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Composition of variable co-efficient singular integral operators

ADIMURTHI and S THANGAVELU

Tata Institute of Fundamental Research Centre, Indian Institute of Science Campus, Bangalore 560012, India

Abstract. In this paper we give a formula for composition of two singular integral operators with variable co-efficients by explicitly calculating the lower order terms. Also we discuss the boundedness of the lower order terms in L^p-spaces.

Keywords. Singular integrals; Fourier transform L^p-spaces; convolutions.

1. Introduction

Singular integral operators, extensively studied by Calderon and Zygmund [1] are well known. They are principal value convolutions of the form

$$Tf(x) = af(x) + \lim_{\delta \to 0} \int_{|y| \geq \delta} f(x-y)K(y)\,\mathrm{d}y \tag{1}$$

where the kernel K is homogeneous of degree $-n$, locally integrable away from the origin and has a mean value zero over the unit sphere. It is known for a long time that the composition of two singular integral operators of the above type is again a singular integral operator of the same type. But the relationship between the kernels is not clearly known.

In a recent paper Strichartz [3] takes a different view of the singular integrals. He considers the kernels being determined uniquely by their restrictions to the hyperplane $x_0 = 1$. Indeed, introducing the notation $|t|_\varepsilon^a = |t|^a (sgnt)^\varepsilon$ for any real t and a and $\varepsilon = 0$ or 1, a kernel $K(x_0, x)$ homogeneous of degree $-n-1$ and of parity ε is uniquely determined by $K(1, x)$ as the identity

$$K(x_0, x) = |x_0|_\varepsilon^{-n-1} K(1, x_0^{-1} x)$$

shows. So it is natural to consider kernels of the form

$$K(x_0, x) = |x_0|_\varepsilon^{-n-1} \phi(x_0^{-1} x)$$

for given ε and ϕ. He considers singular integrals of the form

$$S(\phi, \varepsilon) f(x_0, x) = \lim_{\delta \to 0} \int \int_{|y_0| \geq \delta} f(x_0 - y_0, x - y) K(\phi, \varepsilon)(y_0, y)\,\mathrm{d}y_0\,\mathrm{d}y \tag{2}$$

where

$$K(\phi, \varepsilon)(x_0, x) = |x_0|_\varepsilon^{-n-1} \phi(x_0^{-1} x).$$

This type of singular integrals and those of form (1) differs only by a term of the form af, a being a constant.

The conditions on $K(\phi, \varepsilon)$ are transformed into conditions on ϕ. The local integrability of $K(\phi, \varepsilon)$ away from 0 is equivalent to the integrability of ϕ and the mean value zero condition requires $\int \phi = 0$ when $\varepsilon = 0$ and is automatically satisfied when $\varepsilon = 1$. Given two integrable functions ϕ and ψ, and two parities ε_1 and ε_2, he considers the composition of the two singular integral operators $S(\phi, \varepsilon_1) S(\psi, \varepsilon_2)$ and proves that the composition is given by

$$S(\phi, \varepsilon_1) S(\psi, \varepsilon_2) f = c_0 f + S(A, \varepsilon_1 + \varepsilon_2) f$$

where c_0 is a constant and A is given in terms of ϕ and ψ by

$$A(x) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda + 1|^n_{\varepsilon_1 + \varepsilon_2}}{|\lambda|_{\varepsilon_2}} \phi(x + \lambda(x - y)) \psi(y) d\lambda\, dy \tag{3}$$

In this paper, we try to obtain similar results for the composition of variable coefficient singular integral operators. As is well-known, the product of two such singular integrals is not a singular integral of the same form. So we expect that in the composition formula lower order terms will creep in. We consider singular integral operators whose Kernels are of the form

$$K(x_0, x, z_0, z) = b(x_0, x) |z_0|_\varepsilon^{-n-1} \phi(z_0^{-1} z), \tag{4}$$

where b is a given function and ϕ is integrable with $\int \phi = 0$ when $\varepsilon = 0$. The singular integral operator T with kernel $K(x_0, x, z_0, z)$ is then defined, as before, by

$$Tf(x_0, x) = \lim_{\delta \to 0} \int \int_{|z_0| \geqslant \delta} K(x_0, x, z_0, z) f(x_0 - z_0, x - z) dz_0\, dz \tag{5}$$

Thus we repose our problem as follows: Given parities ε_1 and ε_2, integrable functions ϕ, ψ and functions a and b what will be the form of the composition $aS(\phi, \varepsilon_1) bS(\psi, \varepsilon_2)$? Without loss of generality, we assume that $a \equiv 1$ and the following is the main result of this paper.

THE MAIN THEOREM

Assume that $\phi, \psi \in D(R^n)$ satisfy $\int \phi = 0$ if $\varepsilon_1 = 0$ and $\int \psi = 0$ if $\varepsilon_2 = 0$ and that the function b in $C^{[(n+1)/2]+1}$ satisfy the following conditions:

(i) $D^{\alpha^*} b$ is bounded for any α^* with $|\alpha^*| \leqslant m = \left[\dfrac{n+1}{2}\right]$

(ii) $|D^{\alpha^*} b| \leqslant c(1 + |x_0|^2 + |x|^2)^{-1}$ for $|\alpha^*| = m + 1$ for all $(x_0, x) \in R^{n+1}$.

Then we have the following formula for any $f \in D(R^{n+1})$

$$\begin{aligned} S(\phi, \varepsilon_1) bS(\psi, \varepsilon_2) f(x_0, x) &= a(x_0, x) f(x_0, x) + b(x_0, x) S(A, \varepsilon_1 + \varepsilon_2) f(x_0, x) \\ &+ \sum_{1 \leqslant |\alpha^*| \leqslant m} g_{\alpha^*}(x_0, x) T_{\alpha^*} f(x_0, x) + \sum_{|\alpha^*| = m+1} \frac{m+1}{\alpha^*!} I_{\alpha^*} f(x_0, x) \end{aligned} \tag{6}$$

where

$$a(x_0, x) = \left(-4\int_2^\infty t^{-1}\log(t+1/t-1)\,dt\right)(\textstyle\int\phi)(\int\psi)\,b(x_0, x)$$

$$-4\int_0^2 t^{-1}\log(1+t)\,dt(\textstyle\int\phi)(\int\psi) \tag{7}$$

$$A(x) = \lim_{\delta\to 0}\int\int_{\delta\leqslant|\lambda|\leqslant\delta^{-1}} \frac{|\lambda+1|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda|_{\varepsilon_2}}\phi(x+\lambda(x-y))\psi(y)\,d\lambda\,dy$$

$$g_{\alpha^*}(x_0, x) = \frac{(-1)^{|\alpha^*|}}{\alpha^*!}(D^{\alpha^*}b)(x_0, x) \tag{8}$$

$$T_{\alpha^*}f(x_0, x) = \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_{\alpha^*}(z_0^{-1}z) f(x_0-z_0, x-z)\,dz_0\,dz \tag{9}$$

with

$$A_{\alpha^*}(z) = \lim_{\delta\to 0}\int\int_{\delta\leqslant|\lambda|\leqslant\delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^k y^\alpha\phi(y)\psi(z+(\lambda-1)$$

$$\times(z-y))\,d\lambda\,dy$$

(where we have put $\alpha^* = (\alpha_0, \alpha)$) and

$$I_{\alpha^*}f(x_0, x) = \int_0^1 (1-t)^m G_{\alpha^*}f(x_0, x)\,dt \tag{10}$$

with

$$G_{\alpha^*}f(x_0, x) = \iint K_{\alpha^*}(x_0, x, z_0, z) f(x_0-z_0, x-z)\,dz_0\,dz$$

and $K_{\alpha^*}(x_0, x, z_0, z)$ is given by

$$K_{\alpha^*}(x_0, x, z_0, z) = \lim_{\delta\to 0}\int\int_{\delta\leqslant|y_0-z_0|\leqslant\delta^{-1}} |y_0-z_0|^{-n-1}_{\varepsilon_2}|y_0|^{-n-1}_{\varepsilon_1} y_0^{\alpha 0}y^\alpha$$

$$\phi(y_0^{-1}y)(D)^{\alpha^*}b)(x_0-ty_0, x-ty)\psi(z-y/z_0-y_0)\,dy_0\,dy. \tag{11}$$

Further T_{α^*} is bounded from L^p into L^q where $1<p<q<\infty$, $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{|\alpha^*|}{n+1}$ and I_{α^*} is bounded from L^p into L^q where $1<p<q<\infty$, $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{m+1}{n+1}$.

