Existence and Nonexistence of Positive Radial Solutions of Neumann Problems with Critical Sobolev Exponents

ADIMURTHI & S. L. YADAVA

Communicated by J. SERRIN

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\alpha \in C^\infty(\Omega)$. For $\lambda > 0$, $p > 1$, $n \geq 3$ we consider the following problem

$$-\Delta u = u^p + \lambda \alpha(x) u \quad \text{in} \quad \Omega,$$

$$u > 0 \quad \text{in} \quad \Omega,$$

$$\frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \partial \Omega.$$

When $p < \frac{n + 2}{n - 2}$ and $\alpha(x) = -1$, this problem has been discussed extensively in the works of Ni [12], Lin & Ni [10] and Lin, Ni & Takagi [11]. They have proved that there exist positive constants $\lambda_0$ and $\lambda_1$, with $\lambda_0 \leq \lambda_1$, such that (1.1) admits a non-constant solution for $\lambda \geq \lambda_1$ and does not admit any non-constant solution for $\lambda < \lambda_0$. In view of their results, it was conjectured by Lin & Ni [10] that a similar result holds even for $p \geq \frac{n + 2}{n - 2}$.

When $p = \frac{n + 2}{n - 2}$, Brezis [7] posed the question of finding conditions on $\alpha$ and $\Omega$ for which (1.1) admits a solution. Clearly when $\alpha(x) \geq 0$, (1.1) does not admit any solution. Therefore we have to consider two cases: (i) $\alpha(x)$ changes sign, (ii) $\alpha(x) \leq 0$.

In case (i) some partial results have been obtained in [3] by using the variational methods of Brezis & Nirenberg [8]. To describe the results of [3], we further assume that $\int \alpha(x) \, dx < 0$, that there exists an $x_0 \in \partial \Omega$ such that $\alpha(x_0) > 0$, and that $\partial \Omega$ is flat at $x_0$ of order at least four. Under these assumptions, it was shown that for $n \geq 4$ there exists a $\lambda^* > 0$ such that (1.1) admits a solution if and only if $\lambda \in (0, \lambda^*)$. 
In case (ii) the standard variational arguments do not seem to work. On the other hand, in this situation it is easy to construct an example (see Remark 2 at the end of Section 4) such that for any $\Omega$ we can find a negative function $\sigma(x)$ for which (1.1) admits a solution. In view of this and the results of Lin, Ni & Takagi [11], we shall consider the very restricted case of problem (1.1) when $\lambda \sigma(x) \equiv -1$, $\Omega$ is a ball and the solution is radial.

Let $B(R)$ denote the ball of radius $R$ with center at the origin and let $\mu_1(R)$ be the first non-zero eigenvalue of the radial problem

$$-\Delta \varphi = \mu \varphi \quad \text{in} \quad B(R),$$
$$\frac{\partial \varphi}{\partial r} = 0 \quad \text{on} \quad \partial B(R).$$

We consider the problem

$$-\Delta u = u^{(n+2)/(n-2)} - u \quad \text{in} \quad B(R),$$
$$u > 0, \quad u \text{ is radial in} \quad B(R),$$
$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad \partial B(R).$$

and prove the following

**Theorem.** Let $p = (n + 2)/(n - 2)$. The following conclusions hold:

(a) If $n \geq 3$ and $p - 1 > \mu_1(R)$, then (1.3) admits a solution which is radially increasing.

(b) If $n \in \{4, 5, 6\}$ and $p - 1 < \mu_1(R)$, then (1.3) admits a solution which is radially decreasing.

(c) If $n = 3$, then there exists an $R^* > 0$ such that for $0 < R < R^*$, (1.3) does not admit any nonconstant solution.

Here we remark that part (a) of the theorem has been proved by Ni [12] and Lin & Ni [10], and that part (b) gives a counter-example to a part of the conjecture of Lin & Ni [10].

Since we are looking for radial solutions, (1.3) reduces to studying the first turning point $R_1(\gamma)$ of $v(r, \gamma)$, where $v$ satisfies

$$-v'' - \frac{n - 1}{r} v' = v^{\frac{n+2}{n-2}} - v,$$
$$v'(0) = 0, \quad v(0) = \gamma > 0$$

and $R_1(\gamma)$ is defined by

$$R_1(\gamma) = \sup \{r; v'(s, \gamma) \neq 0 \forall s \in (0, r)\}.$$
Let $n = 6$, $\gamma > 1$, $\eta = v(R_1(\gamma), \gamma)$ and $w = v - \eta$. Then $w$ satisfies

$$-\Delta w = w^2 + (2\eta - 1) w + \eta(\eta - 1) \quad \text{in } B(R_1(\gamma)),
$$

$$w > 0 \quad \text{in } B(R_1(\gamma)),
$$

$$w = \frac{\partial w}{\partial v} = 0, \quad \text{on } \partial B(R_1(\gamma)).$$

Hence by Pohožaev's identity we have

$$2(2\eta - 1) \int_{B(R_1(\gamma))} w^2 \, dx + 8\eta(\eta - 1) \int_{B(R_1(\gamma))} w \, dx = 0.$$ 

