Instead of generating optimal clones [30, 31] and sending them to arbitrary number parties which require several bipartite entangled resource states, one can achieve the same task in quantum telecloning protocol with less resources by a single measurement provided the optimal multipartite state is apriori shared between the senders and the receivers. Since the protocol involves a projective measurement at the sender's side which destroys quantum correlations between the sender and the receivers, the shared state cannot be used for any other purpose in later time.

At this point, the natural question arises – *for some reasons, if the single or the multiple receivers do not complete the protocol, can one design a protocol in such a way that the shared entangled state can be reused for telecloning again*? In case of projective measurement performed by the sender, the answer is immediately negative. If we now assume that the measurement process is not perfect, i.e., instead of projective measurement, unsharp (weak) measurements are performed at the sender's side, the answer can be affirmative. We will now concentrate on the scenario where the shared state can be reused for the purpose of telecloning (see Fig. 1 for schematics).

Using weak measurements, such sequential implementations of protocols involving two parties have recently been observed in several directions which include violation of Bell inequalities [34–39], detecting entangled states with the help of steering inequalities [40], and entanglement witnesses [41], the scenario of bi-nonlocal inequalities [42–44], and reusing states for quantum teleportation [45] to name a few. Specifically, it was found that in these sequential scenarios, at most two observers can share the nonlocality [34, 35] and maximum twelve observers can witness entanglement at one side [41] while maximum six receivers can reuse the shared state for teleportation [45]. In multipartite settings, a limited number of works has also been reported in the direction of detecting multipartite entangled states [46–48].

We report here that by using optimal telecloned state, the shared state can be reused *thrice* with the aid of weak measurement at the sender's port, when two receivers are involved while the maximum number for attempting the scheme reduces to *two*, when there are three receivers. Interestingly, we observe that for a fixed number of receivers, maximal attempting number remains unaltered even with the nonoptimal shared state, provided the parameters are adjusted appropriately. Moreover, we find that recycling of state becomes impossible when there are more than three receivers. We also establish a relation between the fidelity obtained in each round of the protocol and the entanglement content of the bipartite reduced state [49] as well as with the monogamy score of entanglement [50–52].

Sec. II illustrates the scenario of recycling the shared multipartite state used in telecloning. In Sec. III, we present the main results, show the maximum number of recycling possible when the optimal state for telecloning is shared between a single sender and two receivers and establish a connection between entanglement content of the shared state and the fidelity obtained in this process while Sec. IV deals with the recycling protocol involving a single sender and an arbitrary number of receivers. In contrast, Sec. V shows the maximum recycling number even when the nonoptimal state is shared between the sender and receivers. We summarize in Sec. VI.

II. PICTURE OF RECYCLING IN NETWORK

Teleportation is the transfer of an arbitrary qubit from a sender, Alice, to a receiver, Charu, with the help of two-bits of classical communication and a shared entangled state between Alice and Charu [2]. When the shared state is maximally entangled, it is always possible to teleport an arbitrary qubit to Charu while in case of a non-maximally entangled state, the fidelity between an arbitrary qubit to be teleported and the state created at Bob's end has to be maximized after optimizing over measurement at Alice's side and the rotation at Bob's part [53].

Let us now consider a scenario involving a single sender and multiple receivers where the sender, Alice, wants to send an unknown qubit to M number of spatially separated receivers, Charus. We call the Alice's end as port, denoted by P. Alice can make multiple clones at her node locally and sends each of them to each Charu via teleportation protocol. However, this is not an efficient protocol and requires much more resource than the scenario when the shared state is multipartite entangled which is known as telecloning protocol. Suppose Alice and M Charus, denoted by $C_1, \dots, C_M$ share a multipartite entangled state, $\rho_{PC_1C_2\dots C_M}$ used as a multi-receiver teleportation channel and the input state which is in possession with

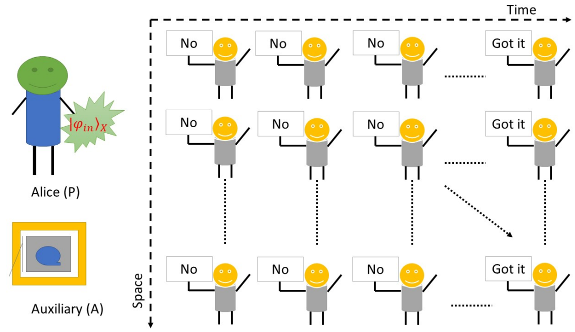

FIG. 1. **Schematics of sequential telecloning protocol.** A single sender, Alice, called port, share an entangled state with M Charus who do not perform the prescribed rotation at their end for n times. To send an arbitrary qubit, Alice performs a weak measurement which creates a possibility to reuse the state after Charus refusal for the next round of telecloning. We assume that M Charus are spatially separated and n attempts to finish the protocol occurs at different time.

Alice to be teleported denoted as $|\phi_{in}\rangle$ at site X. Alice performs a measurement in the Bell-basis, $\{|B_{1,2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), |B_{3,4}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\}$ jointly on her part of the entangled state and the input state. Depending on the classical communication about Alice's measurement outcome, different Charus perform their corresponding unitary operations at their node. In this way, Alice can simultaneously teleport the unknown qubit to different receivers conclusively with an optimal fidelity, which cannot reach to unity due to the no-cloning theorem [23] and is bounded above by the optimal fidelity obtained from approximate universal cloning machine [19, 30, 31].

We now present the entire protocol mathematically. Let us define a string $\{c\} = \{c_1, c_2, \dots, c_M\}$ where $c_i = 1$ if ith Charu C_i is participating in the protocol, and applies local unitary while $c_i = 0$, otherwise. If the initial state can be represented as

$$\rho_{in} = |\phi_{in}\rangle_X\langle\phi_{in}| \otimes \rho_{PC_1C_2\dots C_M}, \tag{1}$$

the reduced state at one of the Charu's end, i, after the measurement and applying the corresponding unitary operation, can be written as

$$\rho_{C_i} = T^i(\rho_{in}) = Tr_{\{\overline{C}_i, X, P\}}(\textstyle\sum_k \mu_k \rho_{in} \mu_k^\dagger) \tag{2}$$

where $\overline{C}_i = \bigcup_{j=1, j\neq i}^{M} C_j$ and $\mu_k = \sqrt{\mathcal{M}_k} \otimes_{j=1}^{M} (U_k)^{c_j}$ with $\sum_k \mathcal{M}_k = \mathbb{I}$ being the set of positive operator valued measurements at Alice's part. In the projective Bell-measurement scenario, $\mathcal{M}_k = |B_k\rangle\langle B_k|$ and the set of unitaries at node i for the output is $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$. For a particular channel used between Alice and multiple Charus, the average fidelity of a teleported state at i-th receiver's node over all possible input states can be defined as

$$F^{i}(\rho_{PAC_1 \ldots C_M}) = \int \langle \phi_{in} | T^{i}(\rho_{in}) | \phi_{in} \rangle d\phi_{in}. \quad (3)$$

Notice that the fidelity obtained here is same even when the measurement performed at Alice's port is the unsharp measurement.