2. The composition formula

For a given integrable function ϕ on R^n and a parity ε the kernel $K(\phi, \varepsilon)$ is defined on R^{n+1} by

$$K(\phi, \varepsilon)(x_0, x) = |x_0|^{-n-1}_\varepsilon \phi(x_0^{-1}x).$$

Then this function $K(\phi, \varepsilon)$ is homogeneous of degree $-n-1$ and equals ϕ on the hyperplane $x_0 = 1$. When the parity ε is even, we assume $\int \phi(x)\mathrm{d}x = 0$. The singular integral $S(\phi, \varepsilon)$ corresponding to the kernel $K(\phi, \varepsilon)$ is defined by

$$S(\phi, \varepsilon) f(x_0, x) = \lim_{\delta \to 0} \int \int_{|y_0| \geqslant \delta} K(\phi, \varepsilon)(y_0, y) f(x_0 - y_0, x - y)\,\mathrm{d}y_0\,\mathrm{d}y \quad (12)$$

for functions $f \in D(R^{n+1})$.

To find exactly how the composition $S(\phi, \varepsilon) b S(\psi, \varepsilon_2)$ will look like, we first assume that ϕ and ψ are in $D(R^n)$. Assume that ϕ, ψ and b satisfy all the conditions of the theorem. Then we have

LEMMA 2.1.

Define

$$K_\delta(x_0, x, z_0, z) = \int \int_{\substack{|y_0| \geqslant \delta \\ |y_0 - z_0| \geqslant \delta}} |y_0|_\varepsilon^{-n-1} |z_0 - y_0|_\varepsilon^{-n-1} \phi(y_0^{-1} y)$$
$$b(x_0 - y_0, x - y) \psi(z - y / z_0 - y_0)\,\mathrm{d}y_0\,\mathrm{d}y \quad (13)$$

Then for any $f \in D(R^{n+1})$ we have

$$S(\phi, \varepsilon_1) b S(\psi, \varepsilon_2) f = \lim_{\delta \to 0} K_{\delta *} f \quad (14)$$

both pointwise and in the L^2 norm.

The proof of Lemma 2.1 is routine and we omit it here. (See [3] Lemma 2.1).

Now we write $K_\delta = A_\delta + B_\delta$ where

$$\begin{aligned} A_\delta(x_0, x, z_0, z) &= K_\delta(x_0, x, z_0, z), \text{ if } |z_0| \geqslant 2\delta \\ &= 0, \text{ if } |z_0| < 2\delta \end{aligned} \quad (15)$$

and

$$\begin{aligned} B_\delta(x_0, x, z_0, z) &= K_\delta(x_0, x, z_0, z), \text{ if } |z_0| < 2\delta \\ &= 0, \text{ if } |z_0| \geqslant 2\delta. \end{aligned}$$

We will consider the convolution with each term separately. For the convolution $B_{\delta *} f$ we have:

LEMMA 2.2.

$$\lim_{\delta \to 0} B_{\delta *} f = a\left(\int \phi\right)\left(\int \psi\right) b f$$

in the L^2 norm where the constant a is given by

$$a = -4 \int_0^2 t^{-1} \log(1 + t)\,\mathrm{d}t.$$

Proof. For $|z_0| < 2\delta$ we have after a change of variables

$$B_\delta(x_0, x, z_0, z) = \delta^{-n-1} \int \int_{\substack{|y_0| \geq 1 \\ |\delta^{-1}z_0 - y_0| \geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |\delta^{-1}z_0 - y_0|_{\varepsilon_2}^{-n-1}$$

$$b(x_0 - \delta y_0, x - \delta y)\, \phi(y/y_0) \psi\left(\frac{\delta^{-1}z - y}{\delta^{-1}z_0 - y_0}\right) \mathrm{d}y_0\, \mathrm{d}y.$$

Therefore,

$$|B_\delta(x_0, x, z_0, z)| \leqslant c\delta^{-n-1} L(\delta^{-1}z_0, \delta^{-1}z)$$

where the function L is given by

$$L(z_0, z) = \int \int_{\substack{|y_0| \geq 1 \\ |z_0 - y_0| \geq 1}} |y_0|^{-n-1} |z_0 - y_0|^{-n-1} |\phi(y/y_0)|$$

$$\psi(z - y/z_0 - y_0)\, \mathrm{d}y_0\, \mathrm{d}y \qquad \text{if } |z_0| < 2\delta$$

$$= 0, \qquad \text{if } |z_0| \geqslant 2\delta.$$

It then follows that L is an integrable function and hence

$$\lim_{\delta \to 0} \iint B_\delta(x_0, x, z_0, z) f(x_0 - z_0, x - z) \mathrm{d}z_0\, \mathrm{d}z$$

$$= \left[\lim_{\delta \to 0} \iint B_\delta(x_0, x, z_0, z) \mathrm{d}z_0\, \mathrm{d}z\right] f(x_0, x).$$

Thus it remains to show

$$\lim_{\delta \to 0} \iint B_\delta(x_0, x, z_0, z)\, \mathrm{d}z_0\, \mathrm{d}z = a(\textstyle\int \phi)(\int \psi)\, b(x_0, x).$$

Now

$$\iint B_\delta(x_0, x, z_0, z)\, \mathrm{d}z_0 \mathrm{d}z$$

$$= \int \int_{-2\delta}^{2\delta} \mathrm{d}z_0 \mathrm{d}z \int \int_{\substack{|y_0| \geq \delta \\ |z_0 - y_0| \geq \delta}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} b(x_0 - y_0, x - y)$$

$$\phi(y/y_0) \psi(z - y/z_0 - y_0)\, \mathrm{d}y_0 \mathrm{d}y$$

$$= \int \int_{-2}^{2} \mathrm{d}z_0 \mathrm{d}z \int_{\substack{|y_0| \geq 1 \\ |z_0 - y_0| \geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} b(x_0 - \delta y_0, x - \delta y)$$

$$\phi(y/y_0) \psi\left(\frac{z - y}{z_0 - y_0}\right) \mathrm{d}y_0 \mathrm{d}y.$$