This implies that $\eta > 1/2$ and hence $v(r, \gamma) > 1/2$ for all $r \in (0, R_1(\gamma))$. Now the asymptotic analysis of Atkinson & Peletier [5] suggests that we can find positive constants $\delta, C_1, C_2, C_3$ and $\gamma_0$ such that, for $\gamma > \gamma_0$ and $R(\gamma) = C_1\gamma^{-1/6}$,

$$R(\gamma) < R_1(\gamma),$$

$$1 - v(R(\gamma), \gamma) \geq \delta,$$

$$C_1\gamma^{1/6} \leq |v'(R(\gamma), \gamma)| \leq C_2/\gamma^{1/6}. \quad (1.8)$$

Integrating (1.4) from $R(\gamma)$ to $R_1(\gamma)$ and using (1.6)–(1.8), we obtain for $C = C_1C_2$ that

$$C/\gamma \geq -R(\gamma)^5 v'(R(\gamma), \gamma) = \int_{R_1(\gamma)}^{R(\gamma)} r^2 v(1 - v) \, dr \geq \delta/12(R_1(\gamma)^6 - C_1/\gamma).$$

Hence

$$R_1(\gamma)^6 \leq \left(\frac{12C}{\delta} + C_1\right)/\gamma \to 0 \quad \text{as } \gamma \to \infty. \quad (1.9)$$

When $n \leq 5$ it may not be true that $v(R_1(\gamma), \gamma)$ is bounded away from zero as $\gamma \to \infty$, whereas estimates similar to (1.6)–(1.8) still hold. Therefore in this case we have to adopt a different procedure to study $R_1(\gamma)$ as $\gamma \to \infty$.

The paper is divided into two parts. In the first part (Section 3), we study the behavior of $R_1(\gamma)$ as $\gamma \to 0, 1$. In the second part (Section 4), following the techniques developed in Atkinson & Peletier [5], we obtain estimates similar to (1.6)–(1.8). Using these (see Section 2) we obtain the proof of the theorem.

In a forthcoming paper we shall study problem (1.3) when $-\Delta$ is replaced by the $p$-Laplacian for $p \leq n$.

While revising this paper, we learned of a recent result of Budd, Knaap & Peletier [9], which discusses the question of existence and non-existence of solutions of (1.3) when $u^{(n+2)/(n-2)} - u$ is replaced by $u^{(n+2)/(n-2)} - u^q$ for $1 < q < 4/(n - 2)$. This problem, for $q = 4/(n - 2)$, has also been treated by Adimurthi, Knaap & Yadava [4].

Recently, Adimurthi & Manconi [11] have tackled this problem in an arbitrary domain using variational techniques. We learned from Prof. J. Serrin that X. J. Wang [13] has also found related results.
2. Proof of the Theorem

In order to prove the theorem, we make use of the standard substitutions,

\[ t = \left( \frac{n-2}{r} \right)^{n-2}, \quad k = \frac{2(n-1)}{n-2}, \quad p = \frac{n+2}{n-2} = 2k - 3, \quad y(t, \gamma) = v(r, \gamma), \]

introduced in [5]. Then from (1.4), \( y \) satisfies the Emden-Fowler equation

\[ -y'' = t^{-k}(y^p - y), \quad y(\infty) = \gamma > 0, \quad y'(\infty) = 0. \]  

(2.1)

Let \( S_1(\gamma) \) be the first turning point of \( y(t, \gamma) \), defined by

\[ S_1(\gamma) = \inf \{ t; y'(s, \gamma) \neq 0 \ \forall \ s \in (t, \infty) \}. \]  

(2.2)

Let \( \varphi \) be the solution of

\[ -\varphi'' = t^{-k} \varphi \quad \text{in} \ (0, \infty), \]

\[ \varphi(\infty) = 1, \quad \varphi'(\infty) = 0 \]  

(2.3)

and let \( \tau_0 \) and \( \tau_1 \) respectively be the first zero and first turning point of \( \varphi \), i.e.,

\[ \tau_0 = \inf \{ t; \varphi(s) > 0 \ \text{for} \ s > t \}, \]

\[ \tau_1 = \inf \{ t; \varphi'(s) > 0 \ \text{for} \ s > t \}. \]  

(2.4)

Then we have

Lemma A. Let \( \gamma \neq 0, 1 \). Then

(i) \( S_1(\gamma) \) exists and \( y(S_1(\gamma), \gamma) > 0 \).

(ii) If \( \gamma \in (0, 1) \), then \( y \) is decreasing, with

\[ \lim_{\gamma \to 0} S_1(\gamma) = 0, \]  

\[ \lim_{\gamma \to 1} S_1(\gamma) = (p - 1)^{1/(k-2)} \tau_1. \]  

(2.5)

(2.6)

(iii) If \( \gamma > 1 \), then \( y \) is increasing, with

\[ \lim_{\gamma \to 1} S_1(\gamma) = (p - 1)^{1/(k-2)} \tau_1. \]  

(2.7)

This result is contained in the works of Ni [12] and Lin & Ni [10]. For the sake of completeness, we present the proof in Section 3.

