1980SoPh...66...71A

# STABILITY OF A STEADY VERTICAL FLOW IN A VISCOUS FLUID

H. M. ANTIA and S. M. CHITRE
*Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India*

(Received 31 May; in revised form 2 August, 1979)

**Abstract.** The one-dimensional non-linear equations for a viscous fluid with finite thermal conductivity are solved to get an exact solution for a steady vertical flow. The stability of such a steady flow is examined to find that the viscosity has a very pronounced stabilizing influence on convective and acoustic modes.

## 1. Introduction

The stability of a viscous fluid layer heated from below was first investigated by Rayleigh (1916) who found that the stability of the fluid layer is determined by the numerical value of the non-dimensional parameter,

$$R^* = \frac{g\alpha\beta}{\kappa\nu} d^4 ,$$

where $g$ denotes the acceleration due to gravity, $d$ the depth of the layer, $\beta(= |\mathrm{d}T/\mathrm{d}z|)$ the uniform adverse temperature gradient maintained across the layer, and $\alpha$, $\kappa$, and $\nu$ are respectively the coefficients of volume expansion, thermometric conductivity and kinematic viscosity; $R^*$ as defined here is called the Rayleigh-number. Rayleigh further showed that instability must set in when $R^*$ exceeds a certain critical value $R_c^*$, and that when $R^*$ just exceeds $R_c^*$ a stationary pattern of motion will prevail. The problem of determining the critical value $R_c^*$ has been discussed extensively by Chandrasekhar (1961) within the framework of the Boussinesq approximation. Later Spiegel (1965) considered the problem of onset of convection in a compressible atmosphere and derived the equations for time-independent convection in a plane-parallel layer of perfect gas with constant viscosity and thermal conductivity. Vickers (1971) and Gough *et al.* (1976) have solved these equations numerically for a simple polytropic atmosphere. Recently Graham and Moore (1978) have derived the equations for onset of convection in a fluid with an arbitrary equation of state and any arbitrary conductivity and viscosity that can be expressed as a function of temperature and pressure. They find that the form of the viscosity has little effect upon the degree of instability of the convective layer, and that there is no falling off of velocity with depth as density increases. There is also no sign of any tendency for motions on a scale other than that of the depth of the entire unstable region.

The foregoing studies are restricted to the onset of steady convection in a fluid layer, and do not attempt to give any information about the growth rates to be expected in an unstable situation. Further they do not consider the overstabilization

*Solar Physics* **66** (1980) 71–78. 0038–0938/80/0661–0071$01.20.

of acoustic modes, which is likely to prevail in a compressible fluid. The over-stabilization of acoustic modes in an inviscid polytropic fluid has been considered by Spiegel (1964), Jones (1976), and Antia *et al.* (1978, hereafter referred to as Paper I). Goldreich and Kelley (1977) have examined the effect of turbulent viscosity on the stability of acoustic modes in a model solar atmosphere to find that all the acoustic modes are stabilized. Their study does not include the effect of turbulent convection in the equation of motion, and they have computed the effect of turbulent viscosity using a perturbation theory.

In this paper we propose to examine the effect of a velocity field on the stability of convective and acoustic modes in a viscous fluid. To keep our analysis simple we assume that the steady state quantities depend only on the vertical coordinate and that the steady flow is in the upward direction. With these simplifying assumptions we solve the system of nonlinear equations governing the motion of a viscous fluid to get an exact solution for various physical quantities in a steady state. We then examine the linear stability of such a fluid flow to find that both convective and acoustic modes are stabilized.

## 2. The Steady State

We shall adopt the usual hydrodynamical equations for the conservation of mass, momentum and energy as being applicable to a compressible fluid layer stratified under constant gravity. Using cartesian coordinates and the usual summation convention, these equations are:

$$
\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho v_j)=0\,,\\
&\rho\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}=\rho g_i-\frac{\partial P}{\partial x_i}+\\
&\qquad\qquad+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\tfrac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\},\\
&\rho\frac{\partial}{\partial t}(C_vT)+\rho v_j\frac{\partial}{\partial x_j}(C_vT)=\frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)-P\frac{\partial v_j}{\partial x_j}+\Phi\,,\\
&P=R\rho T\,.
\end{aligned}
\tag{1}
$$