Since

$$\int\int_{\substack{|y_0|\geqslant 1\\ |z_0-y_0|\geqslant 1}} |y_0|_{\varepsilon_1}^{-n-1}|z_0-y_0|_{\varepsilon_2}^{-n-1}b(x_0-\delta y_0,\, x-\delta y)\phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right)\mathrm{d}y_0\,\mathrm{d}y$$

converges to

$$\int\int_{\substack{|y_0|\geqslant 1\\ |z_0-y_0|\geqslant 1}} |y_0|_{\varepsilon_1}^{-n-1}|z_0-y_0|_{\varepsilon_2}^{-n-1}b(x_0,x)\phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right)\mathrm{d}y_0\,\mathrm{d}y$$

as $\delta\to 0$ and is bounded by an integrable function, dominated convergence theorem shows that

$$\begin{aligned}\lim_{\delta\to 0}\iint B_\delta(x_0,x,z_0,z)\,\mathrm{d}z_0\mathrm{d}z &= b(x_0,x)\int\int_{-2}^{2}\mathrm{d}z_0\mathrm{d}z\int\int_{\substack{|y_0|\geqslant 1\\ |z_0-y_0|\geqslant 1}}|y_0|_{\varepsilon_1}^{-n-1}|z_0-y_0|_{\varepsilon_2}^{-n-1}\\ &\qquad \phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right)\mathrm{d}y_0\mathrm{d}y\\ &= b(x_0,x)\left(\textstyle\int\phi\right)\left(\textstyle\int\psi\right)\int_{-2}^{2}\int_{\substack{|y_0|\geqslant 1\\ |z_0-y_0|\geqslant 1}}|y_0|_{\varepsilon_1}^{-1}|z_0-y_0|_{\varepsilon_2}^{-1}\mathrm{d}y_0\,\mathrm{d}z_0.\end{aligned}$$

We have to consider only the case $\varepsilon_1=\varepsilon_2=1$ and in that case the change of variable $y_0\to(y_0+\frac{1}{2})z_0$ gives

$$\begin{aligned}&\int_{-2}^{2}\int_{\substack{|y_0|\geqslant 1\\ |z_0-y_0|\geqslant 1}} y_0^{-1}(z_0-y_0)^{-1}\,\mathrm{d}y_0\mathrm{d}z_0\\ &= 4\int_0^2\int_{\frac{1}{2}+\frac{1}{z_0}}^{\infty}(\tfrac{1}{2}+y_0)^{-1}(\tfrac{1}{2}-y_0)^{-1}z_0^{-1}\,\mathrm{d}y_0\,\mathrm{d}z_0\\ &= -4\int_0^2 z_0^{-1}\log(1+z_0)\,\mathrm{d}z_0.\end{aligned}$$

Q.E.D.

Turning our attention to the next term, we have for $|z_0| \geqslant 2\delta$

$$A_\delta(x_0, x, z_0, z) = \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} \int |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} b(x_0 - y_0, x - y)\phi(y/y_0)\psi\left(\frac{z-y}{z_0 - y_0}\right) \mathrm{d}y_0 \, \mathrm{d}y$$

Expanding $b(x_0 - y_0, x - y)$ in Taylor series about (x_0, x)

$$b(x_0 - y_0, x - y) = b(x_0, x) + \sum_{1 \leqslant |\alpha^*| \leqslant m} \frac{(-1)^{|\alpha^*|}}{\alpha^*!} y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0, x) + \sum_{|\alpha^*| = m+1} \frac{m+1}{\alpha^*!} \int_0^1 (1-t)^m y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0 - t y_0, x - t y) \, \mathrm{d}t, \tag{16}$$

where $\alpha^* = (\alpha_0, \alpha)$ is a multi index and $m = [\frac{1}{2}(n+1)]$. Thus we have for $|z_0| \geqslant 2\delta$

$$A_\delta(x_0, x, z_0, z) = b(x_0, x) A_{0,\delta}(x_0, x, z_0, z) + \sum_{1 \leqslant |\alpha^*| \leqslant m} \frac{(-1)^{|\alpha^*|}}{\alpha^*!} (D^{\alpha^*} b)(x_0, x) A_{\alpha^*,\delta}(x_0, x, z_0, z) + \sum_{|\alpha^*| = m+1} \frac{m+1}{\alpha^*!} \int_0^1 (1-t)^m A_{\alpha^*,\delta,t}(x_0, x, z_0, z) \, \mathrm{d}t \tag{17}$$

where we have set

$$A_{0,\delta}(x_0, x, z_0, z) = \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} \int |y_0|_{\varepsilon_1}^{-n-1} \; |z_0 - y_0|_{\varepsilon_2}^{-n-1} \phi(y/y_0)\psi\left(\frac{z-y}{z_0 - y_0}\right) \mathrm{d}y_0 \, \mathrm{d}y \tag{18}$$

$$A_{\alpha^*,\delta}(x_0, x, z_0, z) = \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} \int |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} \phi(y/y_0)\psi\left(\frac{z-y}{z_0 - y_0}\right) \mathrm{d}y_0 \, \mathrm{d}y, \tag{19}$$

$$A_{\alpha^*,\delta,t}(x_0, x, z_0, z) = \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} \int |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0 - t y_0, x - t y)\phi(y/y_0)\psi\left(\frac{z-y}{z_0 - y_0}\right) \mathrm{d}y_0 \, \mathrm{d}y. \tag{20}$$

Then the following Lemma 2.3 follows immediately from Lemma 2.3 of [3].

LEMMA 2.3.

$$\lim_{\delta \to 0} A_{0,\delta^*} f = S(A, \varepsilon_1 + \varepsilon_2) f + a'(\textstyle\int \phi)(\int \psi) f$$

both pointwise and in L^2 norm where

$$a' = -4 \int_2^{\infty} t^{-1} \log\left(\frac{t+1}{t-1}\right) \mathrm{d}t$$

and

$$A(x) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda+1|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda|_{\varepsilon_2}} \psi(y)\phi(x+\lambda(x-y)) \mathrm{d}\lambda\, \mathrm{d}y$$

the limit existing both pointwise and in L^1 norm.

Next we consider the terms $A_{\alpha^*,\delta^*} f$ for $1 \leqslant |\alpha^*| \leqslant m$. Setting $A_{\alpha^*,\delta} = A_{k,\delta}$ for convenience when $|\alpha^*| = k$, $1 \leqslant k \leqslant m$ we have the following proposition concerning the convolution

$$\lim_{\delta \to 0} A_{k,\delta} * f.$$

PROPOSITION 2.1.

$$\lim_{\delta \to 0} \iint A_{k,\delta}(x_0, x, z_0, z) f(x_0 - z_0, x - z)\, \mathrm{d}z_0\, \mathrm{d}z$$
$$= \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1} z) f(x_0 - z_0, x - z)\, \mathrm{d}z_0\, \mathrm{d}z,$$

where the function $A_k(z)$ is given by

$$A_k(z) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda - 1)^k y^\alpha \phi(y)$$
$$\psi(z + (\lambda - 1)(z - y))\, \mathrm{d}\lambda \mathrm{d}y \tag{21}$$

where the limit exists in the L^2 norm. Furthermore, the operator

$$f \to \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1} z) f(x_0 - z_0, x - z)\, \mathrm{d}z_0\, \mathrm{d}z$$

is bounded from L^p into L^q for $1 < p < q < \infty, \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{k}{n+1}$

Proof. The proof follows in several steps.