Lemma B. Let \( \gamma \in (1, \infty) \). Then

(i) For \( t \geq S_1(\gamma) \),

\[ y(t, \gamma) \geq Z_1(t, \gamma), \]  

(2.8)
Neumann Problems with Critical Sobolev Exponents

where

\[ Z_1(t, \gamma) = \gamma t \left\{ \frac{t^{k-2} + (\gamma^{p-1} - 1)/(k-1)}{(k-1)(k-2)} \right\}^{1/(k-2)}. \]

(ii) If \( 3 \leq n \leq 6 \), there exist positive constants \( \delta, C_1, C_2, C_3, C_4 \) and \( \gamma_0 \) such that, for all \( \gamma \geq \gamma_0 \) and \( S(\gamma) = C_1\gamma^{1/(k-1)} \),

\[ S_1(\gamma) < S(\gamma), \]

\[ 1 - y(S(\gamma), \gamma) \geq \delta, \]

\[ C_3/\gamma \leq y'(S(\gamma), \gamma) \leq C_2/\gamma, \]

\[ \lim_{\gamma \to \infty} S_1(\gamma) \geq C_4. \]

Assuming the validity of Lemmas A and B, we first complete the proof of the theorem. Since Lemma A gives the behavior of \( S_1(\gamma) \) as \( \gamma \to 0, 1 \), to prove the theorem we must study its behavior at \( \infty \). For this we need three further lemmas.

**Lemma 2.1.** Let \( Z_1 \) be as defined in (2.8). Then

\[ -Z_1'' = \left( \frac{\gamma^{p-\gamma}}{\gamma^p} \right) t^{-k} Z_1^p \text{ in } (0, \infty), \]

\[ \lim_{t \to \infty} Z_1 = \gamma, \]

\[ \gamma - Z_1(t, \gamma) + tZ_1'(t, \gamma) = \left( \frac{\gamma^{p-\gamma}}{\gamma^p} \right) \int Z_1^p s^{-k+1} ds, \]

\[ tZ_1'(t, \gamma) - Z_1(t, \gamma) = \left( \frac{-t^{k-1}}{t^{k-2} + (\gamma^{p-1} - 1)/(k-1)(k-2)} \right). \]

This lemma follows easily from the definition of \( Z_1 \).

**Lemma 2.2.** If \( n = 3 \) \( (k = 4) \), then

\[ \lim_{\gamma \to \infty} S_1(\gamma) < \infty. \]

**Proof.** Let \( \beta(t) = t \cosh \frac{1}{t} \). It is easy to verify that \( \beta \) satisfies

\[ \beta'' = t^{-4} \beta \text{ in } (0, \infty), \]

\[ \lim_{t \to 0} \beta(t) = \infty, \quad \beta(t) = t + C(t), \]

where \( C(t) \geq 0 \). Let \( T_0 \) be such that \( \beta'(T_0) = 0 \). Then the lemma follows if we can show that

\[ \lim_{\gamma \to \infty} S_1(\gamma) \leq T_0. \]
Let $W = (\gamma \beta' - \beta')$. Then $W(\infty) = \gamma$ and $W'(t) = t^{-4} \gamma^5 \beta$. Integrating $W'$ from $S_1(\gamma)$ to $\infty$ and using (2.8), (2.18), (2.15) and (2.16), we obtain

$$y(S_1(\gamma), \gamma) \beta'(S_1(\gamma)) = \gamma - \int_{S_1(\gamma)}^{\infty} t^{-4} \gamma^5 \beta \, dt$$

$$\leq \gamma - \int_{S_1(\gamma)}^{\infty} t^{-3} Z_1 \, dt \leq \gamma - \left( \frac{\gamma^5}{(\gamma^5 - \gamma)} \right) \frac{\gamma S_1(\gamma)^3}{\{S_1(\gamma)^2 + \frac{1}{2} (\gamma^4 - 1)^{3/2} \}}. \quad (2.20)$$

From (2.9) it follows that $S_1(\gamma) = 0(\gamma^{1/3})$ as $\gamma \to \infty$; hence we have

$$\left( \frac{\gamma^5}{(\gamma^5 - \gamma)} \right) \frac{\gamma S_1(\gamma)^3}{\{S_1(\gamma)^2 + \frac{1}{2} (\gamma^4 - 1)^{3/2} \}} = o \left( \frac{1}{\gamma^4} \right)$$

as $\gamma \to \infty$. This together with (2.20) and (i) of Lemma A implies that $\beta'(S_1(\gamma)) < 0$ for $\gamma$ large, and so $S_1(\gamma) \leq T_0$. This proves (2.19) and hence the lemma.