Here $K$ is the thermal conductivity, $\mu$ is the coefficient of dynamic viscosity, and $\Phi$ is the rate of viscous dissipation given by

$$\Phi=2\mu e_{ij}^2-\tfrac{2}{3}\mu(e_{jj})^2\,,$$

where $e_{ij}$ is the strain tensor given by

$$e_{ij}=\tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right).$$

We assume the gas constant $R$, acceleration due to gravity $g$, and specific heat at constant volume $C_v = R/(\gamma - 1)$ to be constants. Here $\gamma$ is the ratio of specific heats. Further we take the $z$-axis to be in vertical direction and for simplicity assume that velocity in the steady state is of the form $v = (0, 0, W(z))$. We shall assume that all the physical quantities in steady state are functions of $z$ only. Using the convention of denoting the steady state quantities (except the vertical component of velocity $W$) by a subscript zero, Equation (1) yields

$$\frac{\mathrm{d}(\rho_0 W)}{\mathrm{d}z} = 0, \qquad P_0 = R\rho_0 T_0\,,$$

$$\rho_0 W\frac{\mathrm{d}W}{\mathrm{d}z} = g\rho_0 - \frac{\mathrm{d}P_0}{\mathrm{d}z} + \tfrac{4}{3}\mu_0\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + \tfrac{4}{3}\frac{\mathrm{d}\mu_0}{\mathrm{d}z}\frac{\mathrm{d}W}{\mathrm{d}z}\,, \tag{2}$$

$$\rho_0 C_v W\frac{\mathrm{d}T_0}{\mathrm{d}z} - RT_0 W\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\left(K_0\frac{\mathrm{d}T_0}{\mathrm{d}z}\right) + \tfrac{4}{3}\mu_0\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2.$$

Equation (2) defines a system of nonlinear equations for $\rho_0$, $T_0$, $P_0$, $W$, $\mu_0$, and $K_0$, and it can be verified that the following solution satisfies the system of Equation (2):

$$T_0 = \frac{g}{R(m+1)}z, \qquad \rho_0 = az^m, \qquad P_0 = \frac{ag}{m+1}z^{m+1}\,,$$

$$W = bz^{-m}, \qquad \mu_0 = -\frac{3ab}{4m}z\,, \tag{3}$$

$$K_0 = Rab\left(\frac{1}{\gamma-1} - m\right)z - \left[\frac{R(m+1)ab^3}{2g}\right]z^{-2m} + K_0^*\,,$$

where $a$, $b$, $m$, and $K_0^*$ are constants. To ensure positivity of the temperature we must have $g > 0$, and to ensure positivity of $\mu_0$ we must have $bm < 0$. Thus the $z$-axis is directed vertically downwards and for $m > 0$, the velocity is in the vertically upward direction. It can be seen that for $m > 0$, this solution is identical to the polytropic solution of Paper I with index $\mathrm{d}\ln P_0/\mathrm{d}\ln\rho_0 = \Gamma = (m+1)/m$, except for the fact that in the present case the thermal conductivity is not constant. This solution is thus rendered suitable for comparison with our results of the inviscid case discussed in Paper I.

## 3. Linear Perturbation Theory

We consider a polytropic fluid layer confined between two horizontal planes situated at $z = d_l$ and $z = d_u$ ($d_l > d_u$). The various physical quantities in the steady state are given by Equation (3). At this stage it is convenient to express all the physical quantities in a dimensionless form. For this purpose we use the values of corresponding physical quantities at the base of the layer ($z = d_l$) as the units, and we further choose a length scale of $RT_0(d_l)/g$ and time scale of $\sqrt{RT_0(d_l)}/g$. Then the steady state quantities in the dimensionless form are given by

$$T_0 = 1 + \frac{z - d_l}{m+1}, \qquad \rho_0 = T_0^m, \qquad P_0 = T_0^{m+1},$$

$$W = -G_W T_0^{-m}, \qquad \mu_0 = \tfrac{3}{4} G_W \frac{m+1}{m} T_0, \tag{4}$$