For the telecloning protocol, we require to replace the initial multiparty state between Alice and Charus, with a particular type of multiparty entangled channel from the universal optimal cloning machine [18, 19]. We choose the shared $2M$-partite entangled resource state to be

$$|\psi\rangle_{PAC} = \frac{1}{\sqrt{2}} \left(|0\rangle_P |\phi_0\rangle_{AC} + |1\rangle_P |\phi_1\rangle_{AC} \right), \quad (4)$$

where

$$|\phi_0\rangle_{AC} = \sum_{j=0}^{M-1} \alpha_j \left|A_j\right\rangle_A \otimes |\{0, M-j\}, \{1, j\}\rangle_C ,$$
$$|\phi_1\rangle_{AC} = \sum_{j=0}^{M-1} \alpha_j \left|A_{M-j-1}\right\rangle_A \otimes |\{0, j\}, \{1, M-j\}\rangle_C , \quad (5)$$

with

$$\left|A_j\right\rangle = |\{0, M-j-1\}, \{1, j\}\rangle$$
$$\alpha_j = \sqrt{\frac{2\,(M-j)}{M\,(M+1)}}. \quad (6)$$

Here $|\{0, a\}, \{1, b\}\rangle$ is the symmetric and normalized state of $(a+b)$ qubits with $'a'$ no. of qubits in state $|0\rangle$ and remaining $'b'$ no of qubits are in orthogonal state $|1\rangle$. A refers to the $M-1$ qubit auxiliary system, which is also at Alice's side by convention, although it can be at a different location. In this case, the shared state between Alice and Charus reads as

$$\rho_{in} = |\phi_{in}\rangle_X \langle \phi_{in}| \otimes \rho_{PAC_1 \ldots C_M}$$
$$\rho_{PAC_1..C_M} = |\psi\rangle_{PAC_1..C_M} \langle \psi| \,. \quad (7)$$

Let us now discuss the sequential scenario of the telecloning protocol.

- Suppose only M' number of Charus agree to apply their unitary operations and receive the teleported states. In this situation, we define a map to get back the recycled channel of $(2M-M')$-party state, where M' charus are traced out and finally averaging is performed over uniformly generated input states so that the recycled channel does not depend on a particular input state. Note that finally averaging over the input states is equivalent to initially taking the input state as the average state $\int |\phi_{in}\rangle \langle \phi_{in}| d\phi_{in} = \frac{\mathbb{I}}{2}$ [45]. The recycled channel, in this case, is given by

$$\begin{aligned} & R^{\{c\}}(\rho_{PAC_1 \ldots C_M}) \\ &= \int Tr_{\{\{C_i \forall i; c_i=1\}, X\}} (\sum_i \mu_i \rho_{in} \mu_i^{\dagger}) d\phi_{in} \\ &= Tr_{\{\{C_i \forall i; c_i=1\}, X\}} (\sum_i \mu_i ((\int |\phi_{in}\rangle \langle \phi_{in}| d\phi_{in}) \otimes \rho_{PAC_1 \ldots C_M}) \mu_i^{\dagger} \\ &= Tr_{\{\{C_i \forall i; c_i=1\}, X\}} (\sum_i \mu_i (\frac{\mathbb{I}}{2} \otimes \rho_{PAC_1 \ldots C_M}) \mu_i^{\dagger}. \end{aligned} \quad (8)$$

Note that the map defined above does not act on the auxiliary system, i.e., identity operators only act on them.

- Depending on the type of measurement performed on Alice's part, the recycled channel becomes useful for teleportation in the next round. Instead of projective measurement, if Alice performs an unsharp Bell measurement, given by

$$\mathcal{M}_i^{\lambda} = \lambda \, |B_i\rangle \, \langle B_i| + \frac{1-\lambda}{4} \mathbb{I}_4, \quad (9)$$

where λ is the unsharp parameter, we will show that the channel can further be used for more rounds depending on the residual entanglement. Similarly, we can redefine, $R^{\{c\},\lambda}, F^{i,\lambda}, T^{i,\lambda}$ by replacing μ_i with $\mu_i^{\lambda} = \sqrt{\mathcal{M}_i^{\lambda}} \otimes_{j=1}^{M} (U_i)^{c_j}$ in previously defined $R^{\{c\}}, F^i$, and T^i consecutively.

- Suppose upto round $n-1$, all the receivers refuse to collaborate in the telecloning protocol. The recycled channel through $n-1$ round can be reused in the next round, n and the corresponding average fidelity in the round n can be calculated as

$$\begin{aligned} & F^{i,\lambda_n}(\rho'_{PAC_1 \ldots C_M}) = \\ & F^{i,\lambda_n}(R^{\{0\},\lambda_{n-1}} \cdot R^{\{0\},\lambda_{n-2}} \ldots R^{\{0\},\lambda_1} \cdot (\rho_{PAC_1 \ldots C_M})), \end{aligned} \quad (10)$$

with the definition $\{0\} = \{0, 0, \ldots, 0\}$ where all $c_i = 0$ and λ_i is the unsharp parameter of Alice's measurement in the round i.

- Let us assume that in previous $(n-1)$ rounds, not all Charus are refusing. Hence the bit string of the round, k, $\{c\}_k$ with the information which Charus have refused , the recycling map acts accordingly and we get the required fidelity in the round n as

$$\begin{aligned} & F^{i,\lambda_n}(\rho'_{PAC_1 \ldots C_K}) = \\ & F^{i,\lambda_n}(R^{\{c\}_{n-1},\lambda_{n-1}} \cdot R^{\{c\}_{n-2},\lambda_{n-2}} \ldots R^{\{c\}_1,\lambda_1} \cdot (\rho_{PAC_1 \ldots C_M})) \end{aligned}$$

where $K < M$ and total $(M-K)$ receivers receive the state in all $(n-1)$ rounds and go out of the protocol in further round. We will evaluate all these situations in the succeeding section for a given state.

