OVERSTABILIZATION OF ACOUSTIC MODES IN A POLYTROPIC ATMOSPHERE

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Abstract. The overstability of sound waves in a polytropic atmosphere is examined for disturbances of arbitrary optical thickness. It is concluded that the Cowling-Spiegel mechanism can operate in the solar convective zone, although the κ-mechanism is predominantly responsible for the observed five-minute oscillations.

1. Introduction

An extensive amount of work has been done to study the instabilities occurring in unstable atmospheres in the framework of the Boussinesq approximation (cf. Spiegel, 1971). These investigations have been largely undertaken with possible applications to outer convection zones in late-type stars. However, it has been shown by Spiegel and Veronis (1960) that the Boussinesq approximation is strictly valid only when the layer depth is small compared to the local density scale-height and as a result it is inadmissible in the study of instabilities which can be excited in these regions of rapidly varying density. Furthermore, the acoustic modes are also filtered out in this approximation. It is for this reason that the full effect of compressibility must be included in any reasonable study of instabilities arising in stellar convective zones.

The main objective of the present paper is to study the growth rates of acoustic modes which can be excited in an unstable compressible layer where thermal dissipation operates. It is proposed to examine the nature of these growth rates and to test their sensitivity to a variety of boundary conditions. The simplest inhomogeneous model incorporating the variation of pressure and density with height is the polytropic atmosphere which has a linear temperature-profile. The polytropic model may be an idealization of the stellar convective zone. Nevertheless, it can include a substantial density variation across the layer and is thus valuable in bringing out the full effects of compressibility in promoting acoustic instabilities.

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It has been recognized that the radiative transfer plays an important role in the behaviour of acoustic modes in the presence of an unstable temperature gradient. Spiegel (1964) has shown that acoustic waves are overstabilized in an unstable polytropic atmosphere provided thermal dissipation obeying Newton's law of cooling operates. The work of Spiegel and the later work of Moore and Spiegel (1966), brought out the essential physical role played by the pressure forces, analogous to the Coriolis forces or the magnetic forces (Cowling, 1957; Chandrasekhar, 1961); this demonstrated that the compressibility can provide the restoring force capable of rendering a layer overstable in the presence of radiative exchange, provided the prevailing temperature gradient is sufficiently superadiabatic. Recently, Graff (1976) investigated the problem for a finite polytropic layer to conclude that the Cowling-Spiegel mechanism makes a non-negligible contribution to the overstabilization of acoustic waves. Such a mechanism has a natural bearing on the observed oscillations in the solar atmosphere and the associated problem of heating the overlying chromospheric layers. The earlier studies of Ulrich (1970) and Leibacher and Stein (1971) were largely motivated to account for the observed five-minute oscillations in terms of the non-propagating response of the solar atmosphere to the trapped acoustic waves excited by overstable oscillations. The acoustic overstability in all the foregoing investigations was worked out in the optically thin approximation. Chitre and Gokhale (1975) demonstrated numerically the existence of acoustic overstability in an unstable polytropic layer for optically thick disturbances. But full problem of non-radial oscillations of the Sun was recently solved by Ando and Osaki (1975) in the framework of the Eddington approximation and it was shown that the driving of the five-minute oscillations is due mainly to the so-called κ-mechanism (cf. Cox and Giuli, 1968) operating in the hydrogen ionization zone.

The work of Ando and Osaki should be considered as a complete analysis of the non-radial modes in the non-adiabatic approximation and it provides a highly plausible explanation of the observed five-minute oscillations. The purpose of the present work is to undertake a hydrodynamical study in order to isolate the Cowling-Spiegel mechanism and to understand the circumstances under which the acoustic waves can be overstabilized for disturbances of arbitrary optical thickness under different boundary conditions. It is concluded that this mechanism does indeed contribute to the self-excitation of acoustic waves in the solar convection zone when there is a large density variation across the layer and provided there is an efficient radiative transfer. Thus, even though the κ-mechanism is dominant for the five-minute oscillations, it is nevertheless of interest to study the problem of self-excitation of sound waves by the effects of compressibility under a variety of boundary conditions. Clearly such a mechanism has applications to the dynamics of pulsating stars and possibly to the wide spectrum of oscillations observed at the solar surface (Deubner, 1976).

The remainder of the paper is arranged in the following way: the basic equations and boundary conditions are set out in Section 2, along with numerical scheme to handle the generalized eigenvalue problem to determine the complex eigenvalues. The numerical results for optically thin and optically thick approximations are described in Section 3. Finally, the discussion and conclusions on the results are given in Section 4.

2. Basic Equations

We consider a polytropic fluid layer confined between two parallel planes situated at $z=0$ and $z=d$ and which is stratified under constant gravity acting in the negative z-direction. The governing equations are the usual hydrodynamical conservation equations for mass, momentum and energy together with the equation of state:

$$\frac{\partial \rho}{\partial t}+\mathrm{div}\,(\rho \mathbf{v})=0\,,$$

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=-\mathrm{grad}\,P+\rho\mathbf{g}\,,$$

$$\rho C_V\frac{\mathrm{d}T}{\mathrm{d}t}+P\,\mathrm{div}\,\mathbf{v}=Q\,,$$

$$P=R\rho T\,.$$

In the energy equation Q is the radiative source term. Throughout we shall assume the gas constant R, the acceleration due to gravity g, the specific heat at constant volume C_V, and the radiative conductivity K to be constants. Let us denote the temperature at the base of the layer by T_{base}. We shall nondimensionalize all the physical quantities with respect to the scale height $H=RT_{\mathrm{base}}/g$, the sound travel time $\sqrt{RT_{\mathrm{base}}}/g$, the pressure and temperature at the base of the layer. We adopt the convention of denoting the unperturbed quantities by a subscript 0 and the perturbed quantities by a subscript 1. The unperturbed temperature is then given by $T_0=1-(1-1/\Gamma)z$, and the unperturbed density $\rho_0=T_0^{1/(\Gamma-1)}$, where the polytropic index

$$\Gamma=\frac{\mathrm{d}\ln P_0}{\mathrm{d}\ln \rho_0}=\frac{m+1}{m}\,.$$

These equations are linearized by writing all physical quantities as $f=f_0(z)+f_1(z)\exp\,(i\alpha_X-\omega t)$, where $\alpha_X=k_XH$ is the dimensionless horizontal wave number and ω the dimensionless eigenvalue which may be complex.

