INSTABILITIES IN A PENETRATIVE ATMOSPHERE

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Abstract. The destabilization of convective, gravity and acoustic modes in a compressible atmosphere consisting of a stable layer overlying an unstable layer is investigated in the optically thin approximation. It is shown that penetration into the stable layer promotes instability under suitable conditions.

1. Introduction

A considerable amount of effort has gone into the study of instabilities of various types of modes in a polytropic atmosphere in an attempt to interpret the observed velocity fields at the surface of stars (see review articles by Stein and Leibacher, 1974; Schatzman and Souffrin, 1967 and also Jones, 1976; Graff, 1976; Ando and Osaki, 1975 and Antia et al., 1978). In such instability calculations the choice of appropriate boundary conditions is rather crucial and, at the same time, it is also somewhat difficult to meet the physically acceptable situations prevailing in stellar atmospheres. In most of the investigations it has been assumed invariably that the convection zone is bounded by rigid or free boundaries. This assumption is rather unrealistic in the sense that it does not permit the propagation of wave motion into the adjacent stable layers. Kato and Unno (1960) have rightly pointed out the importance of penetration into stable layers in view of the existence of a vigorous convective zone just below the visible solar surface and with motion penetrating into the chromosphere. We feel, therefore, that the influence of penetration into overlying stable layers on the instability of various modes present in the unstable layer below is worth investigating and hope that it would bring a better harmony between theoretical interpretations and observations of oscillatory and non-oscillatory velocity fields observed at the solar atmosphere.

Unno (1957) had discussed the problem of penetration to explain the observed anisotropy of motion in the photosphere. He found that the degree of penetration into a layer decreases as the degree of stability of this layer is increased. Kato and Unno (1960) have further studied this problem and found that the penetration is appreciable even if the degree of stability of the stable layer is of the same order of magnitude as the degree of instability of the unstable zone. This work indicated that the penetration is less effective as the thickness of the unstable layer is increased.

Souffrin and Spiegel (1967) studied the instability of gravity waves in a thermally conducting model atmosphere with a convectively unstable layer underlying a stable

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layer in the framework of the Boussinesq approximation. It was shown that gravity modes are overstabilized provided the destabilizing effect of conduction in the unstable layer dominates the stabilizing effect of conduction in the stable layer. Earlier, Souffrin (1966) had pointed out that the gravity waves are highly susceptible to radiative damping and they exist only for \( N_{\text{BV}} \tau \gg 1 \) (\( N_{\text{BV}} \) is the Brunt–Väisälä frequency of the stable layer and \( \tau \) is the radiative cooling time (1/\( q \))). This criterion rules out the existence of gravity waves in the region of atmosphere where \( N_{\text{BV}} \tau \ll 1 \). Clark and Clark (1973) have considered the effect of radiative damping on trapped gravity waves (Thomas \textit{et al.}, 1971), taking into account the variation of \( N_{\text{BV}} \), and have tried to show that the buoyancy provided by even a small region of the atmosphere – in which \( N_{\text{BV}} \tau > 1 \) – is sufficient enough to drive the entire layer into an oscillatory mode. They have argued that the strong radiative damping region present just above the convection zone in the solar atmosphere will favour the convective penetration which they believe to be responsible for generating gravity waves in the region of the solar atmosphere where \( N_{\text{BV}} \tau \gg 1 \).

Recently, on a realistic solar atmospheric model, Ando and Osaki (1977) studied the influence of the chromosphere and corona on solar atmospheric oscillations and found that the transmission of energy does not appreciably affect the stability of acoustic modes.

In the present paper we wish to study the influence of penetration into an overlying stable layer on the degree of instability of convective, gravity and acoustic modes. For this purpose we have considered a plane-parallel model atmosphere consisting of convectively unstable and stable layers of finite depths. The radiative dissipation is taken into account through Newton's law of cooling, applicable to optically thin atmospheres. We have succeeded in showing that the effect of penetration is to promote the instability of convective, gravity and acoustic modes under suitable conditions.

The mathematical description of the problem – perturbation equations and boundary conditions – is presented in Section 2. In Section 3 the results of our numerical calculations are discussed and, finally, a brief summary of the results is given in Section 4.