III. REATTEMPTING VIA OPTIMAL TELECLONED STATE

In this section, we will mainly concentrate on the sequential telecloning protocol which starts with a tripartite entangled state shared between a single sender and two receivers along with the auxiliary state. After the unsharp measurement by Alice, we consider two situations – (1) when both the Charus do not perform the unitary operations for a few rounds, (2) when one of the Charus wishes to finish the protocol while the other one refuses. We are also able to connect the fidelity obtained in each round with the entanglement content of the state in that round.

A. Sequential telecloning with a single sender and two receivers

Let us illustrate this protocol for the simplest scenario having a single sender and two receivers. From Eq. (4), the optimal state in this case can be written as

$$|\psi\rangle_{PAC_1C_2} = \frac{1}{\sqrt{2}}\left(|0\rangle_P\,|\phi_0\rangle_{AC_1C_2} + |1\rangle_P\,|\phi_1\rangle_{AC_1C_2}\right), \tag{11}$$

where

$$|\phi_0\rangle_{AC_1C_2} = \sqrt{\frac{2}{3}}\,|000\rangle_{AC_1C_2} + \sqrt{\frac{1}{6}}\,|101\rangle_{AC_1C_2} + \sqrt{\frac{1}{6}}\,|110\rangle_{AC_1C_2}, \tag{12}$$

and

$$|\phi_1\rangle_{AC_1C_2} = \sqrt{\frac{2}{3}}\,|111\rangle_{AC_1C_2} + \sqrt{\frac{1}{6}}\,|001\rangle_{AC_1C_2} + \sqrt{\frac{1}{6}}\,|010\rangle_{AC_1C_2}. \tag{13}$$

The state between the sender, P, Charus, C_1 and C_2 and the auxillary system along with the state to be teleported can be represented as

$$\rho_{in} = |\phi_{in}\rangle_X\langle\phi_{in}| \otimes \rho_{PAC_1C_2}, \tag{14}$$

$$\text{where } \rho_{PAC_1C_2} = |\psi\rangle\langle\psi|_{PAC_1C_2},$$
$$\text{and } |\phi_{in}\rangle_X = \alpha\,|0\rangle + \beta\,|1\rangle.$$

After the first round of unsharp measurement in Eq. (9), the teleported state at any one of the receiver's side, say C_1, reduces to

$$\rho_{C_1} = T^{1,\lambda}(\rho_{PAC_1C_2}) = Tr_{\{C_2,X,P,A\}}(\sum_i \mu_i^\lambda \rho_{in}\mu_i^{\lambda\dagger}) \tag{15}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{\lambda}{3}\left(|\alpha|^2 - |\beta|^2\right) & \frac{2}{3}\alpha\beta^*\lambda \\ \frac{2}{3}\alpha^*\beta\lambda & \frac{1}{2} + \frac{\lambda}{3}\left(|\beta|^2 - |\alpha|^2\right) \end{pmatrix}$$
$$= \frac{2}{3}\lambda\,|\phi_{in}\rangle\,\langle\phi_{in}| + \frac{3-2\lambda}{6}\mathbb{I}_2. \tag{16}$$

The shared state is symmetric in both the receiver's end, and hence the second Charu, C_2 also obtains the same state. Therefore, after the first round, the expression of required fidelity as a function of the sharpness parameter λ can be computed for a receiver, say, C_1 as

$$\begin{aligned} f_1 &= F^{1,\lambda}(\rho_{C_1}) \\ &= \int \langle\phi_{in}|\,T^{1,\lambda}(\rho_{in})\,|\phi_{in}\rangle\, d\phi \\ &= \int \langle\phi_{in}|\,\rho_{C_1}\,|\phi_{in}\rangle\, d\phi \\ &= \frac{1}{2} + \frac{\lambda}{3}. \end{aligned} \tag{17}$$

It clearly demonstrates that there exists a range of λ above which any arbitrary state can be telecloned to both the receivers with a fidelity more than the classical one, i.e., $2/3$ [54] while the maximal fidelity is in accordance with optimal cloning machine [19, 30, 31], i.e., $f_1 = \frac{5}{6}$ is achieved when the measurement is projective.

1. *Unable to complete the protocol by both the receivers*

Let us now consider the situation when both C_1 and C_2 refuse to perform the corresponding unitary operations required to complete the protocol. Since unsharp measurement is performed at P's node, even after the refusal, the shared resource state can possibly be used for another round of telecloning protocol. As discussed before, the optimal channel has to be recycled and to be used in the second round. The average fidelity for $C_1(C_2)$ in this round can be calculated as

$$\begin{aligned} f_2 &= F^{1,\lambda_2}(R^{\{0\},\lambda_1}(\rho_{PAC_1C_2})) \\ &= F^{1,\lambda_2}(\sum_i \mu_i^{\lambda_1}(\frac{\mathbb{I}}{2}\otimes\rho_{PAC_1C_2})\mu_i^{\lambda_1\dagger}) \\ &= \frac{1}{2} + \frac{P(\lambda_1)}{3}\lambda_2 \\ \mu_i^{\lambda_1} &= \sqrt{\mathcal{M}_i^{\lambda_1}} \otimes_{i=1}^{M}(\mathbb{I}). \end{aligned} \tag{18}$$

The subscript, i in λ_i denotes the round in which measurement is performed. In a similar fashion, if both the Charus refuse to finish the process till round $n-1$, the average fidelity of $C_1(C_2)$ in the round n is found to be

$$\begin{aligned} &f_n \\ &= F^{1,\lambda_n}(R^{\{0\},\lambda_{n-1}}\cdot R^{\{0\},\lambda_{n-2}}\dots R^{\{0\},\lambda_1}\cdot(\rho_{PAC_1C_2})) \\ &= \frac{1}{2} + \frac{P\,(\lambda_1)\,P\,(\lambda_2)\,.....P\,(\lambda_{n-1})}{3}\lambda_n, \end{aligned} \tag{19}$$

where $P(\lambda) = \frac{1}{2}\left[1-\lambda+\sqrt{(1-\lambda)(1+3\lambda)}\right]$. Note here that with the weakness parameter λ_i, the action of the recycled map, $R^{\{0\},\lambda_i}$ depends on the predecided fidelity in the previous rounds. It is due to the fact that Alice performs the measurement having a prefixed λ value according to this predecided fidelity, after which Charus refuse to act. By fixing the fidelity, $f_i > 2/3$, we can calculate the range of unsharp parameter λ_i for $i = 1, \ldots, n-1$. If one demands the high fidelity in the previous round, the weakness parameters also approaches to unity, thereby reducing the quantum correlations in the recycled channel.