Steps. A change of variables in (19) shows that

$$A_{k,\delta}(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_{k,\delta|z_0|^{-1}}(1, z_0^{-1} z) \tag{22}$$

where

$$A_{k,\delta}(1,z)=\int\limits_{\substack{|y_0|\geqslant\delta\\|1-y_0|\geqslant\delta}}\int |y_0|_{\varepsilon_1}^{-n-1}|1-y_0|_{\varepsilon_2}^{-n-1}y_0^{\alpha_0}y^{\alpha}\phi(y/y_0)$$
$$\times\psi\left(\frac{z-y}{1-y_0}\right)\mathrm{d}y_0\,\mathrm{d}y. \qquad (23)$$

By changing y into $y_0 y$ and then putting $\lambda=y_0/(1-y_0)$ we obtain

$$A_{k,\delta}(1,z)=\iint\limits_{T_\delta}\frac{|\lambda+1|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda|_{\varepsilon_1}}\lambda^{k}(\lambda+1)^{-k}y^{\alpha}\phi(y)\psi(z+\lambda(z-y))\,\mathrm{d}\lambda\,\mathrm{d}y,$$

where T_δ is the union of the intervals

$$\left(\frac{\delta}{1-\delta},\frac{1-\delta}{\delta}\right)\text{ and }\left(-\frac{1+\delta}{\delta},-\frac{\delta}{1+\delta}\right).$$

Changing $\lambda+1$ into λ we obtain

$$A_{k,\delta}(1,z)=\iint\limits_{T_\delta}\frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}y^{\alpha}\phi(y)\psi(z+(\lambda-1)(z-y))\,\mathrm{d}x\,\mathrm{d}y$$

with the region T_δ given by

$$T_\delta=\left(\frac{1}{1-\delta},\frac{1}{\delta}\right)\cup\left(-\frac{1}{\delta},\frac{1}{1+\delta}\right).$$

Considering only $\delta<\frac{1}{2}$ we take $T_\delta=R_\delta\cup S_\delta$ where

$$R_\delta=\left(-2,\frac{1}{1+\delta}\right)\cup\left(\frac{1}{1-\delta},2\right)$$

and

$$S_\delta=\left(-\frac{1}{\delta},-2\right)\cup\left(2,\frac{1}{\delta}\right).$$

We then write

$$A_{k,\delta}(1,z)=\iint\limits_{R_\delta}\frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}y^{\alpha}\phi(y)\psi(z+(\lambda-1)(z-y))\,\mathrm{d}\lambda\,\mathrm{d}y$$
$$+\iint\limits_{S_\delta}\frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}y^{\alpha}\phi(y)\psi(z+(\lambda-1)(z-y))\,\mathrm{d}x\,\mathrm{d}y \qquad (24)$$
$$=B_{k,\delta}(1,z)+C_{k,\delta}(1,z).$$

Adding and subtracting appropriate terms we obtain

$$B_{k,\delta}(1,z)=\iint\limits_{R_\delta}\frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}y^{\alpha}\phi(y)\{\psi(z+(\lambda-1)(z-y))$$
$$-\psi(z)\}\,\mathrm{d}\lambda\,\mathrm{d}y+h_{k,\delta}\left(\int y^{\alpha}\phi\right)\psi(z), \qquad (25)$$

where

$$h_{k,\delta} = \int_{R_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \, d\lambda.$$

Likewise we obtain

$$C_{k,\delta}(1,z) = \iint_{S_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \psi(z+(\lambda-1)(z-y))[y^\alpha\phi(y) - z^\alpha\phi(z)] \, d\lambda \, dy + r_{k,\delta}\left(\int\psi\right) z^\alpha\phi(z), \tag{26}$$

where

$$r_{k,\delta} = \int_{S_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k |\lambda-1|^{-n} \, d\lambda. \tag{27}$$

Define a function $A_k(z)$ by

$$A_k(z) = \int \int_{|\lambda|\leq 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha\phi(y)[\psi(z+(\lambda-1)(z-y)) - \psi(z)] \, d\lambda \, dy + \int \int_{|\lambda|\geq 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \psi(z+(\lambda-1)(z-y)) [y^\alpha\phi(y) - z^\alpha\phi(z)] \, d\lambda \, dy + h_k\left(\int y^\alpha\phi\right)\psi(z) + r_k\left(\int\psi\right) z^\alpha\phi(z) \tag{28}$$

with

$$h_k = \int_{|\lambda|\leq 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \, d\lambda$$

and

$$r_k = \int_{|\lambda|\geq 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \, d\lambda.$$

Step 2. Assume that both ϕ and ψ are supported in the ball $\{x: |x| \leq R\}$. Then

$$|A_k(z) - A_{k,\delta}(1,z)| \leq c_k\delta \quad \text{for} \quad |z| \leq 3R$$
$$A_k(z) = A_{k,\delta}(1,z) \quad \text{for} \quad |z| > 3R.$$

Writing

$$A_k(z) = A_{1,k}(z) + A_{2,k}(z) + h_k\left(\int y^\alpha\phi\right)\psi(z) + r_k\left(\int\psi\right)z^\alpha\phi(z). \tag{29}$$

We have

$$A_k(z) - A_{k,\delta}(1,z) = (A_{1,k}(z) - B_{k,\delta}(1,z)) + (A_{2,k}(z) - c_{k,\delta}(1,z)) + (h_k - h_{k,\delta})\left(\int y^\alpha\phi\right)\psi(z) + (r_k - r_{k\delta})\left(\int\psi\right)z^\alpha\phi(z), \tag{30}$$

where

$$A_{1,k}(z) - B_{k,\delta}(1,z) = \int \int_{1/(1+\delta)}^{1/(1-\delta)} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)$$

$$[\psi(z+(\lambda-1)(z-y)) - \psi(z)]\, d\lambda\, dy, \tag{31}$$

$$A_{2,k}(z) - c_{k,\delta}(1,z) = \int \int_{|\lambda| \geq \frac{1}{3}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k (y^\alpha \phi(y)$$

$$- z^\alpha \phi(z))\psi(z+(\lambda-1)(z-y))\, d\lambda\, dy, \tag{32}$$

$$h_k - h_{k\delta} = \int_{1/(1+\delta)}^{1/(1-\delta)} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k\, d\lambda, \tag{33}$$

and

$$r_k - r_{k\delta} = \int_{|\lambda| \geq \frac{1}{3}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k |\lambda-1|^{-n}\, d\lambda. \tag{34}$$

When $|z| > 3R$, we have to consider only terms of the form $\phi(y)\psi(z+(\lambda-1)(z-y))$. For this to be nonzero, we must have $|y| \leq R$ and $|z+(\lambda-1)(z-y)| \leq R$. These inequalities imply

$$|\lambda||z| = |\lambda z| \leq |\lambda z + (1-\lambda)y| + |(\lambda-1)y| \leq (1+|\lambda-1|)R,$$

and since $|z| > 3R$ we obtain

$$3|\lambda| \leq (1+|\lambda-1|).$$

But this is true only when $-1 \leq \lambda \leq \frac{1}{2}$ and our region of integration excludes this. Hence $A_k(z) = A_{k,\delta}(1,z)$ when $|z| > 3R$ follows. Next assuming $|z| \leq 3R$, to estimate (31), we apply mean value to theorem to get an estimate

$$|\psi(z+(\lambda-1)(z-y)) - \psi(z)| \leq c|\lambda-1||z-y|.$$

Thus the integral (31) is dominated by

$$c \int_{|y| \leq R} \int_{1/(1+\delta)}^{1/(1-\delta)} |\lambda|^{n-k}|\lambda-1|^k|z-y|\, d\lambda\, dy \leq c\delta$$

since $(\frac{1}{1+\delta}, \frac{1}{1-\delta}) \subset (\frac{2}{3}, 2)$ and the integrand is bounded there.