2. Mathematical Description of the Problem

For the purpose of this investigation we consider a double-layer compressible atmosphere stratified under gravity acting in the positive \( z \) direction. It consists of a convectively unstable layer extending from \( z = 0 \) to \( z = d_i \), underlying a stable layer which extends from \( z = 0 \) to \( z = -d_u \). At \( z = 0 \) (interface), where the two layers are conjoined, all unperturbed physical quantities, such as temperature, density and pressure, are taken to be continuous. A polytropic law of the form \( p \sim \rho^n \) is assumed in the unstable layer to incorporate an appreciable density variation across the layer,
and the upper layer is taken to be isothermal. The radiative dissipation in both layers
is treated in the optically thin approximation.

The governing hydrodynamical equations in the usual notations are:

(i) Conservation of mass: \( \frac{\partial q}{\partial t} + \nabla \cdot (q \nabla) = 0, \)

(ii) Conservation of momentum: \( q \frac{dV}{dt} = -\nabla p + qg, \) \( (1) \)

(iii) Conservation of energy: \( q C_v \frac{dT}{dt} + p \nabla \cdot \nabla = Q, \)

(iv) Equation of state: \( p = RqT, \)

where \( Q \) represents the radiative exchange term. The mean molecular weight, the gas
constant \( R, \) the specific heat at constant volume \( C_v \) and the acceleration due to
gravity \( g \) are assumed constant. We shall denote the temperature at the interface
\( (z = 0) \) as \( T^* \) and use the scale height \( H = RT^*/g \) as the unit of length and \( \sqrt{RT^*}/g \)
as the unit of time to express the resulting equations in a dimensionless form. Other
physical quantities, such as pressure, density and temperature, are expressed in terms of
their respective values at the interface. Further, we denote the unperturbed and perturbed
physical quantities by the subscripts 0 and 1 respectively.

The temperature, density and pressure distributions in dimensionless form for the
unperturbed state can be written as

(i) Unstable layer \( (0 \leq z \leq d_i) \):

\[
T_0(z) = 1 + \frac{(\Gamma - 1)}{\Gamma} z,
\]

\[
\varrho_0(z) = T_0^m(z),
\]

\[
p_0(z) = T_0^{m+1}(z),
\]

where \( \Gamma = [(d \ln p_0)/(d \ln \varrho_0)], \) the polytropic index of the layer, and \( m = 1/(\Gamma - 1). \)

(ii) Stable layer \( (-d_u \leq z \leq 0) \):

\[
T_0(z) = \text{constant} = 1,
\]

\[
p_0(z) = \varrho_0(z) = \exp(z).
\]

We now introduce an infinitesimal perturbation about this equilibrium model and
assume the perturbed quantities to be of the form

\[
f_1(x, z, t) = f_1(z) \exp(ik_x x + \omega t),
\]

where \( k_x \) is the horizontal wave number and \( \omega \) is a complex eigenvalue.
The linearized perturbation equations for the lower layer can be expressed in the form
\[
\frac{d}{dz} X_l(z) = -\left[\frac{\omega(1 - \gamma_l/\Gamma)}{T_0(\gamma_l\omega + q_l)} + \frac{1}{2T_0\Gamma}\right] X_l(z) - \frac{k_x^2}{\omega} \frac{\omega(q_l + q_u)}{T_0(\gamma_l\omega + q_l)} Y_l(z),
\]
\[
\frac{d}{dz} Y_l(z) = \frac{(1 - \gamma_l/\Gamma)}{T_0(\gamma_l\omega + q_l)} - \omega X_l(z) + \left[\frac{(\omega + q_l)}{T_0(\gamma_l\omega + q_l)} - \frac{1}{2T_0\Gamma}\right] Y_l(z).
\] (4)

For the upper layer the linearized perturbation equations can be cast in the form
\[
\frac{d^2}{dz^2} X_u(z) - \beta^2 X_u(z) = 0,
\] (5)

where
\[
\beta^2 = \frac{1}{4} + k_x^2 + \frac{k_x^2(\gamma_u - 1)}{\omega(\gamma_u\omega + q_u)}.
\] (6)

Also, \( X = q_0^{1/2}W, \ Y = p_{1/2}q_0^{1/2}, \ p_1 \) and \( W \) are the perturbations on pressure and vertical velocity respectively, and subscripts \( u \) and \( l \) refer to the quantities in the upper and lower layers, respectively.