Maximum attempting number. Let us now define the maximal number of rounds that a channel can be used such that the quantum advantage in the fidelity can be obtained – we call it as the maximum attempting number (MAN). It is well known that the classically achievable bound in teleportation is $f_{cl} = \frac{2}{3}$ [54] and hence quantum enhancement is guaranteed when the fidelity is above $\frac{2}{3}$. If we demand the lower bound of fidelity in each round to be $f_i \geq f_l \forall i$, we will reach to a round, n_{cr} for which

$$0 < \lambda_i \leq 1 \ \forall i \in \{1, 2, ..., n_{cr}-1\}, \qquad (20)$$
$$\text{and } \lambda_{n_{cr}} > 1.$$

Therefore, the round $(n_{cr}-1)$ signifies the maximal attempting number since $\lambda > 1$ is not a valid measurement. E.g., let us fix $f_l = 0.675$. To satisfy this, we get a range of possible λ values in each round which leads to a fidelity more than f_l for a shared state in Eq. (11), i.e., $\lambda_1 \geq 0.525$, $\lambda_2 \geq 0.664158$, $\lambda_3 \geq 0.992511$ but $\lambda_4 > 1$. In Table. I we report the relation between this lower bound in fidelity f_l and the MAN for the optimal telecloned state (see Fig. 2).

Range of f_l	MAN
$0.6667-0.6754$	3
$0.6755-0.7222$	2
$0.7223-0.8333$	1

TABLE I. Maximal attempting number, MAN, when the fidelity of each round is greater than or equal to f_l.

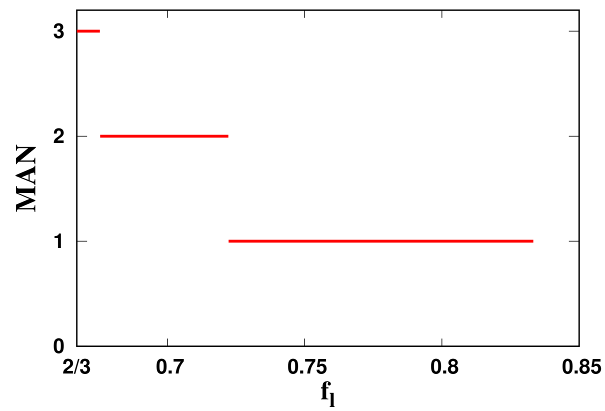

FIG. 2. (Color online.) Maximal attempting number (ordinate) vs. fidelity (f_l). When we demand that each round has fidelity just above the classical one, the maximal attempting number by Charus becomes three while it decreases with the increase of the fidelity in each round. Both the axes are dimensionless.

2. *Completion of protocol by a single receiver*

Let us now assume an asymmetric situation, i.e., any one of the receivers, say C_1, performs the unitary operation communicated by the sender, thereby finishing the telecloning task while C_2 does not finish the protocol. We will now address the question whether the acceptance by C_1 can have any affect on the reuseability of the channel with respect to Alice and C_2. We find that the answer is negative, i.e., reusability of the channel between port and C_2 does not depend on whether C_1 has completed its telecloning protocol or not.

In this picture, C_1 applies the unitary accordingly and leaves the protocol while C_2 does not perform the unitary. Hence in the second round, the entire multipartite protocol reduces to a standard teleportation with a single sender-receiver pair. The fidelity achieved by C_1 can be calculated to be same as before, i.e.,

$$f_1^{C_1} = F^{1,\lambda_1}(\rho_{PAC_1C_2}) = \frac{1}{2} + \frac{\lambda_1}{3}. \qquad (21)$$

Using the reduced scenario involving single sender - single receiver teleportation channel, C_2 can achieve fidelity in the round n as

$$f_n^{C_2} = F^{2,\lambda_n}(R^{\{0\},\lambda_{n-1}} \cdot R^{\{0\},\lambda_{n-2}} \cdot \cdot R^{\{c\}_1,\lambda_1}(\rho_{PAC_1C_2})) = \frac{1}{2} + \frac{P(\lambda_1)P(\lambda_2).....P(\lambda_{n-1})}{3}\lambda_n, \qquad (22)$$

where $\{c\}_1 = \{1,0\}$ carries information that C_1 has finished while C_2 has not. Comparing Eqs. (20) and (23), we can confirm that the maximum attempting number still remains three provided that each round has fidelity just above the classical one. Moreover, we notice that the achievable fidelity by C_2 in each round does not depend on C_1's refusal or acceptance on the first round.

B. Connecting entanglement with fidelity in a sequential scenario

We will establish a connection between entanglement content of the shared state used in each round and the

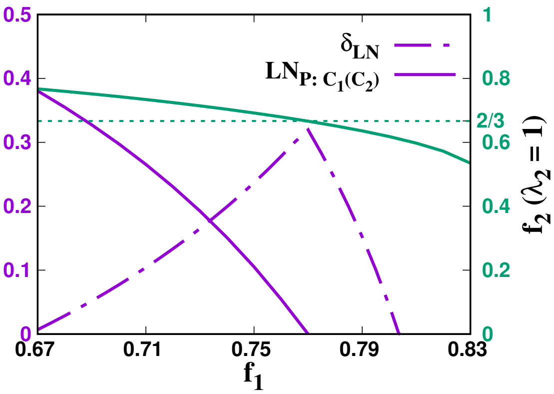

FIG. 3. (Color online.) Maximum achievable fidelity, f_2 (ordinate) with $\lambda_2 = 1$, i.e., with the projective measurement in the second round against f_1 (abscissa) in the first round of the protocol. Entanglement in the bipartition $LN_{P:C_1(C_2)}$ and monogamy score, δ_{LN} (ordinate) defined in Eq. (C2) obtained in the second round, i.e., of the first recycled state shared between port, auxiliary system and two receivers with respect to f_1 (abscissa). Here we assume that in the first round, unsharp measurement is performed and both the Charus have not performed the unitary operations. All the axes are dimensionless.

fidelity obtained in that round. To quantify bipartite entanglement, we choose logarithmic negativity (LN) in Eq. (B2) for the reduced state between the sender (port) and one of the receivers. On the other hand, we also relate the shareability of entanglement in each round which we characterize via monogamy of entanglement (see Appendix C).