$$\frac{K_0}{C_v} = G_W(m+1)[1 - m(\gamma - 1)]T_0 - (\gamma - 1)\frac{m+1}{2} G_W^3 T_0^{-2m} + K_0^*,$$

where the Mach number

$$G_W = \frac{|W(d_l)|}{\sqrt{RT_0(d_l)}}$$

is the dimensionless value of velocity at the base of the layer and the constant $K_0^*$ is determined from the prescribed value of diffusivity at the base of the layer,

$$G_k = \frac{K_0(d_l)g}{\rho_0(d_l)C_v[RT_0(d_l)]^{3/2}}$$

$$= G_W(m+1)[1 - m(\gamma - 1)] - (\gamma - 1)\frac{m+1}{2} G_W^3 + K_0^*.$$

We linearize the basic equations by writing all physical quantities in the form

$$f(x, y, z, t) = f_0(z) + f_1(z) \exp(\omega t + ik_x x + ik_y y)$$

and neglecting the higher powers of perturbed quantities. Because of the symmetry about $z$-axis there will be no loss of generality in assuming $k_y = 0$. We treat thermal dissipation in the optically thick approximation, and do not consider perturbations in thermal conductivity or the coefficient of viscosity. After a certain amount of algebraic manipulation we get the following system of differential equations in dimensionless form:

$$\mu_0 \frac{\mathrm{d}^2(iv_x)}{\mathrm{d}z^2} = (\omega\rho_0 + \tfrac{4}{3}\mu_0 k_x^2)(iv_x) + (\rho_0 W - \mu_0')\frac{\mathrm{d}(iv_x)}{\mathrm{d}z} +$$

$$+ k_x \mu_0' v_z + \tfrac{1}{3}k_x\mu_0 \frac{\mathrm{d}v_z}{\mathrm{d}z} - k_x T_0 \rho_1 - k_x \rho_0 T_1,$$

$$\tfrac{4}{3}\mu_0 \frac{\mathrm{d}^2 v_z}{\mathrm{d}z^2} - T_0 \frac{\mathrm{d}\rho_1}{\mathrm{d}z} =$$

$$= \tfrac{2}{3}k_x \mu_0'(iv_x) - \tfrac{1}{3}k_x\mu_0 \frac{\mathrm{d}(iv_x)}{\mathrm{d}z} + (\omega\rho_0 + \rho_0 W' + \mu_0 k_x^2)v_z +$$

$$+ (\rho_0 W - \tfrac{4}{3}\mu_0')\frac{\mathrm{d}v_z}{\mathrm{d}z} + (WW' - 1 + T_0')\rho_1 + \rho_0' T_1 + \rho_0 \frac{\mathrm{d}T_1}{\mathrm{d}z}, \tag{5}$$

$$-W\frac{\mathrm{d}\rho_1}{\mathrm{d}z} = k_x\rho_0(iv_x) + \rho_0'v_z + \rho_0\frac{\mathrm{d}v_z}{\mathrm{d}z} + (\omega + W')\rho_1\,,$$

$$(\gamma-1)WT_0\frac{\mathrm{d}\rho_1}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{K_0}{C_v}\frac{\mathrm{d}T_1}{\mathrm{d}z}\right) =$$

$$= \tfrac{4}{3}(\gamma-1)k_x\mu_0W'(iv_x) + (\rho_0T_0' - (\gamma-1)T_0\rho_0')v_z - \tfrac{8}{3}(\gamma-1)\mu_0W'\frac{\mathrm{d}v_z}{\mathrm{d}z} +$$

$$+ (WT_0' - \omega(\gamma-1)T_0)\rho_1 + (\omega\rho_0 - (\gamma-1)W\rho_0' + \frac{K_0}{C_v}k_x^2)T_1 +$$

$$+ \rho_0W\frac{\mathrm{d}T_1}{\mathrm{d}z}\,.$$

Here $v_x$ and $v_z$, respectively denote the perturbations to the $x$ and $z$ components of velocity, and the primes denote the derivatives of corresponding unperturbed quantity with respect to $z$.