The foregoing sets of equations have to be supplemented by appropriate boundary conditions to complete the statement of the problem. We adopt the following set of boundary conditions (cf. Unno, 1957):

(i) Rigid boundary conditions at the bottom of the lower layer and at the top of the upper layer which give
\[
q_0 W = 0 \quad \text{at} \quad z = d_l, \quad z = -d_u.
\] (7)

(ii) At the interface we demand the continuity of mass flux and momentum flux. This requirement gives
\[
[q_0 W] = 0 \quad \text{and} \quad [p_1] = 0 \quad \text{at} \quad z = 0,
\] (8)

where \([\ ]\) denotes the jump in quantity across the interface.

Equation (5) can now be solved analytically since \( \beta \) is a constant, and the boundary conditions (7), (8) can be put in the form
\[
\left[\omega(q_u + q_u) + k_x^2(\gamma_u\omega + q_u)\right] Y_l(0) + \left[\omega(\gamma_u\omega + q_u)\beta \coth(\beta d_u) + \frac{1}{2}\omega(\gamma_u\omega + q_u) - \omega^2(\gamma_u - 1)\right] X_l(0) = 0,
\] (9a)
\[
X_l(d_l) = 0.
\] (9b)
Equation (4), together with the two boundary conditions (9), define an eigenvalue problem. Owing to the mathematical complexity in solving it analytically, we have resorted to its numerical solution, treating this as a generalized eigenvalue problem with $\omega (= \omega_R + i \omega_I)$ as its complex eigenvalue for a given $k_x$. For this purpose we have adopted the numerical technique devised by Antia (1979) for obtaining eigenvalues for ordinary differential equations with complex coefficients.

3. Results and Discussion

A. CONVECTIVE MODES

In order to investigate the effect of penetration on the instability of convective modes excited in an unstable layer, we have computed the convective growth rates of the first four harmonics (1, 2, 3, 4) for different sets of parameters characterizing the two layers. We have compared the results for the corresponding rigid boundary cases – i.e., when rigid boundary conditions are imposed at the top of the unstable layer. Figure 1 shows the dependence of the convective growth rate on the horizontal wavenumber ($k_x$) for an illustrative case, as indicated there. We note that the growth rate increases monotonically with increasing horizontal wavenumber and approaches a finite value asymptotically as $k_x \to \infty$ for both penetrative and rigid boundary cases. The important thing to notice is that the growth rates are somewhat larger for the penetrative boundary case than for the corresponding rigid case. The results clearly bring out the effect of penetration – namely, the penetration of the convective motions in the unstable layer into an overlying stable layer tends to increase the degree of instability of convective modes – in qualitative agreement with the results of previous workers (Kato and Unno, 1960). This result immediately follows from the oscillation theorem in differential equations (cf. Eckart, 1960). In fact, the penetration of convective flows into the stable layer in effect increases the thickness of the unstable zone, and hence the degree of instability in the convective modes is increased.

Convective elements originating in the unstable layer will penetrate into the stable layer due to the imposed continuity conditions for mass and momentum flux across the interface. To describe this mathematically, we may define ‘penetration depth’ as the distance into the stable layer at which the velocity amplitude of the rising element falls to $(1/e)$ of its value at the interface, and is equal to the inverse of $\beta$ defined in Equation (6). Our numerical results show that the penetration depth decreases with increasing horizontal wavenumber $k_x$ and also as the degree of stability of the stable layer (measured by $(\gamma_u - 1)$) is increased. It can be seen from Equation (6) that an increase in the thickness of the unstable zone leads to a decrease in the penetration depth (cf. Kato and Unno, 1960).