Let us first calculate the entanglement after the unsharp measurement is performed by the sender in the first round and both C_1 as well as C_2 decline to perform the unitary operations. In this situation, the fidelity, f_1 is given in Eq. (18). For a fixed fidelity, the corresponding entanglement of the first recycled state between the port and one of the receivers, say C_1 can be computed, $LN_{P:C_1}$ after tracing out the auxiliary qubit, and C_2. It takes the form as

$$\begin{aligned} &LN_{P:C_1} = \\ &\log_2(\frac{1}{6}|(0.5 + 3f_1 - \sqrt{2.5 - 3f_1}\sqrt{9f_1 - 3.5} - 2\sqrt{2} \\ &\times \sqrt{(2.5 - 3f_1)(3f_1 - 0.5\sqrt{2.5 - 3f_1}\sqrt{9f_1 - 3.5}))}| + 1). \end{aligned} \tag{23}$$

We find that by demanding high fidelity in the first round, the entanglement in the recycled state decreases with the increase of f_1 and vanishes for a certain f_1 value, i.e., $f_1 = 0.7697$ (as shown in Fig. 3). If one uses this recycled state to perform another telecloning scheme with the projective measurement at the port's end, the fidelity also decreases and goes below the classical limit at the same point where entanglement vanishes as expected.

Let us now analyze the monogamy score of LN, δ_{LN} after the first round by taking port, P as the nodal observer. It specifies the distribution of entanglement between different sites with respect to the port. We observe that δ_{LN} actually reaches maximum at the same point where $LN_{P:C_1}$ vanishes as depicted in Fig. 3.

We now examine the situation after the second round. In this case, in both first and second rounds, unsharp measurements are performed and Charus do not perform the unitary operations. If we now study the behavior of entanglement of the recycled state between the port and C_1 or C_2 in Eq. (B3), we find that it is non-vanishing only when we demand fidelities, f_1 and f_2 to be just above the classical bound in previous rounds, thereby giving fidelity in the third round beyond 2/3 (comparing Figs. 4 (a) and (b)).

IV. NO-GO THEOREM FOR RECYCLING OF TELECLONING WITH MULTIPLE RECEIVERS

Let us now move to the telecloning situation which involves a single sender and arbitrary number of receivers, say M. When the projective measurement is allowed at port's end, the fidelity of the telecloned state gradually decreases with the increase of the number of receivers, M and for $M \to \infty$, the optimal average fidelity goes to 2/3, the classical limit. This result suggests that the opportunity to recycle the shared entangled state should also decrease with the increase of M in case of unsharp measurement at the sender's side. The question that we address here – what is the maximum number of receivers allowed, i.e., the maximum M upto which the recycling can happen?

When M receivers refuse to finish the protocol in all the rounds till $n-1$, the average fidelity at the round n with M number of receivers can be computed as

$$f_n = \frac{1}{2} + \left[\frac{M+2}{6M}\right] P(\lambda_1) P(\lambda_2) .....P(\lambda_{n-1}) \lambda_n. \tag{24}$$

Let us now elaborate the picture when there are three receivers along with a sender. The teleported state at one of the receiver's end, say C_1 after the first round with $M = 3$ looks like

$$\begin{aligned} \rho_{C_1} &= \begin{pmatrix} \frac{1}{2} - \frac{5}{18}\lambda + \frac{5}{9}\lambda|\alpha|^2 & \frac{5\lambda}{9}\alpha\beta^* \\ \frac{5\lambda}{9}\alpha^*\beta & \frac{1}{2} - \frac{5}{18}\lambda + \frac{5}{9}\lambda|\beta|^2 \end{pmatrix} \\ &= \frac{5}{9}\lambda\,|\phi_{in}\rangle\,\langle\phi_{in}| + \left(\frac{1}{2} - \frac{5}{9}\lambda\right)\mathbb{I}_2, \end{aligned} \tag{25}$$

and f_n at the round n can be determined as

$$f_n = \frac{1}{2} + \frac{5}{18} P(\lambda_1) P(\lambda_2) .....P(\lambda_{n-1}) \lambda_n. \tag{26}$$

If we now assume that in each round, the lower bound of the fidelity is taken to be 0.67, i.e., just above the classical bound, the maximum attempting number in case of three receivers reduces to *two* provided in the first

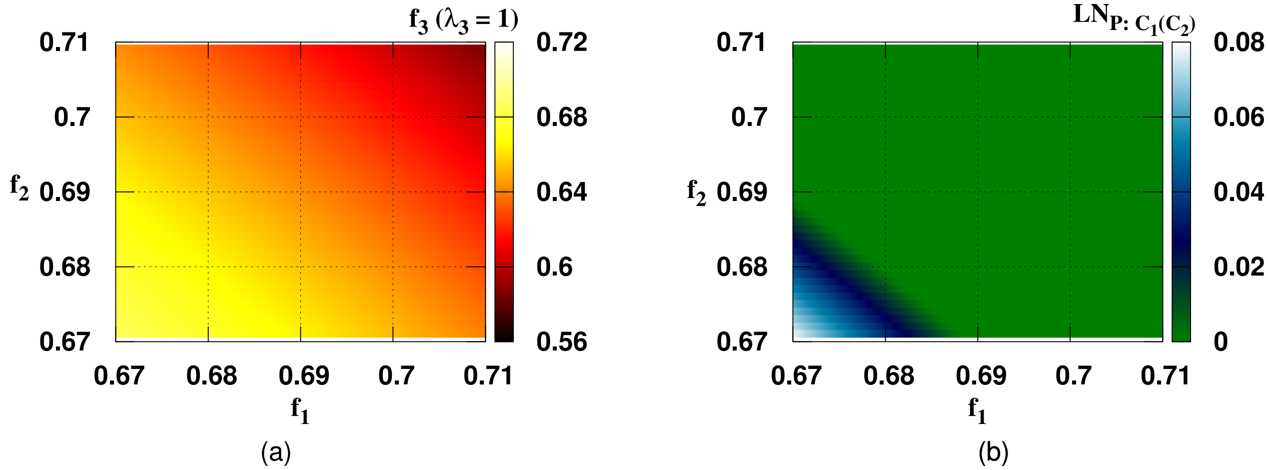

FIG. 4. (Color online.) Features in the second recycled state. (a) Map plot of the fidelity, f_3 with $\lambda_3 = 1$ in the plane of $f_1 - f_2$ (horizontal-vertical axis). (b) The behavior of $LN_{P:C_1(C_2)}$ of the second recycled shared state with f_1 (abscissa) and f_2 (ordinate) which can be used to perform the teleportation in the third round. Both the axes are dimensionless.

round, all the receivers do not perform their unitary operations.