**Lemma 2.3.** If $n \in \{4, 5, 6\}$, then

$$\lim_{\gamma \to \infty} S_1(\gamma) = \infty. \quad (2.21)$$

**Proof.** Suppose (2.21) is not true. Then for a sequence of values $\gamma \to \infty$, we have

$$\lim_{\gamma \to \infty} S_1(\gamma) < \infty. \quad (2.22)$$

For the sequel we use $C$, $C_1$, $C_2$, etc., to denote positive constants independent of $\gamma$. Now from (2.8), (2.9) we have for $t \in (S_1(\gamma), S(\gamma))$,

$$y(t, \gamma) \geq Z_1(t, \gamma) \geq C \frac{t}{\gamma}. \quad (2.23)$$

Let

$$H(t) = \frac{1}{2} t y'^2 - \frac{1}{2} y y' + t^{1-k} \left( \frac{y^{p+1}}{p+1} - \frac{y^2}{2} \right).$$

Then $H(\infty) = 0$ and $H'(t) = \frac{p-1}{2} t^{-k} y^2$. Hence $H(t) \leq 0$. Now integrating $H'(t)$ from $S_1(\gamma)$ to $S(\gamma)$ and using (2.23), we obtain

$$-H(S_1(\gamma)) \geq \frac{p-1}{2} \int_{S_1(\gamma)}^{S(\gamma)} y^2 t^{-k} \, dt$$

$$\geq \frac{C}{\gamma^2} \int_{S_1(\gamma)}^{S(\gamma)} t^{-k+2} \, dt = C \frac{\varphi(\gamma)}{\gamma^2}, \quad (2.24)$$
where
\[ \varphi(\gamma) = \begin{cases} 
\log \frac{S(\gamma)}{S_1(\gamma)} & \text{if } k = 3, \\
(S(\gamma)^{3-k} - S_1(\gamma)^{3-k}) & \text{if } k < 3.
\end{cases} \]

From (2.10), (2.11) and (2.22) we have
\[ C_2/\gamma \geq y'(S(\gamma), \gamma) = \int \frac{S(\gamma)}{S_1(\gamma)} y(1 - y^{p-1}) \frac{t^k}{\gamma} dt \]
\[ \geq \frac{\delta}{k - 1} y(S_1(\gamma), \gamma) \left( \frac{1}{S_1(\gamma)^{k-1}} - \frac{1}{S(\gamma)^{k-1}} \right) \]
\[ \geq C\gamma(S_1(\gamma), \gamma). \]

Hence
\[ -H(S_1(\gamma)) = S_1(\gamma)^{1-k} y(S_1(\gamma), \gamma)^2 \left( \frac{1}{2} - \frac{y(S_1(\gamma), \gamma)^{p-1}}{p + 1} \right) \]
\[ \leq C_3 \frac{S_1(\gamma)^{1-k}}{\gamma^2}. \]

This combined with (2.24) gives
\[ S_1(\gamma)^{k-1} \leq C_4/\varphi(\gamma). \] (2.25)

Since \( S_1(\gamma) \) is bounded by assumption, it follows that \( \varphi(\gamma) \rightarrow \infty \) as \( \gamma \rightarrow \infty \). Therefore from (2.25), \( S_1(\gamma) \rightarrow 0 \) as \( \gamma \rightarrow \infty \), contradicting (2.12). This proves the lemma.

**Proof of the Theorem.** For \( \gamma \neq 0, 1 \), let \( R_1(\gamma) \) and \( u(r, \gamma) \) be defined by
\[ t = \left( \frac{n - 2}{r} \right)^{n-2}, \quad S_1(\gamma) = \left( \frac{n - 2}{R_1(\gamma)} \right)^{n-2}, \]
\[ u(r, \gamma) = y(t, \gamma). \]

Then \( u \) satisfies
\[ -\Delta u = u^{(n+2)/(n-2)} - u \quad \text{in } B(R_1(\gamma)), \]
\[ u > 0 \quad \text{in } B(R_1(\gamma)), \]
\[ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B(R_1(\gamma)). \]

Define \( R_1 = [(n - 2)/\gamma_1]^n \). It is easy to see that \( \mu_1(R_1(\gamma)) = (R_1/R_1(\gamma))^2 \).
Since \( \gamma \rightarrow R_1(\gamma) \) is continuous, (a) follows from (2.5) and (2.6), (b) follows from (2.7) and (2.21), and (c) follows from Lemma 2.2. This proves the theorem.
3. Proof of Lemma A

Let $k > 2$ and let $f: \mathbb{R} \to \mathbb{R}$ be a $C^1$-function. For $\gamma > 0$, let $Y(t, \gamma)$ be the solution of

$$-Y'' = t^{-k} f(Y),$$
$$Y(\infty) = \gamma, \quad Y'(\infty) = 0.$$  \hfill (3.1)

Let

$$F(s) = \int_0^s f(r) \, dr,$$
$$H(t) = \frac{1}{2} t Y'' - \frac{1}{2} Y Y' + t^{1-k} F(Y),$$  \hfill (3.2)
$$H_1(t) = \frac{1}{2} t Y'' - \frac{1}{2} Y Y' + \frac{t^{1-k}}{2(k-1)} Y f(Y).$$  \hfill (3.3)

It is then easy to see that $Y$ satisfies

$$\lim_{t \to \infty} H(t) = \lim_{t \to \infty} H_1(t) = 0,$$  \hfill (3.4)
$$\lim_{t \to \infty} Y'(t, \gamma) t^{k-1} = \frac{f(\gamma)}{(k-1)},$$  \hfill (3.5)
$$H'(t) = \frac{1}{2} t^{-k} [Y f(Y) - 2(k-1) F(Y)],$$  \hfill (3.6)
$$H_1'(t) = \frac{Y't^{1-k}}{2(k-1)} [Y f'(Y) - (2k-3) f(Y)],$$  \hfill (3.7)
$$(Y' Y^{1-k} t^{k-1})' = -2(k-1) t^{k-2} Y^{-k} H_1(t).$$  \hfill (3.8)