Coming to (32), we have, again by an application of mean value theorem,

$$|A_{2,k}(z) - c_{k,\delta}(1,z)|$$

$$\leq c \int \int_{|\lambda| \geq \frac{1}{3}} |\lambda|^{n-k}|\lambda-1|^{k-1}|\psi(z+(\lambda-1)(z-y))||z-y|\, d\lambda\, dy$$

$$\leq c \int \int_{|\lambda-1| \geq \frac{1}{2\delta}} |\lambda-1|^{n-1}|\psi(z+(\lambda-1)(z-y))||z-y|\, d\lambda\, dy.$$

Putting $(\lambda-1)|z-y|=\lambda'$ and writing $|z-y|^{-1}(z-y)=w$, we obtain

$$|A_{2,k}(z)-c_{k,\delta}(1,z)|$$

$$\leqslant c\int\int_{|\lambda'|\geqslant|z-y|/2\delta}|\lambda'|^{n-1}|z-y|^{1-n}|\psi(z+\lambda'w)|\mathrm{d}\lambda'\mathrm{d}y.$$

Note that $|\psi(z+\lambda'w)|=0$ if $|\lambda'|\geqslant 4R$. So we have to integrate only over $|\lambda'|\leqslant 4R$. This gives

$$|A_{2,k}(z)-c_{k,\delta}(1,z)|\leqslant c\int_{|z-y|\leqslant 8R_\delta}|z-y|^{1-n}\mathrm{d}y\int_{|\lambda|\leqslant 4R}|\lambda|^{n-1}\mathrm{d}\lambda\leqslant c\delta.$$

It is easy to check that $|h_k-h_{k,\delta}|\leqslant c\delta$ and for the remaining term we need to consider only the case when $\varepsilon_2=1$ and in that case it is simple to estimate it in terms of δ. This completes step (2).

Step 3. Next we write

$$A_{k,\delta}(1,z)=A_{k,\delta k+1}(1,z)+D_{k,\delta}(1,z), \tag{35}$$

with

$$D_{k,\delta}(1,z)=A_{k,\delta}(1,z)-A_{k,\delta k+1}(1,z).$$

We have to consider the convolutions of f with the kernels defined by

$$\begin{aligned}E_{k,\delta}(z_0,z)&=|z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k}\{A_{k,(\delta|z_0|^{-1})k+1}(1,z_0^{-1}z)\\&\quad-A_k(z_0^{-1}z)\}\quad\text{for}\quad|z_0|\geqslant 2\delta\\&=0,\quad\text{for}\quad|z_0|<2\delta,\end{aligned}\tag{36}$$

$$\begin{aligned}A_k^\delta(z_0,z)&=|z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k}A_k(z_0^{-1}z),\quad\text{for}\quad|z_0|\geqslant 2\delta\\&=0,\quad\text{for}\quad|z_0|<2\delta,\end{aligned}\tag{37}$$

$$\begin{aligned}D_k^\delta(z_0,z)&=|z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k}D_{k,\delta|z_0|^{-1}}(1,z_0^{-1}z),\quad\text{for}\quad|z_0|\geqslant 2\delta\\&=0,\quad\text{for}\quad|z_0|<2\delta.\end{aligned}\tag{38}$$

We will take up the terms one by one. For the first term (36), we claim

$$\lim_{\delta\to 0}\iint E_{k,\delta}(z_0,z)f(x_0-z_0,x-z)\mathrm{d}\lambda\,\mathrm{d}z=0. \tag{39}$$

By step (2), we see that

$$|E_{k,\delta}(z_0,z)|\leqslant c\delta|z_0|^{-n-2}\text{ and }E_{k,\delta}(z_0,z)=0$$

if $|z|\geqslant 3R|z_0|$ or $|z_0|<2\delta$. Defining an integrable function L_k by

$$\begin{aligned}L_k(z_0,z)&=c|z_0|^{-n-2},\quad\text{for}\quad|z|\leqslant 3R|z_0|\text{ and }|z_0|\geqslant 2\\&=0,\quad\text{for }|z|>3R|z_0|\text{ or }|z_0|<2.\end{aligned}\tag{40}$$

We observe that

$$|E_{k,\delta}(z_0,z)|\leqslant\delta^{-n-1}L_k(\delta^{-1}z_0,\delta^{-1}z).$$

Therefore, we get

$$\lim_{\delta\to 0}\iint E_{k,\delta}(z_0, z) f(x_0 - z_0, x - z)\,dz_0\,dz$$

$$= (\lim_{\delta\to 0}\iint E_{k,\delta}(z_0, z)\,dz_0\,dz) f(x_0, x).$$

Write

$$E_{k,\delta}(z_0, z) = |z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k}[A_{k,(\delta|z_0|^{-1})^{k+1}}(1, z_0^{-1}z) - A_k(z_0^{-1}z)]$$

$$= |z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} F_{\delta|z_0|^{-1}}(1, z_0^{-1}z),$$

with

$$F_\delta(1, z) = \int \int_{1/(1+\delta^{k+1})}^{1/(1-\delta^{k+1})} \frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)\psi(\lambda z - (\lambda-1)y)\,d\lambda\,dy$$

$$+ \int_{|\lambda|\geqslant\delta^{-(k+1)}} \frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)\psi(\lambda z - (\lambda-1)y)\,d\lambda\,dy$$

$$= F_\delta^1(1, z) + F_\delta^2(1, z) \text{ (say)}.$$

Now we have

$$\int F^1_{\delta|z_0|^{-1}}(1, z_0^{-1}z)\,dz = |z_0|^n\left(\int y^\alpha\phi\right)\left(\int\psi\right) \int_{(1+(|z_0|^{-1}\delta)^{k+1})^{-1}}^{(1-(\delta|z_0|^{-1})^{k+1})^{-1}} |\lambda|^0_{\varepsilon_1+\varepsilon_2}\lambda^{-k}\frac{(\lambda-1)^k}{|\lambda-1|_{\varepsilon_1}}\,d\lambda$$

and we obtain

$$\left|\int\int_{|z_0|\geqslant 2\delta} |z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} F^1_{\delta|z_0|^{-1}}(1, z_0^{-1}z)\,dz_0\,dz\right|.$$

$$\leqslant c\int_{|z_0|\geqslant 2\delta} |z_0|^{k-1}\,dz_0 \int_{(1+(\delta|z_0|^{-1})^{k+1})^{-1}}^{(1-(\delta|z_0|^{-1})^{k+1})^{-1}} \lambda^{-k}|\lambda-1|^{k-1}\,d\lambda$$

$$\leqslant c\delta^k \int_{|z_0|\geqslant 2} |z_0|^{-2}\,dz_0 \leqslant c\delta^k.$$

For F_δ^2 term we add and subtract the term $z^\alpha\phi(z)$ and both terms can be estimated as in the previous case to yield a bound of δ^k. Thus we obtain

$$\lim_{\delta\to 0}\iint E_{k,\delta}(z_0, z)\,dz_0\,dz = 0$$

and this proves the claim.

Regarding (37), we claim the following:

$$\lim_{\delta\to 0}\int\int_{|z_0|\geqslant 2\delta} |z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} A_k(z_0^{-1}z) f(x_0 - z_0, x - z)\,dz_0\,dz = A_k f,$$

where
$$A_k f(x_0, x) = \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z) f(x_0 - z_0, x - z)\, dz_0\, dz,$$

and A_k defines a bounded linear operator from L^p into L^q for $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{k}{n+1}$.

We first make the observation that
$$|A_k(z)| \leqslant c(1+|z|^2)^{-n/2}.$$
Indeed, we have to prove this only for $A_{k,\delta}(1, z)$ since
$$|A_k(z) - A_{k,\delta}(1, z)| \leqslant c\delta \quad \text{for} \quad |z| \leqslant 3R$$

and they are equal when $|z| > 3R$. From the expression
$$A_{k,\delta}(1, z) = \iint_{T_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)\psi(z + (\lambda-1)(z-y))\, d\lambda\, dy,$$

it is clear that $A_{k,\delta}(1, z)$ is bounded and for $|z| > 3R$ the integrand vanishes unless $|\lambda||z| \leqslant (1 + |\lambda - 1|)R$ and $-1 \leqslant \lambda \leqslant \frac{1}{2}$. Thus in the above, the λ integration is only over the interval $|\lambda| \leqslant \frac{3R}{|z|}$ and this gives the required estimate.