Theorem 1. *No recycling is possible when the number of receivers exceeds three.*

Proof. Analyzing Eq. (24), we realize that with the increase in the number of the receivers, the opportunity for recycling gradually decreases. And when the number of receiver becomes four or more, i.e., $M \geq 4$, leads to a condition on sharpness parameter which is unphysical in the second round. ■

V. CONSEQUENCE OF DISENTANGLING OPERATOR ON ATTEMPTING TELECLONING

Instead of using the optimal state for telecloning, we start the protocol by taking non-optimal state for telecloning as the shared multipartite resource state. Let us first introduce a disentanglement operator $\hat{D}$. The effect of this operator $\hat{D}_i$ on the qubit i in the computational basis is given by

$$\hat{D}_i \left|0\right\rangle_i = \left|0\right\rangle_i, \; \hat{D}_i \left|1\right\rangle_i = \eta_i \left|1\right\rangle_i .$$

Notice that application of this operator on maximally entangled state, say on $|B_3\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$ produces a non-maximally entangled state of the form, $\hat{D}_1 |B_3\rangle = 1/\sqrt{1+|\eta_1|^2}(|01\rangle + \eta_1 |10\rangle)$.

We now apply disentangling operator on each qubit of the two-receiver optimal telecloning state given in Eq. (11) which modifies the state to [33]

$$\begin{aligned}
|\psi(\eta)\rangle_{PAC_1C_2} = B\Big(&|0000\rangle + \frac{\eta_P\eta_{C_1}}{2}|1010\rangle + \frac{\eta_A\eta_{C_1}}{2}|0110\rangle \\
&+ \frac{\eta_P\eta_{C_2}}{2}|1001\rangle + \frac{\eta_A\eta_{C_2}}{2}|0101\rangle \\
&+ \eta_P\eta_A\eta_{C_1}\eta_{C_2}|1111\rangle \Big), \quad (27)
\end{aligned}$$

where

$$\begin{aligned}
B = \Big(1 + &\frac{|\eta_P\eta_{C_1}|^2}{4} + \frac{|\eta_A\eta_{C_1}|^2}{4} + \frac{|\eta_P\eta_{C_2}|^2}{4} \\
&+ \frac{|\eta_A\eta_{C_2}|^2}{4} + |\eta_P\eta_A\eta_{C_1}\eta_{C_2}|^2\Big)^{-\frac{1}{2}}. \quad (28)
\end{aligned}$$

Here the set $\eta = \{\eta_P, \eta_A, \eta_{C_1}, \eta_{C_2}\}$ contains all the disentangling parameters of the port, auxiliary qubit, C_1, and C_2. Note that η_js can be taken to be real without loss of generality [55], each of them varies from 0 to 1 and the unit values of all of them represent the optimal state.

We now study the effect of non-optimal shared state on the maximal attempting number by considering different scenarios which emerge due to the different choices of η_js.

- **Case** 1. Let us take the situation, when $\eta_P = \eta$ while $\eta_{C_1} = \eta_{C_2} = \eta_A = 1$, i.e., when only the port qubit is affected by the disentangling operator. By this action, one expects that the entanglement of the shared state in the bipartition $P : AC_1C_2$ gets reduced, thereby decreasing the performance. The fidelity of the telecloned state in the first round is given by

$$f_1 = \frac{1}{2} + \frac{1+4\eta+\eta^2}{9\left(1+\eta^2\right)}\lambda = \left(\frac{1}{2}+\frac{\lambda}{9}\right) + \frac{4\mathcal{C}\left(\eta\right)}{18}\lambda, \quad (29)$$

with $\mathcal{C}(\eta) = \frac{2\eta}{1+\eta^2}$ is the concurrence [56] of $|B_1\rangle$ after applying the disentangling operator on the first party.

Following the same prescription as discussed in Sec. III, we calculate the fidelity for the round n provided all the Charus have not finished the protocol in previous $(n-1)$ rounds and it is given by

$$f_n = \left(\frac{1}{2} + \frac{P\left(\lambda_1\right)P\left(\lambda_2\right)....P\left(\lambda_{n-1}\right)}{9}\lambda_n\right) + \frac{4\mathcal{C}\left(\eta\right)}{18}P\left(\lambda_1\right)P\left(\lambda_2\right)....P\left(\lambda_{n-1}\right)\lambda_n. \quad (30)$$

In Table. II, we determine the range of η when we choose $f_l = 0.67 > 2/3$, by which the maximum attempting number can be achieved. Interestingly, we find that even in presence of the disentangling operation which reduces the entanglement content of the shared state, there exists a range of η_P by which the maximal attempting number can still remain three as in the optimal shared state, reported in Sec. III.

Range of η_P	MAN
$1-0.7327$	3
$0.7326-0.3675$	2
$0.3674-0.1349$	1

TABLE II. Range of η_P with MAN, when $f_l = 0.67$.

