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On a conjecture of Mahler

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Let R be the field of real numbers. For α in R, let $\|\alpha\|$ be the distance of α from the nearest integer. The following conjecture of Kurt Mahler [*Bull. Austral. Math. Soc.* 14 (1976), 463-465] is proved.

Let m, n be two positive integers $n \geq 2m$. Let S be a finite or infinite set of positive integers with the following properties:

(Q_1) S contains the integers $m, m+1, \ldots, n-m$;

(Q_2) every element of S satisfies

$$\|s/n\| \geq m/n .$$

Then

$$\sup_{\alpha \in R} \inf_{s \in S} \|s\alpha\| = m/n .$$

1. Introduction

Let R be the field of real numbers. For $\alpha \in R$, let $\|\alpha\|$ be the distance of α from the nearest integer. Mahler [3] has proved the following:

THEOREM. *Let S be a finite or infinite set of positive integers with the following two properties:*

(i) S contains the integers $1, 2, \ldots, n-1$;

(ii) S does not contain any multiple of n.

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Then

$$\sup_{\alpha\in R}\inf_{s\in S} \|s\alpha\| = 1/n .$$

Mahler also conjectured the following generalization.

CONJECTURE. Let m, n be two positive integers such that $2m \le n$. Let S be a finite or infinite set of positive integers with the following two properties:

(Q_1) S contains the integers $m, m+1, \ldots, n-m$;

(Q_2) every element of S satisfies the inequality

$$\|s/n\| \ge m/n .$$

Then

$$\sup_{\alpha\in R}\inf_{s\in S} \|s\alpha\| = m/n .$$

Our object in this note is to prove the above conjecture.

2.

We shall use the following:

LEMMA. *Let m be a positive integer. Let K be a convex body in the n-dimensional euclidean space R_n with centre 0 and volume $V(K) > 2^n m$. Then there are m non-zero points $X_1, \ldots, X_m$ of the integral lattice such that*

(i) $X_i \in K$, $1 \le i \le m$,

(ii) $X_i - X_j \in K$, $1 \le i, j \le m$,

(iii) $0 \prec X_1 \prec X_2 \prec \ldots \prec X_m$, *where* $\prec$ *is the lexicographic ordering in* R_n.

REMARK. This result is essentially due to van der Corput [1], but he did not bring out the fact that the points X_i can be chosen to satisfy *(ii)* and *(iii)* also. Here we indicate the necessary modifications to ensure *(ii)* and *(iii)*.

Proof. van der Corput's result rests on the fact that $V(\frac{1}{2}K) > m$ implies the existence of a point $Z \in R_n$ and $(m+1)$ distinct points $Y_0, Y_1, \ldots, Y_m$ such that $Z \in \frac{1}{2}K + Y_i$, $0 \le i \le m$ (see, for example, Lekkerkerker [2], p. 44). We can suppose that the points Y_i are arranged in the lexicographic order

$$Y_0 \prec Y_1 \prec \ldots \prec Y_m .$$

Let

$$X_i = Y_i - Y_0 , \quad 1 \le i \le m .$$

Then $0 \prec X_1 \prec X_2 \prec \ldots \prec X_m$. Also since $\frac{1}{2}K$ is symmetric convex with centre 0, we have

$$X_i = 2 \frac{(Z-Y_0)-(Z-Y_i)}{2} \in K$$

and

$$X_i - X_j = Y_i - Y_j = 2 \frac{(Z-Y_j)-(Z-Y_i)}{2} \in K .$$

This proves the lemma.

3.

Proof of Conjecture. For $\alpha = 1/n$ the condition (Q_2) implies that

$$\inf_{s \in S} \|s\alpha\| \ge m/n .$$

Therefore

$$\sup_{\alpha \in R} \inf_{s \in S} \|s\alpha\| \ge m/n .$$

It remains to prove that for every $\alpha \in R$,

$$\inf_{s \in S} \|s\alpha\| \le m/n .$$

Let $T = \{m, m+1, \ldots, n-m\}$.

Since $S \supset T$ it suffices to prove that there is a $t \in T$ such that

$$\|t\alpha\| \le m/n .$$

If $n = 2m$, then $T = \{m\}$ and $\|m\alpha\| \le \frac{1}{2} = m/n$ for every real number α.

Let $n > 2m$. Let $0 < \varepsilon < (n/2m) - 1$, so that $\frac{m(1+\varepsilon)}{n} < \frac{1}{2}$. Since T is a finite set it is enough to prove that for every such ε, there is a $t \in T$ such that

(1) $$\|t\alpha\| < \frac{m(1+\varepsilon)}{n} .$$

Consider the parallelogram Π with centre 0 defined by

(2) $$|\alpha x - y| < \frac{m(1+\varepsilon)}{n} ,$$

(3) $$|x| < n .$$

If (x, y) is an integral point in Π, then clearly

$$\|x\alpha\| < \frac{m(1+\varepsilon)}{n} .$$

The area of Π is equal to $4m(1+\varepsilon) > 4m$. By the lemma it follows that Π contains m non-zero integral points

$$X_i = (x_i, y_i) , \quad i = 1, 2, \ldots, m , \quad 0 \le x_1 \le x_2 \le \ldots \le x_m ,$$

and

$$X_i - X_j \in \Pi \quad \text{for} \quad 1 \le i, j \le m .$$

We observe that

(i) $x_i > 0$ for each i, because otherwise (2) implies that $|y_i| < \frac{m(1+\varepsilon)}{n} < \frac{1}{2}$ and hence $y_i = 0$;

(ii) $x_i \ne x_j$ when $i \ne j$, because $x_i = x_j$ implies, by (2), that $|y_i - y_j| < \frac{2m(1+\varepsilon)}{n} < 1$ and hence $y_i = y_j$; that is, $X_i = X_j$ so that $i = j$.

Hence

$$1 \le x_1 < x_2 < \ldots < x_m \le n-1 .$$

If $x_i \in T$ for some i, then $\|x_i\alpha\| < \frac{m(1+\varepsilon)}{n}$ and the result follows. Suppose that $x_i \notin T$ for $i = 1, 2, \ldots, m$. Let

$$1 \le x_1 < x_2 < \ldots < x_\gamma < m < n-m < x_{\gamma+1} < \ldots < x_m \le n-1 .$$

Clearly $1 \le \gamma \le m-1$, and $1 \le i \le \gamma$. If the integers $x_i + n - m$, $1 \le i \le \gamma$ are distinct from the integers $x_{\gamma+j}$, $1 \le j < m-\gamma$, then the interval $n-m+1 \le x \le n-1$ contains at least $\gamma + (m-\gamma) = m$ integers, which is impossible because the length of the interval is $m - 2$. Therefore there exist i and j, $1 \le i \le \gamma$, $1 \le j \le m-\gamma$, such that $x_i + n - m = x_{\gamma+j}$. Then $X_{\gamma+j} - X_i = (n-m, y)$ is an integral point in Π. Therefore

$$\|(n-m)\alpha\| < \frac{m(1+\varepsilon)}{n}$$

and $n-m \in T$. Thus the conjecture is proved.

References

[1] J.G. van der Corput, "Verallgemeinerung einer Mordellschen Beweismethode in der Geometrie der Zahlen", *Acta Arith.* 1 (1936), 62-66.

[2] C.G. Lekkerkerker, *Geometry of numbers* (Bibliotheca Mathematica, 8. Wolters-Noordhoff, Groningen; North-Holland, Amsterdam, London; 1969).

[3] Kurt Mahler, "A theorem on diophantine approximations", *Bull. Austral. Math. Soc.* 14 (1976), 463-465.

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