It can be seen that this set of equations is of seventh order in $z$ derivatives and hence for a complete specification of the problem we require in all seven boundary conditions at the bounding surfaces. We assume that the bounding surfaces are maintained at constant temperature and that the $xz$ and $zz$ components of viscous stress tensor vanish at the boundaries. In addition we assume that a constant flux of fluid is maintained at the lower boundary. These conditions can be written as

$$T_1 = 0\,,$$

$$\frac{\mathrm{d}(iv_x)}{\mathrm{d}z} - k_xv_z = 0,$$

$$\tfrac{4}{3}\frac{\mathrm{d}v_z}{\mathrm{d}z} - \tfrac{2}{3}k_x(iv_x) = 0 \qquad \text{at} \quad z = d_l \quad \text{and} \quad z = d_u, \tag{6}$$

and

$$\rho_0v_z + W\rho_1 = 0 \qquad \text{at} \qquad z = d_l\,.$$

The set of Equations (5) supplemented by the boundary conditions (6) is solved numerically by the method described by Antia (1979) to obtain the complex eigenvalues $\omega$ for various values of $\Gamma$, $\gamma$, $G_k$, $G_W$, $T_r$, and $k_x$.

## 4. Discussion and Conclusions

The system of Equations (5) together with boundary conditions (6), for a choice of parameters $\Gamma$, $\gamma$, $G_k$, $G_W$, $T_r$, and $k_x$, give two series of modes. The convective modes which correspond to aperiodic disturbances have real eigenvalues. The fastest growing convective mode (i.e. the one for which $\omega$ is largest) has no nodes in $v_z$ eigenfunctions while for the successive lower modes the number of nodes in

$v_z$ increases by one. The harmonics of convective modes are denoted by C1, C2, C3, .... The C1 mode which has higher growth rate than any other convective mode, dominates over the other modes and hence in our discussion we will consider only the C1 mode. The other series of modes with complex eigenvalues $\omega = (\omega_R + i\omega_I)$ are identified with acoustic modes. These modes which arise due to the effects of compressibility correspond to the acoustic modes of Paper I modified by the presence of the velocity field and viscosity. The lowest (i.e. the one for which the frequency $\omega_I$ is smallest) acoustic mode is denoted by $F$-mode while the successive higher harmonics are denoted by P1, P2, ....

TABLE I

Real and imaginary parts of eigenvalues of the first four harmonics of acoustic modes for $\gamma = 1.1$, $G_k = 0.0355$, $G_W = 0.0$, 0.002 686 and 0.008 955, $T_r = 0.1$ for different values of $\Gamma$ and $k_x$. The numbers in parentheses are the powers of ten

| | Mode | $G_W = 0.0$ | | $G_W = 0.002\,686$ | | $G_W = 0.008\,955$ | |
|---|---|---|---|---|---|---|---|
| | F | −1.89 (−3) | 0.091 | −1.72 (−3) | 0.091 | −1.03 (−3) | 0.091 |
| $\Gamma = 1.33$ | P1 | 1.88 (−2) | 0.870 | −3.39 (−2) | 0.876 | −8.28 (−2) | 0.903 |
| $k_x = 0.10$ | P2 | 2.01 (−2) | 1.421 | −1.23 (−1) | 1.503 | −2.32 (−1) | 1.617 |
| | P3 | −1.07 (−2) | 1.939 | −2.96 (−1) | 2.170 | −4.74 (−1) | 2.417 |
| | F | −8.97 (−3) | 0.907 | −2.07 (−2) | 0.907 | −3.59 (−2) | 0.910 |
| $\Gamma = 1.33$ | P1 | 8.49 (−3) | 1.129 | −6.46 (−2) | 1.144 | −1.38 (−1) | 1.169 |
| $k_x = 1.00$ | P2 | 2.07 (−3) | 1.579 | −1.63 (−1) | 1.687 | −2.86 (−1) | 1.806 |
| | P3 | −1.33 (−2) | 2.052 | −3.25 (−1) | 2.289 | −5.30 (−1) | 2.539 |
| | F | −2.00 (−3) | 0.086 | −2.47 (−3) | 0.085 | −3.11 (−3) | 0.085 |
| $\Gamma = 1.66$ | P1 | 2.01 (−2) | 1.051 | −3.04 (−2) | 0.947 | −1.32 (−1) | 0.967 |
| $k_x = 0.10$ | P2 | 3.68 (−2) | 1.982 | −1.00 (−1) | 1.794 | −2.65 (−1) | 1.893 |
| | P3 | −3.27 (−4) | 2.918 | −2.65 (−1) | 2.628 | −5.77 (−1) | 2.852 |
| | F | −1.25 (−2) | 0.869 | −2.40 (−2) | 0.858 | −4.31 (−2) | 0.865 |
| $\Gamma = 1.66$ | P1 | 7.27 (−3) | 1.281 | −5.24 (−2) | 1.181 | −1.58 (−1) | 1.206 |
| $k_x = 1.00$ | P2 | 2.62 (−2) | 2.109 | −1.28 (−1) | 1.927 | −2.99 (−1) | 2.036 |
| | P3 | −7.38 (−3) | 3.000 | −2.81 (−1) | 2.708 | −6.12 (−1) | 2.940 |