- **Case 2.** Let us now take $\eta_{C_1} = \eta_{C_2} = \eta_C$ and the rest of η_js can be taken as unity, i.e., $\eta_P = \eta_A = 1$. This scenario is in some sense complementary of Case 1 since the disentangling operation in this case acts on the receiver's end. It is interesting to find out which disentangling operations (port or the receivers) have more adverse effects on MAN. In the first round, the optimal fidelity reads as

$$f_1 = \frac{1}{2} + \frac{1}{6}\left(\frac{1+2\eta_C+2\eta_C^3+\eta_C^4}{1+\eta_C^2+\eta_C^4}\right)\lambda, \quad (31)$$

while in the round n, it becomes

$$f_n = \frac{1}{2} + \frac{1}{6}\left(\frac{1+2\eta_C+2\eta_C^3+\eta_C^4}{1+\eta_C^2+\eta_C^4}\right) \times P\left(\lambda_1\right)P\left(\lambda_2\right)\dots P\left(\lambda_{n-1}\right)\lambda_n. \quad (32)$$

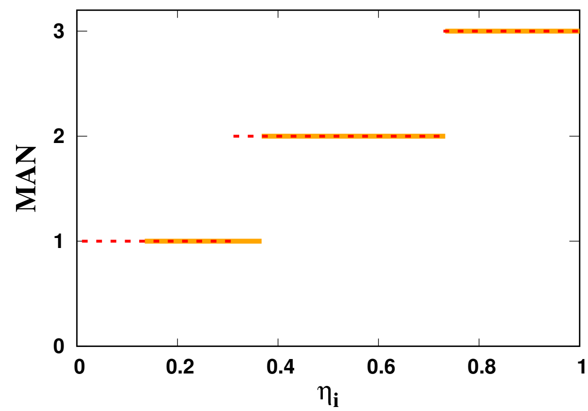

FIG. 5. **MAN vs. disentanglement parameter.** The dashed red line represents the situation with $\eta_i = \eta_{C_1(C_2)} = \eta_C$, while the orange solid line indicates the picture with $\eta_i = \eta_P$. In both the the cases, the maximal attempting number can reach three as obtained in the optimal telecloning case. Both the axes are dimensionless.

We compute the ranges of $\eta_{C_1} = \eta_{C_2} = \eta_C$ (see Table. III) for which the maximal attempting number remains constant to three by choosing $f_l = 0.67$. Comparing Tables II and III, we find that disentangling operation on port has much stronger consequence on the recycling of telecloning process compared to the disentangling operation on the receivers. In Fig. 5, we illustrate the maximal attempting number for the entire range of η_C and η_P when λ is fixed to $= 0.67$. In both the cases, the maximum rounds in which the protocol can be attempted with fidelity more than the classical one goes to three.

Range of η_C	MAN
$1-0.7290$	3
$0.7289-0.3115$	2
$0.3114-0.0101$	1

TABLE III. By choosing f_l just above the classical fidelity, the range of $\eta_{C_1} = \eta_{C_2} = \eta_C$ is listed against MAN.

In this non-optimal scenario, more asymmetry can also be introduced by applying different disentangling operations on different receivers, or in both the port and the receiver's ends and so on. For example, if $\eta_{C_1} \neq \eta_{C_2}$ and $\eta_P = \eta_A = 1$, the fidelity after step n can be written as

$$f_n = \frac{1}{2} + \frac{1}{3}\left[\frac{1+2\eta_{C_1}+2\eta_{C_1}\eta_{C_2}^2+\eta_{C_1}^2\eta_{C_2}^2}{2+\eta_{C_2}^2+2\eta_{C_1}^2+\eta_{C_1}^2\eta_{C_2}^2}\right] \times P\left(\lambda_1\right)P\left(\lambda_2\right)....P\left(\lambda_{n-1}\right)\lambda_n, \quad (33)$$

while the same reduces to

$$f_n = \frac{1}{2} + \left[\frac{1+\eta_C\eta_P\left(2+2\eta_C^2+\eta_C^3\eta_P\right)}{6+3\eta_C^2\left(2\eta_C^2\eta_P^2+\eta_P^2+1\right)}\right] \times P\left(\lambda_1\right)P\left(\lambda_2\right)....P\left(\lambda_{n-1}\right)\lambda_n \qquad (34)$$

when $\eta_{C_1} = \eta_{C_2} = \eta_C$, $\eta_P \neq 1$ and $\eta_A = 1$. By choosing the disentangling parameters suitably, it is always possible to perform the telecloning protocol with quantum advantage for thrice provided all the receivers in previous rounds decline to complete the scheme.

VI. DISCUSSION

Quantum teleportation protocol which illustrates an infinite resource reduction using quantum mechanical systems is one of the main pillars in the field of communication. In particular, to send an arbitrary qubit from a sender to a receiver, quantum protocol requires only two bits of classical communication provided an entangled resource state is shared between them, instead of an infinite amount of classical communication. After the discovery of the theoretical protocol, it has been experimentally verified in several physical systems like photons, continuous variable systems, ion traps etc and has also been extended in several directions. One of the interesting avenues in the field of quantum communication is to build a quantum network involving multiple senders and multiple receivers. In this direction, it was shown that instead of sharing multiple maximally entangled states, genuine multipartite entangled states can have some beneficial role in multipartite quantum communication protocols. For example, Greenberger-Horne-Zeilinger (GHZ) state [57] or W state [58] are found to be useful for both multipartite version of quantum teleportation and dense coding [21, 22, 59].

A prominent example of teleportation in a multipartite domain include the telecloning protocol where a single sender wants to send an arbitrary qubit to multiple receivers, and hence this protocol is restricted by the bounds obtained via the approximate cloning machine. Interestingly, one can find that instead of sharing multipartite entangled states like the GHZ or the W states, the protocol is successful when an optimal state obtained from the cloning machine is shared.

In summary, we investigated the telecloning scheme in the sequential scenario where the resource state can be used in different instances. Such sequential scenarios are applied in different directions, although they are mostly restricted to the identification of entangled states. In this work, we go beyond the detection of entanglement and illustrate the usefulness of unsharp measurement in the multipartite quantum communication protocol. Specifically, unlike projective measurement, the measurement at the sender's side in telecloning is made unsharp so that the quantum correlations between the sender and the receivers do not get destroyed after the measurement and hence the shared state remains useful for some quantum information tasks even after a few rounds of the protocol.

Suppose after the first round, receivers are unable to perform the unitary operations for some reasons, it can be shown that the resulting entangled state due to the unsharp measurement can be reused for another round of the protocol for some suitable range of unsharp parameters. We proved that when the shared state for telecloning is the optimal as well as the nonoptimal multipartite states obtained after applying disentangling operations at the sender's or the receivers' sides or both, the maximum round in which the protocol can be performed with quantum advantage is three. We found that the fidelity obtained in each round is connected with the entanglement content of the shared state between the sender and one of the receivers as well as the monogamy score between the sender and the receivers. We also demonstrate that the reattempting scenario is meaningful in the telecloning scheme only when it involves a single sender and the maximum three receivers.