Now
$$A_k^\delta(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z)$$
is homogeneous of degree $-n-1+k$. By setting $\lambda^2 = |z_0|^2 + |z|^2$ we have
$$A_k^\delta\left(\frac{z_0}{\lambda}, \frac{z}{\lambda}\right) = \lambda^{(n+1-k)} |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z)$$
which shows that
$$\left|A_k^\delta\left(\frac{z_0}{\lambda}, \frac{z}{\lambda}\right)\right| \leqslant \left(1 + \left|\frac{z}{z_0}\right|^2\right)^{\frac{n+1-k}{2}} |A_k(z_0^{-1}z)|.$$
$$\leqslant c$$

Therefore
$$|A_k^\delta(z_0, z)| \leqslant c(|z_0|^2 + |z|^2)^{\frac{-n-1+k}{2}},$$
which is the Riesz-potential of order $n+1-k$. Hence the convolution with $A_k^\delta(z_0, z)$ makes sense and we can pass to the limit as $\delta \to 0$. Also
$$A_k f(x_0, x) = \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z) f(x_0 - z_0, x - z)\, dz_0\, dz$$

defines a bounded linear operator from L^p into L^q whenever $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n+1}$.

Q.E.D.

Finally for the convolution with $D_k^\delta(z_0, z)$ we have the following claim.

$$\lim_{\delta\to 0} \iint D_k^\delta(z_0, z) f(x_0 - z_0, x - z) dz_0 \, dz = 0$$

in the L^p norm for any p satisfying

$$\frac{n+1}{n+2-k} < p < \frac{n+1}{n+1-k}.$$

Proof. We will show that for any p satisfying the above inequalities D_k^δ is in $L^p(R^{n+1})$ and the L^p norm of it tends to zero as $\delta \to 0$ and this will prove the claim.

We first make the observation that $D_k^\delta(z_0, z)$ vanishes for $|z| > 3R|z_0|$. This follows from the fact that

$$D_{k,\delta}(1, z) = A_{k,\delta}(1, z) - A_{k,\delta k+1}(1, z),$$

and this vanishes for $|z| > 3R$. Thus we have

$$\iint |z_0|^{-p(n+1-k)} |D_{k,\delta|z_0|^{-1}}(1, z_0^{-1} z)|^p dz_0 \, dz$$

$$\leqslant \int\limits_{|z| \leqslant 3R|z_0|} \delta^p \int\limits_{|z_0| \geqslant 2\delta} |z_0|^{-p(n+1-k)} |z_0|^{-p} dz_0 \, dz$$

$$\leqslant c\delta^{n+1-p(n+1-k)} \int\limits_{|z_0| \geqslant 2} |z_0|^{n-p(n+2-k)} dz_0.$$

Since $n - p(n+2-k) < -1$ the integral is finite and since $n+1-p(n+1-k) > 0$ the L^p norm of D_k^δ tends to 0.

Q.E.D.

Step 4. From step (1), (2) and (3), the proof of the Proposition (2.1) is complete save for the expression for A_k. We now write $A_k(z)$ in the form

$$A_k(z) = \lim_{\delta\to 0} \Bigg\{ \iint\limits_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y) \psi(z + (\lambda-1)(z-y)) d\lambda \, dy$$

$$+ \Bigg[h_k - \int\limits_{\delta \leqslant |\lambda| \leqslant 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k d\lambda \Bigg] \left(\int y^\alpha \phi\right) \psi(z)$$

$$+ \Bigg[r_k - \int\limits_{2 \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k |\lambda-1|^{-n} d\lambda \Bigg] z^\alpha \phi(z) \left(\int \psi\right) \Bigg\}.$$

Since we have removed the singularities at 0 and ∞ the above integral exists and the bracketed expressions tend to zero as $\delta \to 0$. Thus the above limit exists in the L^2 norm and equals

$$A_k(z) = \lim_{\delta\to 0}\int \int_{\delta\leqslant|\lambda|\leqslant\delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^k y^\alpha\phi(y)$$

$$\psi(z+(\lambda-1)(x-y))\mathrm{d}\lambda\,\mathrm{d}y.$$

This completes the proof of Proposition 2.1.

Now it remains to consider convolution with kernels of the form

$$\int_0^1 (1-t)^m A_{\alpha^*,\delta,t}(x_0, x, z_0, z)\,\mathrm{d}t \text{ with } |\alpha^*| = m+1.$$

Concerning the convolution with

$$A_{\alpha^*,\delta,t}(x_0, x, z_0, z),$$

we will prove now the following Lemma. For the sake of convenience, we will drop t and write $G_{m+1,\delta}$ instead of $A_{\alpha^*,\delta,t}$. With this notation, we have

LEMMA 2.4.

$$G_{m+1}f(x_0, x) = \lim_{\delta\to 0}\iint G_{m+1,\delta}(x_0, x, z_0, z)f(x_0-z_0, x-z)\,\mathrm{d}z_0\,\mathrm{d}z \tag{41}$$

exists and we obtain

$$G_{m+1}f(x_0, x) = \iint K_{m+1}(x_0, x, z_0, z)f(x_0-z_0, x-z)\,\mathrm{d}z_0\,\mathrm{d}z, \tag{42}$$

where the Kernel $K_{m+1}(x_0, x, z_0, z)$ is given by

$$K_{m+1}(x_0, x, z_0, z) = \lim_{\delta\to 0}\int \int_{\delta\leqslant|z_0-y_0|\leqslant\delta^{-1}} |y_0|^{-n-1}_{\varepsilon_1}|z_0-y_0|^{-n-1}_{\varepsilon_2}$$

$$y_0^{\alpha_0}y^\alpha(D^{\alpha^*}b)(x_0-ty_0, x-ty)\phi(y_0^{-1}y)\psi\left(\frac{z-y}{z_0-y_0}\right)\mathrm{d}y_0\,\mathrm{d}y. \tag{43}$$

The limit exists in the L^2 norm and G_{m+1} defines a bounded linear operator from L^p into L^q whenever $1<p<q<\infty$ and $\frac{1}{q}=\frac{1}{p}-\frac{m}{n+1}$ and the norm of G_{m+1} is independent of t.

Proof. Let us write g instead of $(D^{\alpha^*}b)$ and define

$$H_\delta(x_0, x, y_0, y) = y_0^{\alpha 0}y^\alpha|y_0|^{-n-1}_{\varepsilon_1}\phi(y_0^{-1}y)g(x_0-ty_0, x-ty), \text{for } |y_0|\geqslant\delta$$

$$= 0, \quad \text{otherwise.}$$

In the integral

$$(G_{m+1,\delta} * f)(x_0, x)$$
$$= \iint f(x_0 - z_0, x - z)\,dz_0\,dz \int\limits_{\substack{|y_0| \geq \delta \\ |z_0 - y_0| \geq \delta}} \int y_0^{\alpha_0} y^{\alpha} |y_0|_{\varepsilon_1}^{-n-1} \phi(y_0^{-1} y)$$
$$|z_0 - y_0|_{\varepsilon_2}^{-n-1} g(x_0 - ty_0, x - ty) \psi\left(\frac{z - y}{z_0 - y_0}\right) dy_0\,dy$$

changing the order of integration we have

$$(G_{m+1,\delta} * f)(x_0, x) = \iint H_\delta(x_0, x, y_0, y)\, K_2^\delta f(x_0 - y_0, x - y)\,dy_0\,dy$$

where $K_2^\delta f$ is the integral given by

$$(K_2^\delta f)(x_0, x) = \int\limits_{|z_0| \geq \delta} \int |z_0|_{\varepsilon_2}^{-n-1} \psi(z_0^{-1} z) f(x_0 - z_0, x - z)\,dz_0\,dz.$$