We shall consider two types of disturbances: optically thin and optically thick. For optically thin disturbances Newton's law of cooling becomes applicable and the linearized equations take the following dimensionless form:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}(\rho_0 W) = \frac{\omega(1-\gamma/\Gamma)}{T_0(\gamma\omega+q)}(\rho_0 W) - \left(\frac{\alpha_X^2}{\omega} + \frac{\omega(\omega+q)}{T_0(\gamma\omega+q)}\right)p_1\,,\\
&\frac{\mathrm{d}p_1}{\mathrm{d}z} = \left(\frac{(1-\gamma/\Gamma)}{T_0(\gamma\omega+q)} - \omega\right)(\rho_0 W) - \left(\frac{\omega+q}{\gamma\omega+q}\right)\frac{p_1}{T_0}\,,\\
&\rho_0\theta = \frac{1}{(\gamma\omega+q)}((1-\gamma/\Gamma)(\rho_0 W) + \omega(\gamma-1)p_1),\\
&\rho_1 = \frac{1}{T_0}(p_1 - \rho_0\theta)\,.
\end{aligned}
\tag{1}
$$

For the optically thick approximation the perturbed equations become:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}(\rho_0 W) = \frac{\omega}{T_0}(\rho_0\theta) - \left(\frac{\omega}{T_0} + \frac{\alpha_X^2}{\omega}\right)p_1\,,\\
&\frac{\mathrm{d}\theta}{\mathrm{d}z} = \theta'\,,\\
&\frac{\mathrm{d}p_1}{\mathrm{d}z} = -\omega(\rho_0 W) + \frac{\rho_0\theta}{T_0} - \frac{p_1}{T_0}\,,\\
&\frac{\mathrm{d}\theta'}{\mathrm{d}z} = -\frac{(1-\gamma/\Gamma)}{G_k}(\rho_0 W) + \left(\alpha_X^2 + \frac{\omega\gamma\rho_0}{G_k}\right)\theta - \frac{\omega(\gamma-1)}{G_k}p_1\,.
\end{aligned}
\tag{2}
$$

Here p_1, ρ_1, θ and W are the perturbations in the pressure, density, temperature and vertical velocity respectively. The ratio of specific heats is denoted by γ and G_k is the conductivity parameter

$$
\frac{K}{\rho_0(0)C_V(RT_{\text{base}})^{3/2}} = \frac{\kappa}{(RT_{\text{base}})^{3/2}}\,,
$$

where κ is the radiative diffusivity; q is the inverse of the radiative cooling time which is related to the parameter G_k by $q = G_k(H/\text{mfp})$.

The foregoing equations have to be supplemented by the boundary conditions at the top and at the base of the layer. We have adopted three sets of boundary conditions (cf. Jones, 1976).

(a) Fixed boundary conditions in which the perturbations in the vertical velocity and temperature vanish at both the boundaries:

$$
\begin{aligned}
&\rho_0 W = 0, \qquad \theta = 0 \quad \text{at} \quad z = 0\,,\\
&\rho_0 W = 0, \qquad \theta = 0 \quad \text{at} \quad z = d\,.
\end{aligned}
\tag{3}
$$

(b) Adiabatic boundary conditions at both the boundaries:

$$\rho_0 W = 0, \qquad \theta = \frac{\gamma - 1}{\gamma} \frac{p_1}{\rho_0} \quad \text{at} \quad z = 0\,,$$

$$\rho_0 W = 0, \qquad \theta = \frac{\gamma - 1}{\gamma} \frac{p_1}{\rho_0} \quad \text{at} \quad z = d\,. \tag{4}$$

(c) Free boundary conditions at the top which demands the vanishing of the Lagrangian pressure perturbation and the linearization of the radiative flux condition, and adiabatic boundary condition at the base of the layer:

$$\rho_0 W = 0, \qquad \theta = \frac{\gamma - 1}{\gamma} \frac{p_1}{\rho_0} \quad \text{at} \quad z = 0\,,$$

$$\rho_0 W = \omega p_1, \qquad \theta = \frac{\Gamma - 1}{\Gamma \omega} W + \frac{T_0}{4} \frac{\mathrm{d}\theta}{\mathrm{d}T_0} \quad \text{at} \quad z = d\,. \tag{5}$$

We require all the four boundary conditions for complete specification of the problem for the optically thick disturbances, but for optically thin approximation the resulting equation in $(\rho_0 W)$ comes out to be of second order and we do not require the boundary condition involving θ on both the boundaries, and so the fixed and adiabatic boundary conditions become identical.

It should be noted that for the adiabatic case the boundary conditions for optically thick equations agree with those for the adiabatic equation obtained by setting $q = 0$ in optically thin equations.

The governing equations for the optically thick case may be cast in the form

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = AY,$$

where A is a 4×4 matrix whose elements are functions of complex eigenvalue and Y is the column vector $(\rho_0 W, \theta, p_1, \theta')^T$. A suitable choice of grid points z_i $(i = 0, 1, \ldots, N)$ and the replacement of the equations by the corresponding difference equations enables us to write the equations as:

$$\frac{Y_{i+1} - Y_i}{h} = A \frac{Y_{i+1} + Y_i}{2}, \tag{6}$$

where

$$Y_i = (\rho_0 W(z_i), \theta(z_i), p_1(z_i), \theta'(z_i))^T$$

and h is the step length. This yields four equations in each of the N intervals (between the $N+1$ grid points) and hence a total of $4N$ equations. Along with the four prescribed boundary conditions we get $4(N+1)$ linear homogeneous equations in equal number of unknowns.

By writing a $4(N+1)$ dimensional vector $X=(x_1, x_2, \ldots, x_{4N+4})$ with $x_{4m+1}=\rho_0 W(z_m)$, $x_{4m+2}=\theta(z_m)$, $x_{4m+3}=p_1(z_m)$, $x_{4m+4}=\theta'(z_m)$ $(m=0, 1, \ldots, N)$, we can rewrite Equation (6) as $BX=0$. Here B is a block matrix with 8 non-zero elements in each row except the first two and the last two which in general, will contain only four non-zero elements resulting from the chosen set of boundary conditions. Thus for a non-trivial solution of X we demand that det $(B)=0$ which gives the required dispersion relation since the elements of the matrix B are functions of eigenvalue ω. In the actual numerical solution it was found convenient to investigate the adiabatic approximation when ω is purely imaginary and det (B) also turns out to be purely imaginary. Under such a circumstance the eigenvalues can be located by examining the changes of sign. These were used as initial guess for evaluating the complex eigenvalues for the full non-adiabatic case employing Muller's method.