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Appendix A: Optimal channel evolution

The density matrix corresponds to the optimal state in Eq. (11) is given by

$$\rho_{PAC} = |\psi\rangle_{PAC_1C_2}\langle\psi|\,. \qquad \text{(A1)}$$

After performing POVM measurements n times, the recycled state at round n can be written as

$$\begin{aligned}\rho^n_{PAC_1C_2} = p_n\,|\psi\rangle_{PAC}\langle\psi| + \left(\frac{1-p_n}{6}\right)\Bigg[\frac{\mathbb{I}_4}{2}\otimes|B_3\rangle\langle B_3| \\ +\mathbb{I}_2\otimes\Big(|000\rangle\langle 000| + |111\rangle\langle 111|\Big) \\ +\frac{\mathbb{I}_2}{2}\otimes\Big(|0\rangle\langle 1|\otimes\mathbb{I}_2\otimes|0\rangle\langle 1| + |1\rangle\langle 0|\otimes\mathbb{I}_2\otimes|1\rangle\langle 0| \\ +|0\rangle\langle 1|\otimes|0\rangle\langle 1|\otimes\mathbb{I}_2 + |1\rangle\langle 0|\otimes|1\rangle\langle 0|\otimes\mathbb{I}_2\Big)\Bigg]\end{aligned} \qquad \text{(A2)}$$

where $p = P\left(\lambda_1\right)P\left(\lambda_2\right).....P\left(\lambda_n\right)$ and $n = \{1,2\}$.

Appendix B: Logarithmic Negativity for recycled channel

Let us first give the definition of logarithmic negativity to quantify entanglement in a bipartite state, ρ_{AB}. Based on it, we also compute the monogamy score of entanglement whose definition will also be given below. For any operator, the trace norm can be calculated as $\|A\| = \sqrt{tr(A^\dagger A)}$ which is basically the sum of the singular values. We can define negativity, a non-convex entanglement monotone [49, 60] as

$$\mathcal{N}(\rho_{A:B}) = \frac{\|\rho^{\Gamma_A}\| - 1}{2} \tag{B1}$$

where ρ^{Γ_A} is the partial transpose of $\rho_{A:B}$ with respect to party A [61, 62]. Using this, logarithmic negativity is defined as

$$LN(\rho_{A:B}) = \log \left\|\rho^{\Gamma_A}\right\| = \log(2\mathcal{N}+1), \tag{B2}$$

which reduces to the modulus of a negative eigenvalue in a two-qubit case.

To relate entanglement in the multipartite channel with its reusability and the maximum achievable fidelity in each round, we calculate $LN(\rho_{A:B})$ between Alice (P) and one of the receivers $C_1(C_2)$. For the first recycled channel, we compute logarithmic negativity which is given in Eq. (23) while the same for the second recycled state can be found to be

$$\begin{aligned} LN_{P:C_1(C_2)} = \log_2 \Bigg[\frac{1}{12} \Bigg| & 3.5 + 3f_1 - X_1 + \Big(3f_1 - 2.5 - X_1\Big)\Big(X_2 - X_3\Big) \\ & - 4\sqrt{(3f_1 - 2.5)(3f_1 - 0.5 + X_1)\Big(X_3 - 1\Big)\Big(1 + X_2 + X_3\Big)} \Bigg| + 1 \Bigg]. \end{aligned} \tag{B3}$$

The behavior of the above expression is plotted in Fig. 4 (b).

Appendix C: Monogamy score

In contrast to classical correlations, quantum correlations cannot be shared arbitrarily among parties in a multipartite state. Specifically, in a tripartite state, ρ_{ABC}, if A and B are highly entangled, monogamy of entanglement says that the entanglement content between A and C cannot be large [50–52]. More precisely, the monogamy inequality for a bipartite quantum correlation measure, $\mathcal{Q}$, for a N-party state, $\rho_{A_1 A_2 \ldots A_N}$ can be written as

$$\sum_i \mathcal{Q}(\rho_{A_1:A_i}) \leq \mathcal{Q}(\rho_{A:A_2\ldots A_N}). \tag{C1}$$

Based on this inequality, one can define a shareability measure of entanglement, known as monogamy score of entanglement as

$$\delta_{\mathcal{Q}} = \mathcal{Q}(\rho_{A:A_2\ldots A_N}) - \sum_i \mathcal{Q}(\rho_{A_1:A_i}). \tag{C2}$$

In the paper, we choose LN as a measure of entanglement which we denote it as δ_{LN}.

In the telecloning protocol involving a single sender, two receivers and an auxiliary qubit, the summation in δ_{LN} contains three terms. Among them, $LN(\rho_{P:A}) = 0$ while $LN_{PC_1(C_2)}$ are calculated in Eqs. (23) and (B3). And the first term after the second and third round read respectively as

$$\begin{aligned} & LN_{P:AC_1C_2} = \\ & \log_2(\frac{1}{4}|(-0.5 + 3f_1 - \sqrt{2.5 - 3f_1}\sqrt{9f_1 - 3.5} - 2\sqrt{2} \\ & \times \sqrt{(2.5 - 3f_1)(3f_1 - 0.5\sqrt{2.5 - 3f_1}\sqrt{9f_1 - 3.5})})| \\ & + 1), \end{aligned} \tag{C3}$$

and

$$\begin{aligned} LN_{P:AC_1C_2} = \log_2 \Bigg[\frac{1}{8} \Bigg| & 1.5 + 3f_1 - X_1 + \Big(3f_1 - 2.5 - X_1\Big)\Big(X_2 - X_3\Big) \\ & - 4\sqrt{(3f_1 - 2.5)(3f_1 - 0.5 + X_1)\Big(X_3 - 1\Big)\Big(1 + X_2 + X_3\Big)} \Bigg| + 1 \Bigg], \end{aligned} \tag{C4}$$

where

$$X_1 = \sqrt{2.5 - 3f_1}\sqrt{9f_1 - 3.5}, \tag{C5}$$

$$X_2 = \sqrt{1 - \frac{3f_2 - 1.5}{P(f_1)}}\sqrt{1 + \frac{9f_2 - 4.5}{P(f_1)}}, \tag{C6}$$

$$X_3 = \frac{3f_2 - 1.5}{P(f_1)}, \tag{C7}$$

$$P(f_1) = \frac{1}{4}\Big(5 - 6f_1 + \sqrt{(5 - 6f_1)(18f_1 - 7)}\Big). \tag{C8}$$

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