From now on, we assume that $f(0) = f(1) = 0$ and $f'(1) > 0$. Furthermore, we assume that

$$(s - 1) f(s) > 0 \quad \text{for } s > 0 \text{ and } s \neq 1.$$  \hfill (3.9)

For $\gamma > 0$, $\gamma = 1$, put

$$S_0(\gamma, f) = \inf \{t; Y(s, \gamma) = 1, Y'(s, \gamma) = 0 \quad \forall s > t\},$$  \hfill (3.10)
$$S_1(\gamma, f) = \inf \{t; Y(s, \gamma) > 0, Y'(s, \gamma) = 0 \quad \forall s > t\}.$$  \hfill (3.11)

We then have the following

**Lemma 3.1.** For $s \geq 0$, assume that $f$ satisfies

$$sf''(s) - (2k-3) f(s) \geq 0.$$  \hfill (3.12)

Then

$$Y(t, \gamma) \geq \eta_1(t, \gamma)$$  \hfill (3.13)

for $\gamma > 1$ and $t \geq S_1(\gamma, f)$, where

$$\eta_1(t, \gamma) = \frac{\gamma t}{\left( t^{k-2} + \frac{f(\gamma)}{(k-1) \gamma} \right)^{1/(k-2)}}.$$
Proof. Let \( t > S_0(\gamma, f) \). Since \( \gamma > 1 \), it follows from (3.9) that \( Y'(t, \gamma) > 0 \).

Therefore from (3.12) and (3.7), \( H_1'(t) \leq 0 \). Hence \( H_1 \) is increasing and from (3.4), \( H_1(t) \leq 0 \). From (3.8), we have

\[
(Y'Y^{1-k} t^{k-1})' \geq 0.
\]

Integrating this twice from \( t \) to \( \infty \) and using (3.5), we obtain

\[
\frac{1}{Y^{k-2}} - \frac{1}{Y^{k-2}} \leq \frac{\gamma^{1-k} f(\gamma)}{(k-1) t^{k-2}},
\]

which gives

\[
Y(t, \gamma) \geq \frac{\gamma t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1) (\gamma-1)} \right\}^{1/(k-2)}}.
\]

This proves the lemma.

Lemma 3.2. For \( s \geq 0 \), assume that \( f \) satisfies

\[
sf'(s + 1) - (2k - 3)f(s + 1) \leq 0.
\]

Then

\[
Y(t, \gamma) \leq 1 + \eta_2(t, \gamma),
\]

for \( \gamma > 1 \) and \( t \geq S_0(\gamma, f) \), where

\[
\eta_2(t, \gamma) = \frac{(\gamma - 1) t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1) (\gamma-1)} \right\}^{1/(k-2)}}.
\]

Proof. Let \( V = Y - 1 \), \( f_1(s) = f(s + 1) \). Then \( V \) satisfies

\[
-V'' = t^{-k} f_1(V),
\]

\[
V(\infty) = \gamma - 1, \quad V'(\infty) = 0.
\]

Since \( \gamma > 1 \), from (3.9), we get \( Y(t, \gamma) \geq 1 \) and \( Y'(t, \gamma) > 0 \) for \( t \geq S_0(\gamma, f) \). Hence \( V(t) \geq 0 \) and \( V'(t) > 0 \). Therefore for \( t \geq S_0(\gamma, f) \), we have from (3.16), (3.7) and (3.14) that \( H_1'(t) \leq 0 \). So we deduce that \( H_1(t) \geq 0 \) from (3.4) and that

\[
(V'V^{1-k} t^{k-1})' \leq 0
\]

from (3.8). Integrating twice and using (3.5) we obtain for all \( t \geq S_0(\gamma, f) \) that

\[
V(t, \gamma) \leq \frac{(\gamma - 1) t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1) (\gamma-1)} \right\}^{1/(k-2)}} = \eta_2(t, \gamma),
\]

that is, for \( t \geq S_0(\gamma, f) \),

\[
Y(t, \gamma) \leq 1 + \eta_2(t, \gamma).
\]

(3.17)
Since $Y(t, \gamma) \leq 1$ for $t \in [S_1(\gamma, f), S_0(\gamma, f)]$, inequality (3.17) continues to hold for $t \geq S_1(\gamma, f)$. This proves the lemma.

As an immediate consequence of these lemmas we have the following

**Lemma 3.3.** Let $\gamma > 1$ and let $y(t, \gamma)$ satisfy (2.1). For $t \geq S_1(\gamma)$,

(i) $y(t, \gamma) \geq Z_1(t, \gamma)$ if $n \geq 3$. 
(ii) $y(t, \gamma) \leq 1 + Z_2(t, \gamma)$ if $3 \leq n \leq 6$, 

where

$$Z_1(t, \gamma) = \frac{\gamma t}{\left\{t^{k-2} + \frac{\gamma^{2(k-2)} - 1}{(k-1)}\right\}^{1/(k-2)}},$$

$$Z_2(t, \gamma) = \frac{(\gamma - 1) t}{\left\{t^{k-2} + \frac{\gamma^{2(k-2)} - 1}{(k-1)(\gamma - 1)}\right\}^{1/(k-2)}}.$$ 

**Proof.** Let $p = 2k - 3$ and $f(s) = s^p - s$ for $s \geq 0$. Extend $f$ as a $C^1$-function to $\mathbb{R}$. Then clearly $f$ satisfies (3.9), and for $s \geq 0$,

$$sf'(s) - (2k - 3)f(s) = 2(k - 2) s \geq 0.$$ 

Hence, (3.18) follows from Lemma 3.1.