It was found that the lowest adiabatic mode has no nodes in the pressure perturbation as well as in the vertical velocity. For successively higher modes the number of nodes in pressure perturbation increases by one. The number of nodes in the vertical velocity function are same as that for pressure perturbation when $|\omega|<\sqrt{\gamma}\,\alpha_X$ but for $|\omega|>\sqrt{\gamma}\,\alpha_X$, the number of nodes in vertical velocity is one less than that in pressure perturbation. The lowest mode which corresponds to the Lamb mode in isothermal atmosphere is referred in this work as f-mode, while the successively higher eigenvalues are referred to as $P1$, $P2, \ldots$.

3. Numerical Results

3.1. Optically thin approximation

We have made extensive numerical computations to calculate the complex eigenvalues for the optically thin approximation with fixed boundary conditions. We have adopted two values of the polytropic index $\Gamma=1.15$ and 1.66 with the ratio of the specific heats $\gamma=1.01$, 1.05 and 1.1 for $\Gamma=1.15$, and $\gamma=1.05$, 1.1 and 1.2 for $\Gamma=1.66$. For all the cases considered in this work we have calculated the four lowest eigenvalues corresponding to f, P1, P2 and P3 modes. The lowest f-mode which corresponds to the Lamb wave turns out to be stable for all cases considered. The acoustic modes are determined for two values of the dimensionless horizontal wave number $\alpha_X=1$ and 2, but for only one value of the radiative conductivity parameter $q=0.0528$. It is found that the results for other values of q can be obtained by noting that the frequencies (i.e. the imaginary part of ω) for all the modes are left practically unaltered by the variation of q, while the corresponding growth rates come out to be directly proportional to q. It may be noted that this value of the dimensionless inverse of radiative cooling time corresponds to the radiative diffusivity $\kappa=2\times10^{12}\ \mathrm{cm^2\ s^{-1}}$ which is typically obtained at the top of the solar convection zone and we have chosen the maximum value of q permitted for a given G_k by taking the ratio of (H/mfp) to be unity. The base temperature is taken to be 11 500 K and the results are computed for a number of values of the parameter $T_r = T_{top}/T_{base}$, namely 0.01, 0.1 and 0.5. We can thus look at the results as being applicable to a layer whose base temperature is 11 500 K and the parameter T_r determines the temperature at the top. The layer thickness is given by

$$\frac{d}{H} = \frac{1 - T_r}{(\Gamma - 1)/\Gamma}.$$

Let us first discuss the case $\Gamma = 1.15$ (i.e. $m = 6$). For $\alpha_X = 1$, we get acoustic overstability for all the choices of $\gamma = 1.01$, 1.05 and 1.1 for the value of the parameter $T_r = 0.01$. The case $T_r = 0.01$ represents an almost complete polytrope and the density variation across the layer is so large that acoustic overstability occurs for all superadiabatic temperature gradients chosen. However, when T_r becomes of the order of 0.10 the overstability occurs only for $\gamma = 1.01$ and 1.05, that is when the temperature gradient is moderately superadiabatic, while for $T_r = 0.5$ the overstability disappears for $\gamma = 1.05$ and is present only for $\gamma = 1.01$. Such a value of γ is very close to unity and the γ-mechanism is likely to contribute to the overstability of the oscillations. When the wave number is increased to 2, the overstability is present only for $T_r = 0.01$ (almost complete polytrope) and $\gamma = 1.01$ and 1.05.

For the case of $\Gamma = 1.66$ ($m = 1.5$), the overstability for the wave number equal to 1 is obtained for $T_r = 0.01$ for $\gamma = 1.05$, 1.1 and 1.2, but when T_r is increased to 0.1, the temperature gradient has to be sufficiently superadiabatic to drive the oscillations, that is, the overstability vanishes when γ becomes 1.2, but is present for smaller values of γ, and for $T_r = 0.5$, the overstability for the same wave number shows only for $\gamma = 1.01$ (very likely due to γ-mechanism).

All the results which we have discussed so far were obtained with fixed boundary conditions. We relaxed the boundary condition by letting the top surface to be free and found that the frequencies were only mildly affected but the growth rates tended to increase for free boundary condition. However it is found that for $\Gamma = 1.15$ the growth rates are only very mildly affected by the change in the boundary condition. This means that the free boundary conditions tend to destabilise the layer in optically thin approximation, the effect being more pronounced for higher value of Γ.

The overstability arising from the Cowling-Spiegel mechanism for the optically thin approximation seems to occur when the temperature gradient is sufficiently superadiabatic and the parameter T_r is sufficiently small. In other words, in the notation of Jones (1976) a large enough depth parameter favours overstability. It is worthwhile to observe here that the criterion given by Spiegel (1964) for overstability in the limit $\alpha_X \gg 1$ and $T_r = 0$ namely, $\gamma < 2(m+2)/(2m+3)$ is borne out by our detailed numerical results. Graff (1976) has examined the overstabilization of acoustic waves for optically thin disturbances to find that the joint effect of the superadiabatic temperature gradient and of the radiative dissipation can generate acoustic overstability in a simplified model of the solar convection zone.

3.2. Optically thick approximation

The coupled system of four equations obtained for the optically thick approximation is solved numerically for a variety of boundary conditions. Extensive calculations of dimensionless complex eigenvalue ω giving the growth rates (ω_R) and the frequencies (ω_I) are performed for fixed boundary conditions. The calculations are carried out for three values of dimensionless horizontal wave number $\alpha_X = 0$, 1 and 2, and for three values of the conductivity parameter G_k corresponding to the physical variables applicable at the top of the solar convection zone ($\kappa = 2 \times 10^{12}$ cm^2 s^{-1}, $G_k = 0.0528$), at the base of the layer ($\kappa = 5 \times 10^9$ cm^2 s^{-1}, $G_k = 1.32 \times 10^{-4}$) which is approximately two scale-height deep, and an intermediate value of ($\kappa = 10^{11}$ cm^2 s^{-1}, $G_k = 2.64 \times 10^{-3}$). We have selected four values of the parameter T_r, i.e. 0.01, 0.1, 0.2, and 0.5. The polytropic index Γ is taken to be 1.15 and 1.66, the corresponding value of m is 6.0 and 1.5 respectively. A variety of values of the ratio of specific heat γ ranging from 1.01 to 1.2 for $\Gamma = 1.15$ and from 1.01 to 1.8 for $\Gamma = 1.66$ are chosen. For all cases the four lowest eigenvalues corresponding to f, P1, P2 and P3 modes are calculated. The f-mode which corresponds to the Lamb wave always turns out to be stable for all cases considered. This f-mode is absent for $\alpha_X = 0$ and so in that case the four lowest eigenvalues correspond to P1, P2, P3 and P4 modes. Table I gives a sample of numerical results showing the dimensionless growth rates and the frequencies for the case $\alpha_X = 1$, $\Gamma = 1.66$. It is clear from the results that the frequencies are pretty much insensitive to the variation of G_k, in fact for a large variation of G_k by a factor of 400, the frequencies are hardly changed. They are also seen to be only mildly affected by variations in γ, and are monotonically increasing function of γ. They exhibit the expected monotonically increasing behaviour with α_X. The frequencies also increase with Γ and T_r, when other parameters are kept constant. However the growth rates are drastically influenced by all the above parameters. Moreover it is found that there is no fixed monotonic behaviour with respect to most of the parameters.