We also notice that the convective growth rates increase with the thickness of the stable layer, but the increase is reduced as the thickness of the stable layer becomes much larger than the penetration depth. While the radiative dissipation in the stable layer ($q_u$) has a destabilizing effect, that in the unstable layer ($q_l$) tends to annul the
Fig. 1. The dimensionless eigenvalue $\omega$ of convective modes is plotted against the horizontal wave number $k_x$ for $\Gamma = 1.66$, $\gamma_t = 1.1$, $\gamma_u = 1.33$, $T_r = T_{bas} / T^* = 2.0$, $d_a = 1.0$, $q_t = 0.1$, $q_u = 0.1$ for the first four harmonics (1, 2, 3, 4). The full curves (-----) refer to the penetrative boundaries and broken curves (-----) are for rigid boundary conditions.
destabilizing buoyancy force by rapidly smoothing out the temperature fluctuations and, hence, has a stabilizing effect on the convective modes. The increase in the degree of stability of the stable layer brought out by an increase in \((\gamma_u - 1)\) leads to a reduction in the degree of instability of the convective modes. Further, we found that a decrease in the superadiabaticity of the unstable layer (i.e., as the buoyancy factor \((1 - \gamma_i/\Gamma)\) approaches unity) reduces the convective growth rates. This is due to the fact that a decrease in \((1 - \gamma_i/\Gamma)\) makes the unstable layer less unstable and therefore the destabilizing mechanism becomes less efficient.

B. GRAVITY MODES

With a view to analysing the instability of gravity modes in a penetrative atmosphere, we have computed eigenvalues \((\omega = \omega_R + i\omega_I; \omega_R\) is the growth rate and \(\omega_I\) the frequency) for the first five harmonics \((G1, G2, G3, G4, G5)\) as a function of the horizontal wavenumber \(k_x\), depths of stable and unstable zones, for chosen values of other parameters such as \(\Gamma, \gamma_i, \gamma_u, q_I\) and \(q_u\). Our numerical results are shown in Figures 2 and 3 for the typical sets of parameters indicated there.

Figure 2 shows the dependence of the growth rate \((\omega_R)\) and frequency \((\omega_I)\) of gravity waves on \(k_x\) (cf. Souffrin and Spiegel, 1967, Figure 2). We notice that while the frequency increases monotonically with \(k_x\) and approaches the Brunt–Väisälä frequency of the stable layer \((N_{BV} = \sqrt{(\gamma_u - 1)/\gamma_u})\) for large \(k_x\), the growth rate exhibits a maximum which itself is a function of the thickness of the upper layer \((d_u)\) and the order of mode involved. In the model atmosphere considered here, the gravity waves present in the stable layer, because of the continuity conditions at the interface, penetrate into the lower unstable layer and thereby trigger oscillations which are destabilized through the combined action of buoyancy force and thermal conduction (cf. Souffrin and Spiegel, 1967).

To describe the penetration of gravity waves mathematically we may define the penetration depth \((d_{pl})\) as the distance at which the amplitude of the gravity wave falls to \((1/e)\) of its value at the interface. An explicit relation for \(d_{pl}\) can be obtained by demanding a solution of the form \(X_i, Y_i \sim \exp(-\alpha z)\) for the lower layer, assuming the coefficients of Equation (4) to be locally constant. We have (for the adiabatic case \(q_I = 0\))

\[
\alpha^2 = \frac{k_x^2}{4\Gamma^2 T_0^2} + \frac{k_x^2(1 - \gamma_i/\Gamma)}{\omega_I^2\gamma_i T_0} - \omega_I^2,
\]

and, thus, \(d_{pl} = 1/\alpha\).