For $s \geq 1$, $n \leq 6$, let $h(s) = -ps^{p-1} + (p - 1) s + 1$. Since $n \leq 6$ we have $p \geq 2$. Therefore $h''(s) = -p(p-1)(p-2)s^{p-3} \leq 0$ and hence $h$ is concave. Since $h(1) = 0$ and $h'(1) = -(p - 1)^2$, we have $h(s) \leq -(p - 2)^2(s - 1) \leq 0$.

For $s \geq 0$, we have

$$sf'(s + 1) - (2k - 3)f(s + 1) = -p(s + 1)^{p-1} + (p - 1)(s + 1) + 1 = h(s + 1) \leq 0.$$ 

Hence (3.19) follows from Lemma 3.2. This proves the lemma.

For $i = 1, 2$, and $\gamma_i > 0$ let $q_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ be continuous functions. Let $\varphi_i$ satisfy

$$-\varphi_i'' = t^{-k} q_i(t) \varphi_i,$$

$$\varphi_i(\infty) = \gamma_i, \quad \varphi_i'(\infty) = 0.$$ 

Denote by $T_{0,i}$ and $T_{1,i}$ respectively the first zero and first turning point of $\varphi_i$ (see (2.4)). Then

**Lemma 3.4.** (i) Assume that $T_{0,1}$ exists and also that $q_2(t) \geq q_1(t)$ for $t \geq T_{0,2}$. Then $T_{0,2} > 0$ and $T_{0,2} \geq T_{0,1}$.

(ii) Assume that $T_{0,1}$ and $T_{1,1}$ exist and also that $q_2(t) \geq q_1(t)$ for $t \geq T_{1,2}$. Then $T_{1,2} > 0$ and $T_{1,2} \geq T_{1,1}$.
Proof. Let \( W = \varphi_1 \varphi_2 - \varphi_1' \varphi_2 \). Then \( W(\infty) = 0 \) and
\[
W'(t) = t^{-k}(\varphi_2 - \varphi_1) \varphi_1 \varphi_2.
\]
(3.21)
Suppose that (i) is not true. Then \( T_{0,2} < T_{0,1} \) and hence from (3.21), \( W'(t) \geq 0 \) for all \( t \geq T_{0,1} \). Therefore \( W(T_{0,1}) \leq 0 \). But \( W(T_{0,1}) = \varphi_1'(T_{0,1}) \varphi_2(T_{0,1}) > 0 \), which is a contradiction. This proves (i).

Suppose that (ii) is not true. Then \( T_{1,2} < T_{1,1} \). From (i) it follows that \( T_{0,1} < T_{0,2} \).

Using (3.21), we obtain \( W'(t) \geq 0 \) for \( t \in [T_{1,1}, T_{0,1}] \). Therefore we have
\[
0 < -\varphi_1(T_{1,1}) \varphi_2'(T_{1,1}) = W(T_{1,1}) \leq W(T_{0,1})
\]
\[
= \varphi_1'(T_{0,1}) \varphi_2(T_{0,1}) < 0,
\]
which is a contradiction. This proves (ii) and hence the lemma.

Let \( \varphi, \tau_0, \tau_1 \) be as in (2.3) and (2.4). For \( a > 0 \), denote \( \varphi(t, a) = \varphi(at) \) and let \( \tau_{0,a} \) and \( \tau_{1,a} \) be the first zero and first turning point of \( \varphi(\cdot, a) \). Then we have
\[
\frac{\tau_{0,a}}{a} = \frac{\tau_0}{a}, \quad \frac{\tau_{1,a}}{a} = \frac{\tau_1}{a},
\]
\[
-\varphi''(\cdot, a) = a^{2-k} t^{-k} \varphi(\cdot, a),
\]
\[
\varphi(\infty, a) = 1, \quad \varphi'(\infty, a) = 0.
\]
(3.22)
Let \( y(t, \gamma) \) and \( S_1(\gamma) \) be as in (2.1) and (2.2). Define
\[
S_0(\gamma) = \inf \{ t, y(s, \gamma) \neq 1, y'(s, \gamma) \neq 0 \quad \forall s > t \}.
\]
(3.23)
We then have