It is found in all the cases considered that the growth rates are always monotonically decreasing function of γ when all other parameters are kept constant. Two features that are evident from the numerical results are that acoustic overstability is favoured by small values of T_r, and by a sufficiently large superadiabatic temperature gradient, i.e., when γ is sufficiently less than Γ. Also it is found that the values of γ close to unity (i.e., the γ-mechanism) promotes overstability. It is also found that for small values of T_r the overstability is favoured by the smaller values of G_k while for larger value of T_r the larger values of G_k favour overstability. The most remarkable difference between the optically thick and thin results is that in the optically thick case the growth rates are not proprotional to the conductivity parameter G_k and in fact in most cases they change sign as G_k varies. This is in sharp contrast to almost exact proportionality of better than one tenth of a percent even for values of q as high as 0.0528, for the optically thin case. This is probably a

TABLE I

A sample of numerical results showing the dimensionless growth rates and the frequencies for the case $\alpha_X = 1$, $\Gamma = 1.66$

	Mode	$G_k = 5.28 \times 10^{-2}$		$G_k = 2.64 \times 10^{-3}$		$G_k = 1.32 \times 10^{-4}$	
	T_r	0.1	0.5	0.1	0.5	0.1	0.5
$\gamma = 1.01$	P1	2.20(−2) 1.251	−6.56(−3) 2.314	−7.60(−3) 1.260	−3.21(−3) 2.339	−2.31(−3) 1.272	−3.91(−4) 2.340
	P2	5.25(−2) 2.067	5.66(−3) 4.358	−1.58(−3) 2.012	−4.15(−3) 4.384	−3.66(−3) 2.030	−5.31(−4) 4.386
	P3	2.78(−2) 2.940	7.38(−3) 6.473	7.52(−3) 2.865	−5.15(−3) 6.493	−4.31(−3) 2.883	−6.55(−4) 6.496
$\gamma = 1.05$	P1	1.84(−2) 1.264	−2.17(−2) 2.334	−8.69(−3) 1.276	−8.03(−3) 2.378	−3.02(−3) 1.289	−1.48(−3) 2.384
	P2	3.97(−2) 2.090	−2.24(−2) 4.402	−4.02(−3) 2.044	−1.22(−2) 4.462	−5.02(−3) 2.066	−2.14(−3) 4.470
	P3	7.43(−3) 2.964	−3.93(−2) 6.535	2.71(−3) 2.914	−1.71(−2) 6.611	−6.32(−3) 2.937	−2.75(−3) 6.621
$\gamma = 1.10$	P1	1.37(−2) 1.281	−4.02(−2) 2.359	−1.00(−2) 1.295	−1.39(−2) 2.427	−3.87(−3) 1.310	−2.80(−3) 2.437
	P2	2.43(−2) 2.119	−5.64(−2) 4.457	−6.97(−3) 2.084	−2.20(−2) 4.558	−6.64(−3) 2.110	−4.11(−3) 4.573
	P3	−1.73(−2) 2.994	−9.67(−2) 6.612	−3.04(−3) 2.973	−3.13(−2) 6.756	−8.71(−3) 3.002	−5.29(−3) 6.775
$\gamma = 1.50$	P1	−2.52(−2) 1.427		−1.94(−2) 1.460		−1.03(−2) 1.491	
	P2	−8.38(−2) 2.339		−2.75(−2) 2.383		−1.88(−2) 2.437	
	P3	−1.92(−1) 3.226		−4.22(−2) 3.409		−2.63(−2) 3.480	

manifestation of the fact that the adiabatic limit in the case of optically thick equations represents a singularity unlike the optically thin case where the behaviour is smooth in the limit. Some other less pronounced features of our results are that for larger values of Γ the effect of T_r is more evident than for the small value of Γ. Also in most cases the smaller values of α_X favour overstability. However the behaviour of growth rates with respect to α_X or the frequency ω_I (i.e. the various modes) shows no fixed pattern and depends on the combination of other parameters.

As an example consider the case $\Gamma = 1.15$ and $\alpha_X = 1$. For $T_r < 0.1$ the P1-mode is unstable for all three values of G_k when $\gamma < 1.05$. For $\gamma = 1.1$ it is stable for the highest value of G_k (0.0528) while for lower values of G_k it is still unstable. For $T_r = 0.5$ the P1 mode is always stable, however the P2 and P3 modes are unstable for the highest value of G_k (0.0528) and $\gamma = 1.01$, while for other values of γ and G_k all the modes calculated turn out to be stable. This clearly shows the operation of the γ-mechanism. Now consider the case $\Gamma = 1.66$ and $\alpha_X = 1$. For $T_r = 0.01$ the P2 mode is unstable for all values of G_k when $\gamma \lesssim 1.1$, but for $\gamma > 1.1$ it is stable for the highest value of G_k, while for lower values of G_k it is still unstable. For $T_r = 0.1$ all modes calculated are stable for lowest value of G_k (1.32×10^{-4}). However for the highest value of G_k the P2 mode is unstable for $\gamma \lesssim 1.1$ and stable for highest values of γ. In the case of $T_r = 0.5$ only the P2 and P3 modes for highest value of G_k and $\gamma = 1.01$ are unstable, while all other cases are found to be stable.