The frequency (imaginary part of $\omega$) of acoustic modes is essentially unaffected by the velocity field and viscous dissipation. However, the velocity field has a pronounced effect on the growth rates (real part of $\omega$). Table I gives the frequencies and growth rates of the acoustic modes for the following choice of parameters: $\Gamma = 1.66$ and 1.33, $\gamma = 1.1$, $G_k = 0.0355$, $G_W = 0.0$, $2.686 \times 10^{-3}$ and $8.955 \times 10^{-3}$, $T_r = T_0(d_u)/T_0(d_l) = 0.1$ and $k_x = 0.1$ and 1.0. The case $G_W = 0.0$ corresponding to a situation where there is no motion in the steady state is discussed in Paper I. Thus introduction of an outflowing velocity field stabilizes the acoustic modes and in fact none of the modes are overstable even when the velocity is rather small. It can be seen that even such small value of Prandtl number ($\nu/\kappa$) of order 0.1 completely stabilizes the acoustic modes. The coefficient of kinematic viscosity $\nu$ in our case is given by

$$\nu = \mu/\rho = -\tfrac{3}{4}\frac{(m+1)}{m} W \frac{RT_0}{g}$$
$$= -\tfrac{3}{4}\Gamma WH \qquad \text{(note that } W < 0\text{)},$$

where $H$ is the local pressure scale height. This is of the same form as the turbulent viscosity used by Goldreich and Keeley (1977) and our results about the damping of acoustic modes are consistent with their results obtained by using a perturbation theory. The stabilizing influence of viscosity is quite marked and only under very extreme circumstances it is possible to get overstable acoustic modes. Thus when $T_r \lesssim 0.01$ and $G_W \lesssim 2.68 \times 10^{-3}$ we can get some overstable acoustic modes for small values of $k_x$. This corresponds to the case of a layer in which the temperature varies by two orders of magnitude and density varies by even more. Such an almost complete polytrope leads to an unstable situation as we have seen in Paper I.

As expected the degree of instability of convective modes is also decreased by the introduction of a velocity field and it is found that the grwoth rates of convective modes decrease by increasing the velocity. Figure 1 displays the growth rate of convective (C1) mode against $k_x$ for the following choice of parameters: $\Gamma = 1.33$, $\gamma = 1.1$, $G_k = 0.00178$, $G_W = 0.0$, $2.686 \times 10^{-3}$ and $8.955 \times 10^{-3}$ and $T_r = 0.33$. It