Let us now estimate the L^2 norm of $H_\delta(x_0, x, y_0, y)$ for fixed (x_0, x).

$$\iint |H_\delta(x_0, x, y_0, y)|^2\,dy_0\,dy$$
$$\leq \iint |y_0|^{2(m-n)} |\phi(y_0^{-1} y)|^2 |g(x_0 - ty_0, x - ty)|^2\,dy_0\,dy$$
$$\leq \operatorname*{Sup}_{(x_0, x)} (1 + |x_0|^2 + |x|^2)^2 |g(x_0, x)|^2 \iint |y_0|^{2(m-n)} |\phi(y_0^{-1} y)|^2$$
$$(1 + |x_0 - ty_0|^2)^{-2}\,dy_0\,dy$$
$$\leq c(1 + |x_0|^2)^2 t^{2(n-m)-n-1} \|\phi\|_2^2 \int |y_0|^{2(m-n)+n} (1 + |y_0|^2)^{-2}\,dy_0.$$

Since we have chosen $m = [(n+1)/2]$ the integral converges. Thus we see that H_δ is in L^2 and in the L^2 norm it converges to the function H defined by

$$H(x_0, x, y_0, y) = y_0^{\alpha_0} y^{\alpha} |y_0|_{\varepsilon_1}^{-n-1} \phi(y_0^{-1} y) g(x_0 - ty_0, x - ty).$$

Since $K_2^\delta f$ is a truncated singular integral, it converges boundedly to the function $K_2 f$ in the L^2 norm. Therefore, passing to the limit as $\delta \to 0$, we obtain

$$G_{m+1} * f = \iint H(x_0, x, y_0, y) K_2 f(x_0 - y_0, x - y)\,dy_0\,dy.$$

Now suppose m_2 is the multiple associated with the singular integral $K_2 = S(\psi, \varepsilon_2)$. Then it is shown in [2] that we have

$$m_2(z_0, z) = \lim_{\delta \to 0} \int\limits_{\delta \leq |x_0| \leq \delta^{-1}} \int \exp[-i(x_0 z_0 + x \cdot z)] |x_0|_{\varepsilon_2}^{-n-1} \psi(x_0^{-1} x)\,dx_0\,dx$$
$$= \lim_{\delta \to 0} m_2^\delta(z_0, z),$$

where m_2^δ is the Fourier transform of the function which equals

$$|x_0|_{\varepsilon_2}^{-n-1}\psi(x_0^{-1}x) \quad \text{for} \quad \delta \leqslant |x_0| \leqslant \delta^{-1}$$

and zero elsewhere. The relation between K_2 and m_2 is given by $(K_2 f)\hat{} = m_2 \hat{f}$ where $\hat{}$ denotes the Fourier transformation.

In the expression for $G_{m+1} * f$ making the change of variable $y_0 \to y_0 + x_0$, $y \to y + x$, writing $K_2 f$ in terms of $m_2 \hat{f}$ and changing the order of integration, we obtain

$$G_{m+1} * f(x_0, x) = \iint \exp[i(z_0 x_0 + z \cdot x)] m_2(z_0, z) \hat{H}(x_0, x, z_0, z) \hat{f}(z_0, z) \mathrm{d}z_0 \mathrm{d}z$$

the Fourier transform being in the z variables. Writing

$$F^{-1}[m_2 \hat{H}] = K_{m+1}(x_0, x, z_0, z)$$

we obtain the formula

$$G_{m+1} * f(x_0, x) = \iint K_{m+1}(x_0, x, z_0, z) f(x_0 - z_0, x - z) \mathrm{d}z_0 \mathrm{d}z.$$

Since m_2 is a bounded function and $\hat{H}(x_0, x, z_0, z)$ is an L^2 function K_{m+1} is an L^2 function in the z variables. As $m_2^\delta \to m_2$ as $\delta \to 0$, we get

$$K_{m+1}(x_0, x, z_0, z) = \lim_{\delta \to 0} F^{-1}[m_2^\delta \hat{H}](z_0, z).$$

But

$$F^{-1}[m_2^\delta \hat{H}](z_0, z) = (F^{-1} m_2^\delta * \hat{H})(z_0, z)$$

$$= \int\limits_{\delta \leqslant |y_0| \leqslant \delta^{-1}} \int |y_0|_{\varepsilon_2}^{-n-1} \psi(y_0^{-1} y) H(x_0, x, z_0 - y_0, z - y) \mathrm{d}y_0 \mathrm{d}y$$

$$= \int\limits_{\delta \leqslant |z_0 - y_0| \leqslant \delta^{-1}} \int |y_0 - z_0|_{\varepsilon_2}^{-n-1} \psi\left(\frac{z - y}{z_0 - y_0}\right) H(x_0, x, y_0, y) \mathrm{d}y_0 \mathrm{d}y.$$

Thus we have obtained

$$K_{m+1}(x_0, x, z_0, z) = \lim_{\delta \to 0} \int\limits_{\delta \leqslant |z_0 - y_0| \leqslant \delta^{-1}} \int |y_0 - z_0|_{\varepsilon_2}^{-n-1} |y_0|_{\varepsilon_1}^{-n-1} y_0^{\alpha_0} y^\alpha$$

$$.\phi(y_0^{-1} y) g(x_0 - t y_0, x - t y) \psi\left(\frac{z - y}{z_0 - y_0}\right) \mathrm{d}y_0 \mathrm{d}y.$$

Since we have an integral in t variable also and since the L^2 norm of H depends on t in a non-integrable way, we are unable to assert that

$$\int_0^1 (1 - t)^m G_{m+1} * f \,\mathrm{d}t$$

defines a bounded linear operator on L^2. Instead, we have

$$|G_{m+1} * f| \leqslant c \iint |y_0|^{(-n+m)} |\phi(y_0^{-1} y)| |K_2 f(x_0 - y_0, x - y)| \mathrm{d}y_0 \mathrm{d}y$$

and the kernel

$$|y_0|^{-n+m} |\phi(y_0^{-1} y)|$$

is homogeneous of degree $-n+m$ and hence $G_{m+1} * f$ defines a bounded linear operator from L^p into L^q whenever

$$1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{m+1}{n+1}$$

with the norm independent of t.

Q.E.D.

Proof of the theorem:

Combining Lemmas 2.3, 2.4 and Proposition 2.1 together, we immediately obtain the Theorem stated in the introduction.

3. Some estimates

To obtain the composition formula, we have made extremely restrictive assumptions on ϕ and ψ viz ϕ, ψ are in $D(R^n)$. We want now to relax these conditions by getting appropriate estimates.

As far as the term A is concerned, we immediately obtain the following theorem which is proved in [3].

THEOREM 3.1.
Let B denote the Banach space of all functions f for which the norm

$$\|f\|_B^2 = \int_{R^n} |f(x)|^2 (1+|x|^2)^{\frac{n+1}{2}} \, dx < \infty,$$

and let B_0 denote the subspace of B containing those functions satisfying $\int f(x)\,dx = 0$. Then

$$\| A \|_B \leqslant c \| \phi \|_B \| \psi \|_B \text{ for all } \phi, \psi \in D(R^n)$$

satisfying $\int \phi = 0$, $\int \psi = 0$, so the mapping $(\phi, \psi) \to A$ extends to a bounded bilinear mapping from $B_0 \times B_0$ to B_0.

We are now going to show that similar estimates for the terms A_k can be obtained. We are not venturing to get optimal estimates. We just imitate the proof of the above theorem and we would not give a complete proof. We will just give a sketch of the proof.

To start with, we first prove the following Lemma concerning the Fourier transform of A_k.