We shall now examine the sensitivity of the Cowling-Spiegel mechanism to the boundary conditions applied at the top and the base of the layer. We have discussed the results obtained with fixed boundary conditions, i.e. no flux of momentum across the boundary surfaces which are also held at constant temperatures. Such fixed boundary conditions are evidently not realistic when applied to the solar surface layers. In order to get somewhat closer to the solar surface conditions we have considered two more sets of boundary conditions, i.e. adiabatic and the free boundary conditions as mentioned earlier. The results are displayed in Figures 1, 2, 3, 4 for two values of parameters $T_r = 0.1$ and 0.5, and $G_k = 0.0528$ and 1.32×10^{-4}. The dimensionless growth rate for the P1 mode which has one node in the pressure perturbation are plotted against a_X. Clearly the growth rates for acoustic modes are highly sensitive to the boundary conditions. The growth rates obtained by using the fixed and adiabatic conditions at both the boundaries are seen from the plots not to be too different from each other. The character of the growth rates is however, drastically altered by the free surface conditions applied at the top boundary. For the choice of the parameter $T_r = 0.5$, i.e. a small variation of the temperature across the layer, the free boundary conditions have tended to destabilize the acoustic modes for both choices of the polytropic index Γ and the conductivity parameter G_k. The situation is different for $T_r = 0.1$, or a large variation of the temperature (and hence of density). Here for a large value of G_k the free boundary conditions have always given stable acoustic waves, while the other two sets of boundary conditions have yielded instability. But for a small value of G_k

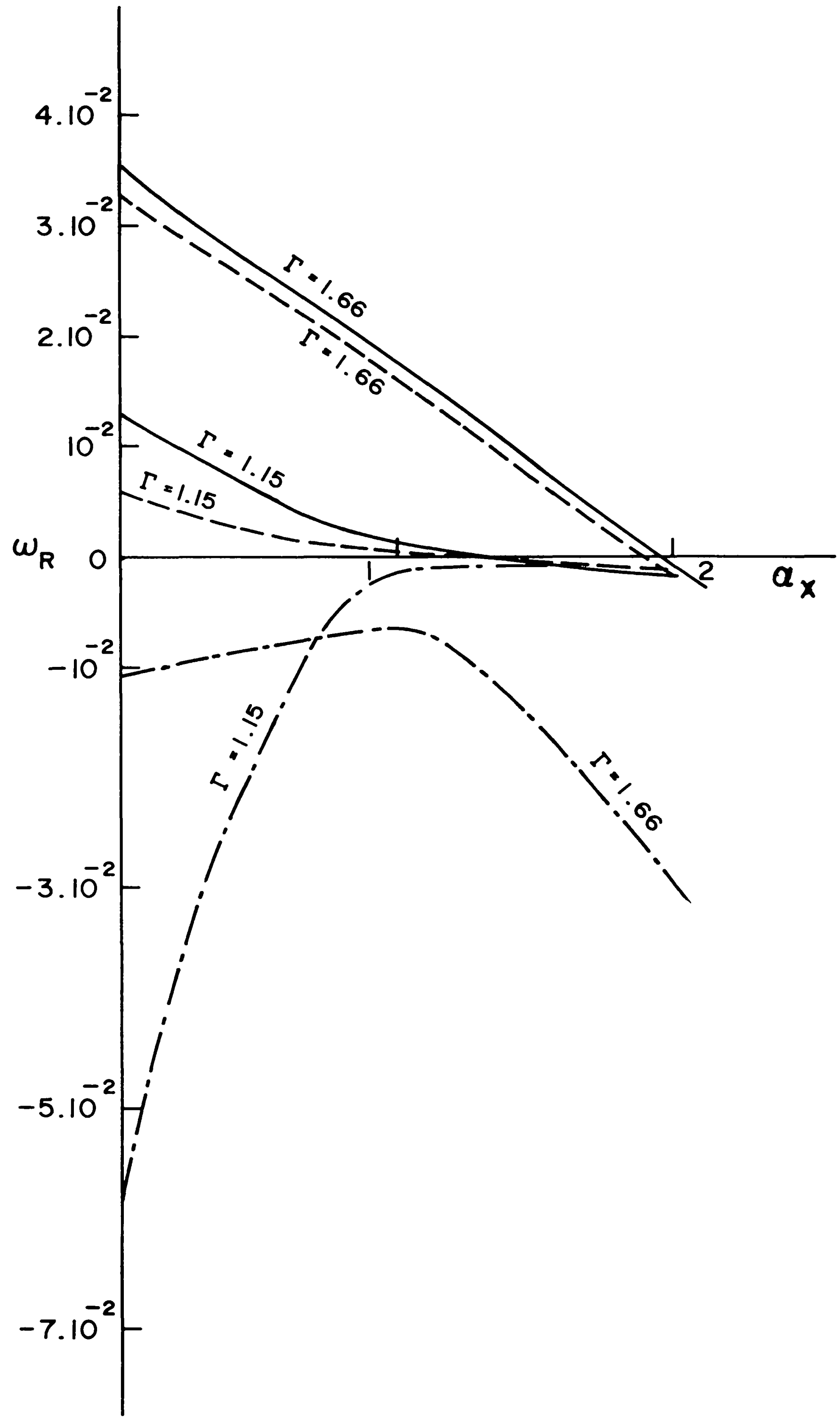

Fig. 1. The dimensionless growth rate ω_R for the P1-mode is plotted against the dimensionless wave number α_X for the value of the parameter $T_r = T_{top}/T_{base} = 0.1$ and $G_k = 5.28 \times 10^{-2}$. The results for the rigid (———), adiabatic (– – – – – –), free (– · – · – · ·) boundary conditions are indicated for two choices of the polytropic index $\Gamma = 1.15$ and 1.66 with $\gamma = 1.05$.