Lemma 3.5. If \( \gamma \neq 0, 1 \), then \( S_0(\gamma) \) exists and
\[
\lim_{\gamma \to 0} S_0(\gamma) = 0.
\]
Proof. First consider the case \( \gamma > 1 \). Let
\[
\varphi_2(t) = y(t, \gamma) - 1,
\]
\[
\varphi_2(t) = \frac{(\varphi_2 + 1)^p - (\varphi_2 + 1)}{\varphi_2}.
\]
Then \( \varphi_2 \) satisfies
\[
-\varphi_2'' = t^{-k} \varphi_2 \varphi_2,
\]
\[
\varphi_2(\infty) = \gamma - 1, \quad \varphi_2'(\infty) = 0.
\]
From (3.23) it follows that \( S_0(\gamma) \) is the first zero of \( \varphi_2 \) and that \( \varphi_2(t) \geq (p - 1) \) for \( t \geq S_0(\gamma) \). Taking \( \varphi_1 = (p - 1)^{1/(k-2)} \), \( \varphi_1(t) = \varphi(t, (p - 1)^{-1/(k-2)}) \) in (i) of Lemma 3.4, we conclude that \( S_0(\gamma) \) exists and
\[
(p - 1)^{1/(k-2)} \tau_0 \leq S_0(\gamma).
\]
(3.24)
Now consider the case $0 < \gamma < 1$. Let
\[
\tilde{\varphi}_2 = 1 - y(t, \gamma),
\]
\[
q_2(t) = \frac{(1 - \varphi_2) - (1 - \varphi_2)^p}{\varphi_2}.
\]
Then $\varphi_2$ satisfies
\[
-\varphi_2'' = t^{-k} q_2(t) \varphi_2',
\]
\[
\varphi_2(\infty) = 1 - \gamma,
\]
\[
\varphi_2'(\infty) = 0,
\]
with $S_0(\gamma)$ as its first zero. By taking $q_1 = \min\{q_2(t), t \geq S_0(\gamma)\}$, $q_1(t) = \varphi(t, \varphi_1^{1/(k-2)})$ in (i) of Lemma 3.4, we obtain the existence of $S_0(\gamma)$. Since $q_2(t) \leq (p - 1)$ for $t \geq S_0(\gamma)$, again from (i) of Lemma 3.4, we obtain
\[
S_0(\gamma) \leq (p - 1)^{1/(k-2)} \tau_0.
\]
(3.25)

Now suppose that $S_0(\gamma)$ does not tend to zero as $\gamma$ approaches zero. Then by going to a subsequence and using (3.25), we have
\[
\lim_{\gamma \to 0} S_0(\gamma) = S_0 > 0.
\]
(3.26)

Since the boundedness of $y$ implies that $y'$ and $y''$ are uniformly bounded in $(S_0(\gamma), \infty)$, the Arzelà-Ascoli Theorem implies that there exists a subsequence such that $y(t, \gamma) \to y_0(t)$ uniformly on compact sets and that $y_0$ satisfies
\[
-y_0'' = t^{-k}(y_0^p - y_0) \quad \text{in } (S_0, \infty),
\]
\[
y_0(\infty) = y_0'(\infty) = 0.
\]
(3.27)

From the uniqueness of the solution of (3.27), $y_0 \equiv 0$. But $y_0(S_0) = 1$. This contradiction proves the lemma.

**Proof of the Lemma A.** From (2.1), it follows that $y$ is increasing for $\gamma > 1$, and $y$ is decreasing for $\gamma < 1$.

First consider the case $\gamma > 1$. Suppose $S_1(\gamma) = 0$. Then $y(t, \gamma) > 0$ for $t > 0$ by (3.18), and $y(t, \gamma)$ is an increasing function by (2.1). Since $y(S_0(\gamma), \gamma') = 1$, from Lemma 3.5 we can find a $C > 0$ such that for $t \in (0, (S_0(\gamma)/2)$,
\[
1 - y^{p-1}(t, \gamma) \geq C.
\]
(3.28)

From (3.18), we can find a $C_1 > 0$ such that for $t \in (0, S_0(\gamma)/2)$,
\[
y(t, \gamma) \geq C_1 t.
\]
(3.29)

Integrating (2.1) and using (3.28) and (3.29), we have
\[
\infty > y'(S_0(\gamma)/2, \gamma') \geq \int_0^{S_0(\gamma)/2} t^{-k} y(1 - y^{p-1}) dt
\]
\[
\geq CC_1 \int_0^{S_0(\gamma)/2} t^{-k+1} dt = \infty,
\]
which is a contradiction. Hence $S_1(\gamma) > 0$; from (3.18), we have $y(S_1(\gamma), \gamma > 0)$. 