LEMMA 3.1.
Under the hypotheses of the main theorem, we have

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int_{\delta \leqslant |1-s| \leqslant \delta^{-1}} |1-s|_{\varepsilon_2}^{-1} |s|_{\varepsilon_1}^{-1} \, s^k (D^\alpha \hat{\phi})(s\xi) \hat{\psi}((1-s)\xi) \, ds$$

Proof. Since we have

$$A_k(z) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)\psi(z+(\lambda-1)(z-y))\,d\lambda\,dy$$

and the limit exists in the L^2 sense we can interchange the limit and the Fourier transform. Thus

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \exp(-i\xi \cdot z) \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)$$
$$\psi(z+(\lambda-1)(z-y))\,d\lambda\,dy.$$

The triple integral is absolutely convergent and so we may put $z' = z + (\lambda - 1)(z - y)$ and then integrate with respect to z' and y to obtain

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^0_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \hat{\psi}\left(\frac{\xi}{\lambda}\right)(D^\alpha \hat{\phi})\left(\frac{\lambda-1}{\lambda}\xi\right)d\lambda.$$

Now the change of variable $(\lambda - 1)/\lambda = s$ gives

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int_{\delta \leqslant |1-s| \leqslant \delta^{-1}} |1-s|^{-1}_{\varepsilon_2} |s|^{-1}_{\varepsilon_1} s^k (D^\alpha \hat{\phi})(s\xi)$$
$$\hat{\psi}((1-s)\xi)\,ds.$$

Q.E.D.

We will now prove the following theorem.

THEOREM 3.2.
For $1 \leqslant k \leqslant m$ let B_k denote the space of all functions f for which the norm

$$\|f\|^2_{B_k} = \int_{R^n} |f(x)|^2 (1+|x|^2)^{\frac{n+1}{2}-k} < \infty.$$

Then $\|A_k\|_{B_k} \leqslant c\|\phi\|_B \|\psi\|_B$ for all $\phi, \psi \in D(R^n)$ satisfying $\int \phi = 0$ and so the mapping $(\phi, \psi) \to A_k$ extends to a bounded bilinear mapping from $B_0 \times B_0$ into B_k.

Proof. As we have remarked earlier, the proof is exactly same as the proof of Theorem 2.6 of [3]. We remark that $f \in B_k$ is equivalent to

$$\hat{f} \in H^{\frac{n+1}{2}-k}$$

and the condition $\int f = 0$ is equivalent to $\hat{f}(0) = 0$. Thus we have to obtain an estimate of the form

$$\|\hat{A}_k\|_{\frac{n+1}{2}-k} \leqslant c\|\hat{\phi}\|_{\frac{n+1}{2}} \|\hat{\psi}\|_{\frac{n+1}{2}}.$$

Again we have to consider two cases.

Case 1: $\dfrac{n+1}{2}$ is an integer.

In this case the $H^{\frac{n+1}{2}-k}$ norm is equivalent to

$$\left(\sum_{|\alpha|\leqslant\frac{n+1}{2}-k} \|D^\alpha f\|_2^2\right)^{1/2}$$

and we have to bound each of $\|D^\alpha \hat{A}_k\|_2$ by the product of $H^{\frac{n+1}{2}}$ norms of $\hat{\phi}$ and $\hat{\psi}$. Our hypotheses allow us to differentiate $\hat{A}_k$ under the integral sign (and even we can put $\delta = 0$ since $\hat{\phi}(0) = 0 = \hat{\psi}(0)$ the integral being absolutely convergent). Thus $D^\beta \hat{A}_k$ is a sum of terms of the form

$$\int_{-\infty}^{\infty} |1-s|_{\varepsilon_2+|\gamma|}^{-1+|\gamma|} |s|_{\varepsilon_1+|\nu|}^{-1+|\nu|} s^k (D^{\alpha+\nu}\hat{\phi})(s\xi)(D^\gamma\hat{\psi})((1-s)\xi)\,\mathrm{d}s$$

where $\nu+\gamma = \beta$.

Noting that $D^\gamma\hat{\psi} \in H^{\frac{n+1}{2}-|\gamma|}$ and $D^{\alpha+\nu}\hat{\phi} \in H^{\frac{n+1}{2}-k-|\nu|}$ we need to show

$$\left\| \int_{-\infty}^{\infty} |s|^{b-1}|1-s|^{a-1} f((1-s)\xi) g(s\xi)\,\mathrm{d}s \right\|_2$$

$$\leqslant c\|f\|_{\frac{n+1}{2}-a}\|g\|_{\frac{n+1}{2}-b}$$

provided $a+b \leqslant \dfrac{n+1}{2}$ and $f(0) = 0$ if $a = 0$. In fact it is sufficient to prove this for $a+b = \dfrac{n+1}{2}$ and $a = 0$, $b = k$ and the proof given in [3] goes without any change.

Case 2. $\dfrac{n+1}{2}$ is not an integer.

In this case the $H^{\frac{1}{2}(n+1)-k}$ norm is equivalent to

$$\left(\|f\|_2^2 + \sum_{|\alpha|=\frac{n}{2}-k} \iint |D^\alpha f(x) - D^\alpha f(y)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+1}}\right)^{1/2}.$$

In this case we need to obtain estimates of the form

$$\iint \left| \int_{-\infty}^{\infty} |s|^{b-1}|1-s|^{a-1}[f((1-s)x)g(sx) - f((1-s)y)g(sy)]\,\mathrm{d}s \right|^2 \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+1}}$$

$$\leqslant c\|f\|^2_{\frac{n+1}{2}-a}\|g\|_{\frac{n+1}{2}-b}$$

again with $a+b = n/2$ and $f(0) = 0$ if $a = 0$. The proof of this goes along similar lines with that of Theorem 2.6 of [3].

Q.E.D.

4. Remarks and comments

(1) Having obtained the above estimates, now we see that A with ϕ, $\psi \in B_0$ defines a singular integral operator $S(A, \varepsilon_1 + \varepsilon_2)$. But what about A_k's? The conditions A_k belongs to B_k is equivalent to the condition that the restriction of the kernel

$$|z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} A_k(z_0^{-1} z)$$

to the unit sphere S^n is square integrable. A close look at the proof of the boundedness of the convolutions with Riesz potentials in [2] reveals that with the condition A_k belongs B_k the kernels

$$|z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} A_k(z_0^{-1} z)$$

indeed defines bounded linear operators between L^p and L^q for appropriate p and q. Again for the convolution with G_{m+1} the above conditions on ϕ and ψ do imply that $G_{m+1} * f$ is bounded from L^p into L^q. Thus our composition formula is valid even when ϕ, $\psi \in B_0$.

(2) The same theorem holds even if we replace the kernels by more general kernels of the form $|z_0|_\varepsilon^{-n-1} \Phi(x_0, x, z_0^{-1} z)$ where Φ satisfies the following conditions.

(i) $|D^{\alpha^*} \Phi(x_0, x, z)| \leqslant c |\phi_{\alpha^*}(z)| \quad \text{for} \quad |\alpha^*| \leqslant m = \left[\frac{n+1}{2}\right]$,

(ii) $|D^{\alpha^*} \Phi(x_0, x, z)| \leqslant c |\phi_{\alpha^*}(z)| (1 + |x_0|^2 + |x|^2)^{-1} \quad \text{for} \quad |\alpha^*| = m + 1$,

with the functions ϕ_{α^*} belonging to $D(R^n)$.

3. The estimates we got are not optimal. Further, we have to go in the Taylor expansion of b upto derivatives of order $m+1$. We do not know whether $1 + \left[\frac{n+1}{2}\right]$ is optimal or not.

References

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