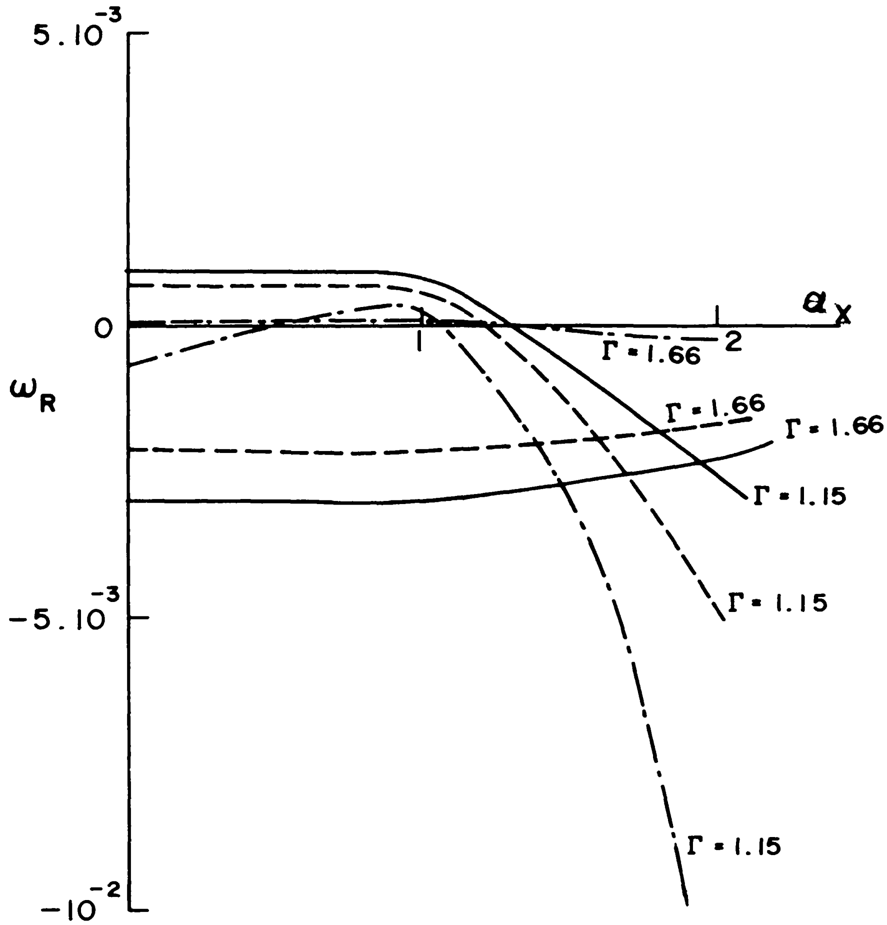

Fig. 2. The dimensionless growth rate ω_R for the P1-mode is plotted against the dimensionless wave number α_X for the value of the parameter $T_r = T_{top}/T_{base} = 0.1$ and $G_k = 1.32 \times 10^{-4}$. The results for the rigid (———), adiabatic (– – – – – –), free (– · – · – · – ·) boundary conditions are indicated for two choices of the polytropic index $\Gamma = 1.15$ and 1.66 with $\gamma = 1.05$.

$(= 1.32 \times 10^{-4})$ the free boundary conditions have tended to overstabilize the acoustic modes for $\Gamma = 1.66$, while for $\Gamma = 1.15$ the situation is reversed. The question of realistic boundary conditions needs a careful examination and we hope to study the effect on the acoustic modes of penetration into an overlying stable layer in a separate communication.

A remarkable feature of the computation is the occurrence of subadiabatic overstability in the optically thick approximation for all sets of boundary conditions and for an almost complete polytrope. It is found that even for $\gamma > \Gamma$ when the prevailing temperature gradient is subadiabatic, some of the modes turn out to be overstable when $T_r \ll 1$ and G_k is small. For example for fixed boundary conditions and $\Gamma = 1.15$, $\gamma = 1.2$ and $T_r = 0.01$ the most unstable mode found is the P2-mode for $G_k = 1.32 \times 10^{-4}$, which has the dimensionless complex eigenvalues as (0.0026, 1.253) for $\alpha_X = 1$, and (0.0045, 1.036) for $\alpha_X = 0$. For $\alpha_X = 2$ all the modes calculated turn out to be stable. Similarly for $\Gamma = 1.66$, $\gamma = 1.7$ and $T_r = 0.01$ several modes turn out to be unstable for the lower values of G_k i.e. 1.32×10^{-4}

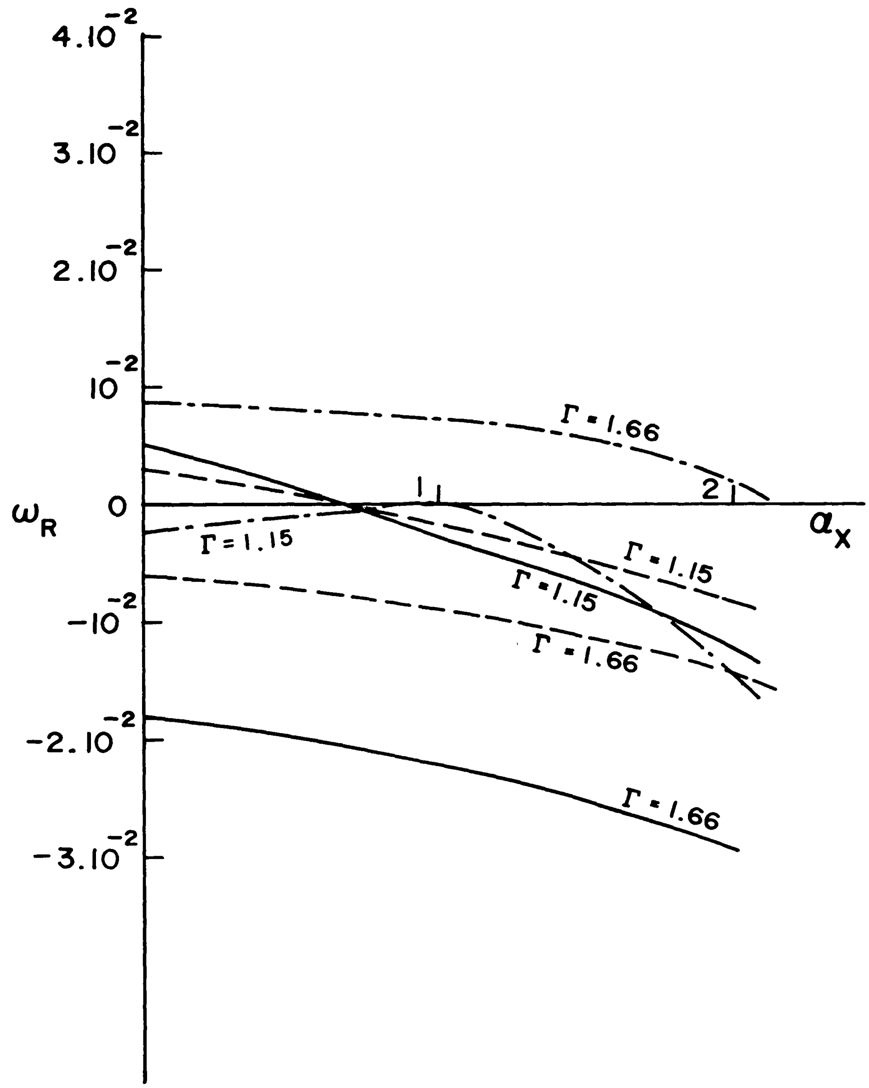

Fig. 3. The dimensionless growth rate ω_R for the P1-modes is plotted against the dimensionless wave number α_X for the value of the parameter $T_r = T_{\text{top}}/T_{\text{base}} = 0.5$ and $G_k = 5.28 \times 10^{-2}$. The results for the rigid (———), adiabatic (– – – – – –), free (– · – · – · –) boundary conditions are indicated for two choices of the polytropic index $\Gamma = 1.15$ and 1.66 with $\gamma = 1.05$.