For large \(k_x\), the frequency \((\omega_I)\) of gravity waves is almost constant and, therefore, it is easy to see from the above expression that \(d_{pl} \sim k_x^{-1}\). That is, at large \(k_x\), the penetration of gravity waves into the unstable layer is small and, consequently, does not excite much oscillation in the unstable layer. As a result, the destabilizing effect of conduction in the unstable layer is weak compared to the stabilizing effect of conduction in the stable layer, leading to a fall-off of growth rates at large \(k_x\). For
Fig. 2. The dimensionless frequency $\omega_I$ (normalized to Brunt-Väisälä frequency $N_{BV}$ of the stable layer) and growth rate $\omega_r$ of gravity modes are plotted against the horizontal wave number $k_x$ for $\Gamma = 1.66$, $\gamma_i = 1.1$, $\gamma_u = 1.33$, $T_r = 2.0$, $d_u = 1.0$, $q_i = 0.01$, $q_u = 10^{-4}$ for the first four harmonics (G1, G2, G3, G4).
small $k_x$, on the other hand, the horizontal scale of motion is large – i.e., motion is almost confined to the horizontal plane which reduces the net work done by the destabilizing buoyancy forces and, hence, diminishes the instability of gravity waves (cf. Souffrin and Spiegel, 1967).

We also notice from Figure 2 that, for higher modes, not only are the growth rates reduced and the curve flattened out, but also the peak is shifted slightly to a higher $k_x$. Similarly, we also found that the peak shifts to a lower $k_x$ for greater thicknesses of the stable layer ($d_u$). If we define the vertical wavenumber $k_z$ as $nπ/d_u$ ($n =$ order of mode), then from our results it is seen that the preferred wavelength at which waves are most unstable is proportional to $d_u$ and $k_x/k_z \simeq$ constant for the preferred disturbances.

In Figure 3 we have plotted growth rates ($\omega_R$) as a function of the thickness ($d_u$) of the stable layer for a set of parameters indicated there. We find that $\omega_R$ displays a maximum which shifts to higher $d_u$ for a higher order of modes and smaller $k_x$. This behaviour is simply a reflection of a standard variation of growth rates $\omega_R$ with a vertical wavenumber $k_z$ discussed earlier. At a large vertical wavenumber (small $d_u$), the vertical scale of motion is very small. In other words, the ratio of horizontal to vertical velocities is very large. At a small vertical scale of motion, the destabilizing mechanism becomes comparatively weak and therefore reduces the growth rate at small $d_u$. On the other hand, for a small vertical wavenumber (large $d_u$) – because the density decreases exponentially with distance in the stable isothermal layer and since $\rho_0 W^2 =$ constant – the amplitude of vertical velocity at the interface would be very small compared to its value in the upper region of the layer. This, in turn, diminishes the work done by the destabilizing forces in the unstable layer and, hence, the growth rate is reduced.

We further notice that for a higher order of modes the growth rate is considerably reduced, the peak becomes rather flat and shifts to a larger value of $d_u$. For a given mode, maximum growth rate should occur at the same value of $k_z$ – i.e., $n/d_u$ should be constant – which implies therefore that the peak should shift to larger $d_u$ for higher harmonics. The flattening of the curve for higher harmonics can be explained if we note the variation of $k_z$ with $d_u$:

$$\frac{dk_z}{dd_u} = -\frac{nπ}{d_u^2} = -\frac{k_z}{d_u},$$

i.e., at large $d_u$, $k_z$ varies rather slowly with $d_u$ and hence flattens the curve.

The underlying unstable layer is expected to play a crucial role in the over-stabilization of gravity waves, and indeed our numerical calculations show that the growth rates of the gravity wave depend on the depth of the unstable zone. It is found that the growth rate $\omega_R$ first increases with increasing $d_u$, reaches a maximum, and finally flattens out at large values of $d_u$. The position of maximum is found to be a function of the order of modes, $k_x$ and $d_u$. That can be understood in terms of the relative values of $d_i$ and $d_{pl}$ (see Equation (10)). For very small $d_i$ – i.e., $d_i \ll d_{pl}$ –
the destabilizing forces in the unstable layer would be rather insensitive to the vertical oscillations stirred by the penetrating gravity waves which will lead to a reduction in the instability of gravity waves. On the other hand, for large $d_i \gg d_{pl}$, there is hardly any vertical oscillation in the deeper regions of the layer to extract energy from the existing destabilizing force and, therefore, the growth rates will remain almost constant for large $d_i$.