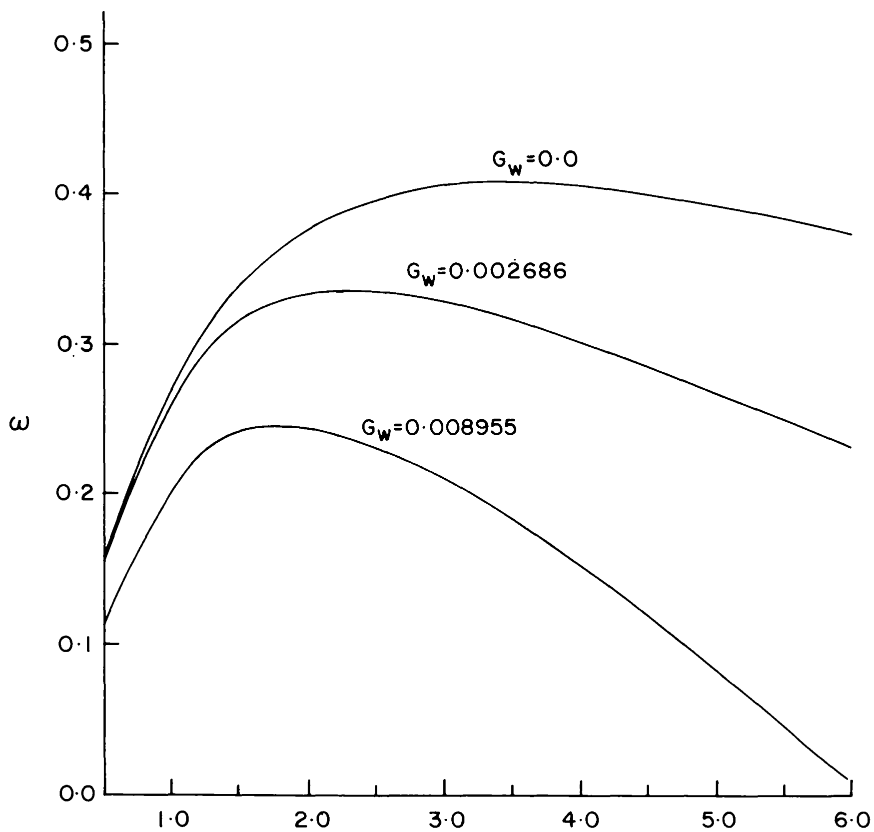


Fig. 1. Eigenvalue $\omega$ for convective (C1) mode are shown as a function of the horizontal wave number $k_x$ for $\Gamma = 1.33$, $\gamma = 1.1$, $G_k = 0.001\,78$, $G_W = 0.0$, $0.002\,686$, and $0.008\,955$ and $T_r = 0.33$.

can be seen that the growth rate of convective modes possess a distinct maximum with respect to $k_x$, thus giving a preferred length scale for convection. However, the value of $k_x$ at which maximum growth rate is attained, decreases by increasing $G_W$ and hence the preferred length scale $(2\pi/k_x)$ of convective modes increases by increasing $G_W$. It is found that for higher values of $G_W$ or $k_x$ there are no growing convective modes, while for small values of $G_W$ and $k_x$ only a limited number of convective modes exist.

We have studied the influence of an outflow in the unperturbed state on the stability of acoustic and convective modes in a polytropic fluid. We find that the introduction of such a field has a pronounced stabilizing influence on the acoustic and convective modes. The preferred length scale of convective modes increases by increasing the magnitude of the vertical velocity. Our results may not be directly applicable to situations arising in the outer layers of stars. Nevertheless our analysis shows that viscosity plays an important role in determining the stability of various modes in stellar atmospheres.

## References

Antia, H. M.: 1979, *J. Computational Phys.* **30**, 283.
Antia, H. M., Chitre, S. M., and Kale, D. M.: 1978, *Solar Phys.* **56**, 275.
Chandrasekhar, S.: 1961, *Hydrodynamic and Hydromagnetic Stability*, Clarendon, Oxford.
Goldreich, P. and Keeley, D. A.: 1977, *Astrophys. J.* **211**, 934.
Gough, D. O., Moore, D. R., Spiegel, E. A., and Weiss, N. O.: 1976, *Astrophys. J.* **206**, 536.
Graham, E. and Moore, D. R.: 1978, *Monthly Notices Roy. Astron. Soc.* **183**, 617.
Jones, C. A.: 1976, *Monthly Notices Roy. Astron. Soc.* **176**, 145.
Rayleigh, Lord: 1916, *Philos. Mag.* **32**, 529.
Spiegel, E. A.: 1964, *Astrophys. J.* **139**, 959.
Spiegel, E. A.: 1965, *Astrophys. J.* **141**, 1068.
Vickers, G. T.: 1971, *Astrophys. J.* **163**, 363.