and 2.64×10^{-3}. The maximum growth rate for $\alpha_X = 0$ is for the P2-mode for $G_k = 2.64 \times 10^{-3}$ and is given by (0.0164, 2.048) and for $\alpha_X = 1$ it is (0.0121, 2.190). For $\alpha_X = 2$ the P3 mode for $G_k = 1.32 \times 10^{-4}$ is the most unstable with a value of (0.0086, 3.291). This shows that we get a significant positive value of growth rates of order of a percent of the frequencies under favourable conditions even for subadiabatic gradients. The mechanism for the subadiabatic overstability is not altogether clear, but it is probable that the κ-mechanism is not completely absent in Equations (2) because of the variation of the opacity even though the conductivity K is held constant.

It would be interesting to compare our results with those of Jones (1976), who has made an analytic study of growth rates for the optically thin as well as optically

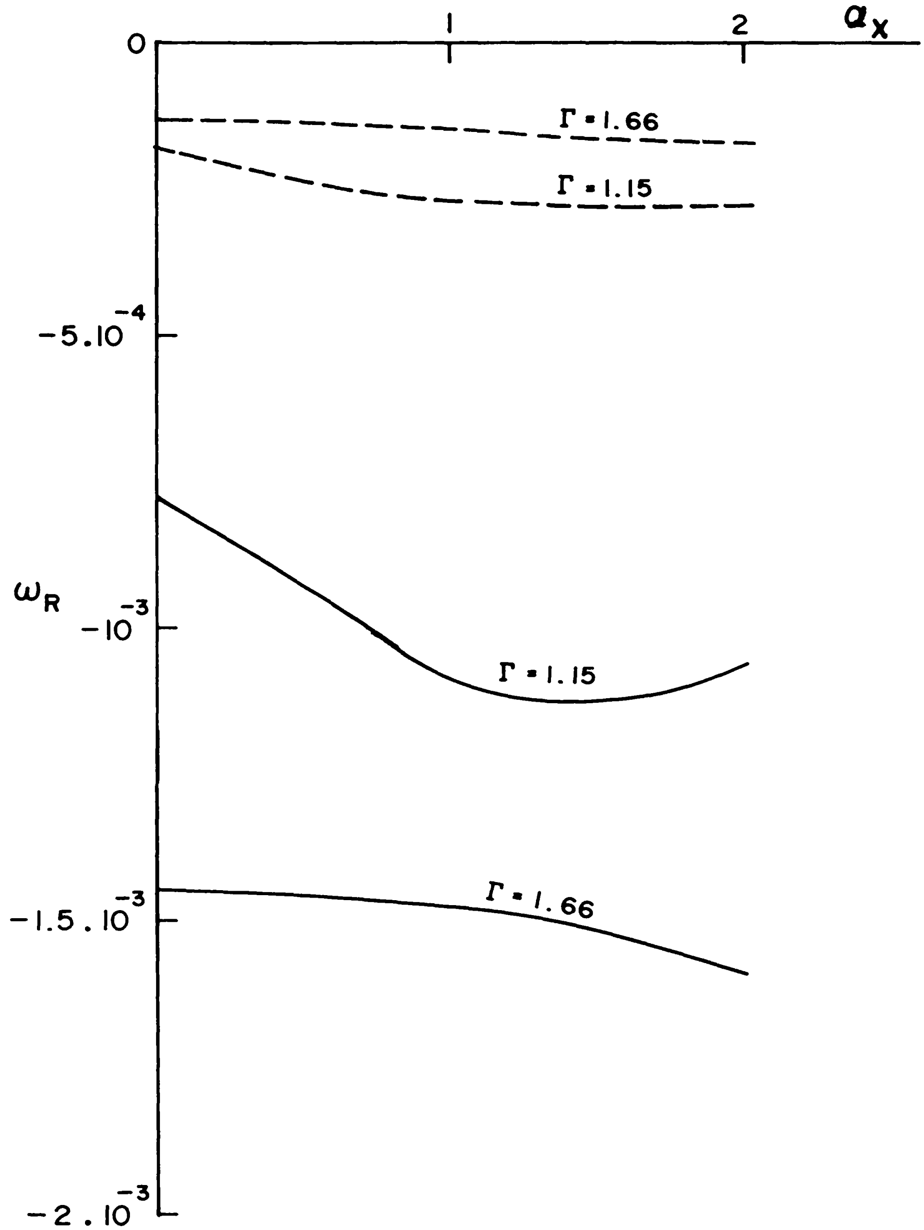

Fig. 4. The dimensionless growth rate ω_R for the P1-modes is plotted against the dimensionless wave number α_X for the value of the parameter $T_r = T_{top}/T_{base} = 0.5$ and $G_k = 1.32 \times 10^{-4}$. The results for the rigid (———) and adiabatic (– – – – – –) boundary conditions are indicated for two choices of the polytropic index $\Gamma = 1.15$ and 1.66 with $\gamma = 1.05$. The growth rates for the free boundary conditions come out to be of the order of 10^{-6} and are not shown in the diagram.

thick disturbances in the framework of the quasi-adiabatic approximation. For optically thin disturbances in the limit $\alpha_X \gg 1$, his results agree with those of Spiegel (1964), and also with our numerical results. For optically thick disturbances he finds that the values of γ close to unity will promote overstability (γ-mechanism). This result is again confirmed by our calculations. Jones has indicated that the large depth parameter (i.e. small values of T_r) promotes instabilities, which is consistent with our calculations. However he has pointed out that in the absence of the

κ-mechanism it is not possible to get overstability for superadiabatic temperature gradients, on the other hand there is overstability occurring for subadiabatic gradients. This is at variance with our numerical results. We would like to point out that our numerical results are in perfect agreement with his numerical results for completely nonadiabatic optically thick equations (Jones, 1977). We do get overstability for subadiabatic gradients when T_r is very small but in that case the overstability is even more pronounced for superadiabatic gradients. For small values of depth parameter ($T_r \lesssim 1$) and small values of G_k we do not get overstability for any value of γ (including the subadiabatic cases). However it should be noted that we do get overstability even for small values of the depth parameter when γ is close to unity and G_k is fairly large. For example for $\gamma = 1.05$ and $T_r = 0.5$, $\Gamma = 1.15$, $G_k = 0.0528$, which is a pronouncedly non-adiabatic situation, some of the modes are found to be unstable for $\alpha_X = 0$ and 1.