It should be emphasized that overstability of gravity wave occurs only when the radiative dissipation in the stable layer is very small compared to that in the unstable
Fig. 4. The dimensionless growth rate $\omega_R$ of acoustic modes are plotted against the thickness of the stable layer $d_u$ for $\Gamma^* = 1.66, \gamma_s = 1.1, \gamma_s = 1.1, T_r = 2.0, q_t = 1.0, q_s = 0.1$ for two values of $k_x$ (0.25 and 0.5) corresponding to P1, P2, P3 modes.
layer; for \( q_u = q_i \) the gravity waves are found to be stable, in agreement with the results of Souffrin and Spiegel (1967).

C. ACOUSTIC MODES

We have computed complex eigenvalues for the first five harmonics of acoustic modes for a number of parameters characterizing the two layers. For the purpose of illustration, in Figure 4 we have plotted growth rates as a function of thickness of stable layer. Other results, such as the variation of \( \omega_I \) and \( \omega_R \) with horizontal wavenumber \( k_x \), are found to be standard and are not presented here.

From Figure 4 we note that the growth rate of acoustic modes displays a maximum, clearly bringing out the way the instability of acoustic waves is affected by overlying stable layers. An interesting feature is the occurrence of relative maxima for different harmonics – viz., the peak shifts to higher \( d_u \) for higher harmonics, the peak being mildly affected by a change in \( k_x \). Similar behaviour, as we have already discussed, is also seen for gravity waves with the difference that, for acoustic waves, the separation between peaks is rather pronounced. This behaviour may be interpreted in the same manner as that of gravity waves, with \( k_x = n\pi/(d_u + d_i) \) instead of \( n\pi/d_u \) because, unlike gravity waves, acoustic waves are propagating in both layers. For the cases shown in Figure 4, we found that at the peak \( \omega_R > 0 \), indicating overstability, while for corresponding rigid boundary cases we did not find overstability. Thus, our numerical calculations indicate that penetration into overlying stable layers tends to increase the degree of instability of acoustic waves.

Further, we have examined the effect of thickness of the unstable zone on the instability of acoustic waves. Our numerical results show that an increase in \( d_i \) does not lead to a monotonic increase in the growth rate, but instead the growth rate attains a maximum at a certain value of \( d_i \) beyond which it decreases very slowly. Unlike the gravity waves, the growth rates of acoustic waves decrease monotonically with increasing horizontal wavenumber \( k_x \).

An increase in the degree of stability of the upper layer – i.e., \( (\gamma_u - 1) \) – and decrease in superadiabaticity of the unstable layer reduces the instability of acoustic waves. While the radiative dissipation in the stable layer has a stabilizing effect, that in the unstable layer has a pronounced destabilizing effect.

Finally, we have included the spatial variation of radiative dissipation in the lower layer to show that this does not appreciably alter the picture presented above, except for a mild change in the growth rates.

4. Concluding Remarks

In the present communication we have attempted to examine the instability of convective, gravity and acoustic modes in a penetrative atmosphere consisting of a convectively unstable layer superposed by a stable layer. The main conclusions are as follows:
(a) The degree of instability of convective modes is increased when the convective motion is allowed to penetrate into an overlying stable layer. The penetration depth diminishes as the thickness of the unstable zone is increased.

(b) The gravity modes in such an atmosphere are overstabilized provided the radiative dissipation in the stable layer is very small compared to that in the unstable layer, in agreement with the results of Souffrin and Spiegel (1967). We have found that for a given set of parameters of the layer, there exist preferred horizontal and vertical wavelengths of disturbance for which a given mode is most unstable.

(c) The dependence of growth rates of acoustic modes on the thickness of the stable layer clearly indicates that penetration increases the degree of instability of acoustic modes. We also found that, as with gravity waves, there exists for a given $k_x$ and other parameters a preferred vertical scale for which a given harmonic is most unstable.

In the present work radiative dissipation has been taken into account in the optically thin approximation, which is a highly simplified representation of the sub-photospheric convective zone of the Sun. Therefore, our calculations cannot be applied directly to the solar case. Nevertheless, we have succeeded in our singular aim to bring out the effects of overlying stable layers on the instabilities of various kinds of modes which are believed to be important in solar atmospheric phenomena.

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