The analytical results derived by Jones (1976) are based on the quasi-adiabatic approximation which we believe to be not good for $T_r \ll 1$. But even for $T_r \lesssim 1$ it appears that the quasi-adiabatic approximation is an oversimplification for optically thick disturbances. Our results show that for optically thin disturbances for all values of T_r the growth rates are very nearly proportional to q, even when q is of order of 0.05. This is exactly what is expected from the quasi-adiabatic approximation. However, for optically thick distrubances it is found that even when G_k is as small as 10^{-3} there is no such proportionality. This clearly shows that numerical results for completely non-adiabatic optically thick equations do not agree with the analytic results obtained on the basis of quasi-adiabatic approximation. In other words the non-adiabatic effects are very significant and cannot be treated in the framework of quasi-adiabatic approximation when the disturbances are optically thick. This is probably due to the fact that the optically thick equations are of fourth order while the adiabatic equations are of second order only and so to treat the non-adiabatic effects as a perturbation over the adiabatic solution is not fully justified. Specially, $T_0 = 0$ is a singular point of both the adiabatic as well as non-adiabatic equations. However the solutions in neighbourhood of this point have different character for the two sets of equations. The fully non-adiabatic fourth order system of equations has solutions of form $\rho_0 W \approx a_1 T_0^{m+1} + a_2 T_0^{m+1} \ln T_0$, while the adiabatic second-order system has solution of form $\rho_0 W \approx b_0 T_0^m$, where a_1, a_2 and b_0 are arbitrary constants. Thus clearly for any value of G_k, however small, these two solutions can never agree for sufficiently small values of T_0, and so the quasi-adiabatic approximation will breakdown for small values of T_0 i.e. for an almost complete polytrope. Even for $T_r \lesssim 1$, it is obvious that adiabatic solutions in general cannot satisfy all the four boundary conditions which are applicable for the full system of equations and so it would be necessary to develop a boundary layer theory (analogous to the boundary layer theory for viscous flow) to incorporate the effects of this thermal boundary layer. In the boundary layers the eigenfunctions will be significantly different from the adiabatic eigenfunctions which in any case cannot satisfy the required boundary condition. This thermal boundary layer clearly makes an appreciable contribution to the eigenvalues, and especially the growth rates turn out to be sensitive to the boundary conditions. It may appear that by choosing adiabatic boundary conditions one can avoid the boundary layer since the adiabatic solution will also satisfy these boundary conditions by taking care of the temperature perturbation θ. However it should be noted that even for this set of boundary conditions the values of the derivatives of θ at the boundaries will not be the same for the two sets of equations and there is no reason to assume that the boundary layer will be absent. To examine this aspect we calculated the eigenfunctions of non-adiabatic as well as adiabatic equations for the adiabatic boundary conditions. It is found that eigenfunctions $\rho_0 W$, θ and p_1 for non-adiabatic equations are not significantly different from the adiabatic eigenfunctions. However it is found that the flux perturbation is significantly different near the boundaries, although in regions away from the boundaries it is nearly the same as adiabatic functions. Thus the boundary layer is important even for adiabatic boundary conditions. We believe that discrepancy between our results and Jones' results is due to this thermal boundary layer which has not been accounted for by Jones.

4. Discussion and Conclusions

The main thrust of the computation was to establish the existence, albeit numerically of the overstable acoustic modes in a polytropic layer for disturbances of arbitrary optical thickness due solely to the Cowling-Spiegel mechanism. It is clear from our numerical results that the acoustic overstability does indeed arise in such a layer provided the thermal dissipation operates efficiently and there is a sufficiently strong superadiabatic temperature gradient driving the instability. The principal source that is responsible for exciting the acoustic waves is the flux of thermal energy that is maintained in the layer. The acoustic overstability arises when the restoring force resulting from the compressible effects is sufficiently strong to induce oscillations caused by the combined influence of buoyancy forces, pressure fields and thermal dissipation. The frequencies are found to be insensitive to boundary conditions, but the growth rates depend in a sensitive manner on the boundary conditions adopted and also on the type of optical disturbance.

We have investigated the mechanism for an idealized model by choosing a polytropic layer. This may not be the true description of the solar surface layers, but the polytropic atmosphere has the analytical simplicity, and at the same time has the essential physical features which are believed to be responsible for the self-excitation of sound waves. We have assumed the radiative conductivity to be a constant in the present calculations, but in the subphotospheric layers the radiative conductivity varies by three orders of magnitude over a couple of scale-heights largely because of the steep increase in the absorption coefficient in the hydrogen ionization zone. This mechanism will almost certainly be very efficient in driving the oscillations in the solar convective zone. Ando and Osaki (1975) have given a global solution for non-adiabatic, non-radial oscillations of the Sun and they find a large variety of overstable acoustic modes with the most unstable modes centred at a period of 300 s with a wide range of associated horizontal wavelengths. The driving of these oscillations takes place mainly in a narrow transition zone in the sub-photospheric layers between the inner convective layers and outer radiative zone. Amongst the two excitation mechanisms providing the strong driving, the Cowling-Spiegel mechanism was found by Ando and Osaki to be less efficient than the κ-mechanism on the basis of the contribution of the work integrals.

Our work merely brings out the existence of acoustic overstability as the self-excitation mechanism for sound waves in the solar convection zone. It is almost certain that the κ-mechanism is dominant for the five minute oscillations, but it is conceivable that the simultaneous operation of several physical mechanisms could be responsible for exciting a variety of oscillatory fields on the solar surface. There is already observational evidence for a spectrum of oscillatory periods varying from 90 s to 50 min excited at various levels in the photosphere and chromosphere. It cannot be ruled out that some of the shorter period oscillations could largely be due to the Cowling-Spiegel mechanism. We would like to conclude by remarking that under suitable conditions it is possible to excite sound waves directly in an unstable compressible atmosphere provided there is efficient radiative transfer. The instability is sustained by the conversion of the thermal flux in the medium into the mechanical flux. The favourable condition for promoting the occurrence of acoustic overstability is a large phase lag between the thermal field and the velocity field brought about by the process of heat transfer. The adjustment of temperature takes time as a result and if, at the phase of greatest compression heat is communicated to the gas and at the phase of greatest rarefaction it is abstracted from it, a condition favourable for the self-excitation of sound waves arises. In the present work we have not included the effect of viscous dissipation on the growth rates of the acoustic modes. This may well turn out to be important.

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