Let \( v = y - 1 \) and \( f_1(s) = (s + 1)^p - (s + 1) \). Then \( v(\infty) = y - 1 \), \( v'(\infty) = 0 \), and \( S_0(\gamma) \) and \( S_1(\gamma) \) respectively are the first zero and first turning points of \( v \). Moreover, \( v \) satisfies
\[
-v'' = t^{-k} \left( \frac{f_1(v)}{v} \right) v. \tag{3.30}
\]
Now integrating (3.30) and using (3.24), we can find a \( C > 0 \) such that, for all \( 1 < \gamma \leq 2 \),
\[
v'(S_0(\gamma)) = \int_{S_0(\gamma)}^\infty t^{-k} \left( \frac{f_1(v)}{v} \right) v \, dt \leq C(\gamma - 1). \tag{3.31}
\]
Since
\[
\sup \left\{ \frac{f_1(v)}{v} ; \; t \geq S_0(\gamma), \; \gamma \in (1, 2] \right\} < \infty,
\]
as a consequence of (i) of Lemma 3.4, \( S_0(\gamma) \) is bounded for \( \gamma \in (0, 2] \). From this and from (3.31), we can find a \( C_1 > 0 \) such that for \( 1 < \gamma \leq 2 \),
\[
|v(S_1(\gamma))| \leq v'(S_0(\gamma)) (S_0(\gamma) - S_1(\gamma)) \leq C(\gamma - 1). \tag{3.32}
\]
This inequality implies that for any \( \varepsilon > 0 \), we can find a \( \delta > 0 \) such that whenever \( \gamma - 1 \leq \delta, \; t \geq S_1(\gamma) \),
\[
(1 - \varepsilon) (p - 1) \leq \frac{f_1(v(t))}{v(t)} \leq (1 + \varepsilon) (p - 1). \tag{3.33}
\]
From (3.22), (3.33) and (ii) of Lemma 3.4, we obtain
\[
[(1 - \varepsilon) (p - 1)]^{\frac{1}{(1-k-2)}} \tau_1 \leq S_1(\gamma) \leq [(1 + \varepsilon) (p - 1)]^{\frac{1}{(1-k-2)}} \tau_1
\]
for \( \gamma \leq 1 + \delta \). This inequality implies that
\[
\lim_{\gamma \to 1} S_1(\gamma) = (p - 1)^{\frac{1}{(1-k-2)}} \tau_1. \tag{3.34}
\]
Now consider the case in which \( 0 < \gamma < 1 \). Suppose \( S_1(\gamma) = 0 \). From Lemma 3.5, \( S_0(\gamma) \) exists and \( y(S_0(\gamma), \gamma) = 1 \). Hence from (2.1),
\[
-y'(t, \gamma) \leq -y'(S_0(\gamma), \gamma) \tag{3.35}
\]
for all \( t \in (0, (S_0(\gamma)) \). Also we can find a \( C > 0 \) such that for \( t \in (0, S_0(\gamma)/2) \),
\[
y^p(t, \gamma) - y(t, \gamma) \geq C. \tag{3.36}
\]
Integrating (2.1) and using (3.35) and (3.36), we have
\[
-y'((S_0(\gamma), \gamma) \geq -y'(t, \gamma) \geq \int_{S_0(\gamma)/2}^{S_0(\gamma)/2} s^{-k}(y^p - y) \, dt
\]
\[
\geq C \left[ \frac{1}{t^{k-1}} - \left( \frac{2}{S_0(\gamma)} \right)^{k-1} \right] \to \infty
\]
as \( t \to 0 \), which is a contradiction. This implies that \( S_1(\gamma) \) exists.
Let \( v = 1 - y \) and \( f_1(s) = (1 - s) - (1 - s)^\gamma \). Then \( v(\infty) = 1 - \gamma \), \( v'(\infty) = 0 \), \( S_0(\gamma) \) and \( S_1(\gamma) \) are the first zero and first turning points of \( v \). Moreover, \( v \) satisfies

\[
-v'' = t^{-k} \left( \frac{f_1(v)}{v} \right) v. \tag{3.37}
\]

Since

\[
\inf \left\{ \frac{f_1(v)}{v} ; t \geq S_0(\gamma), \frac{1}{2} \leq \gamma < 1 \right\} > 0,
\]

by Lemma 3.4(i) and by (3.22) we have

\[
\inf \{ S_0(\gamma) ; \frac{1}{2} \leq \gamma < 1 \} > 0.
\]

Therefore by integrating (3.37), we have for some constant \( C > 0 \),

\[
v'(S_0(\gamma)) = \int_{S_0(\gamma)}^{\infty} t^{-k} \left( \frac{f_1(v)}{v} \right) v \, dt \leq C(1 - \gamma). \tag{3.38}
\]

From (3.25), (3.38) and the mean value theorem, we can find a \( C_1 > 0 \) such that

\[
|v(S_1(\gamma))| \leq |v'(S_0(\gamma))| (S_0(\gamma) - S_1(\gamma)) \leq C_1(1 - \gamma).
\]

This implies that for every \( \epsilon > 0 \), we can find a \( \delta > 0 \) such that

\[
(1 - \epsilon) (p - 1) \leq \frac{f_1(v)}{v} \leq (1 + \epsilon) (p - 1) \tag{3.39}
\]

whenever \( 1 - \gamma \leq \delta \) and \( t \geq S_1(\gamma) \). From (3.22), (3.39), and Lemma 3.4(ii) we obtain

\[
[(1 - \epsilon) (p - 1)]^{1/(k-2)} \tau_1 \leq S_1(\gamma) \leq [(1 + \epsilon) (p - 1)]^{1/(k-2)} \tau_1
\]

for \( 1 - \gamma \leq \delta \). This inequality implies that

\[
\lim_{\gamma \to +1} S_1(\gamma) = (p - 1)^{1/(k-2)} \tau_1. \tag{3.40}
\]

Since \( S_1(\gamma) < S_0(\gamma) \), from Lemma 3.5 we have \( \lim_{\gamma \to 0} S_1(\gamma) = 0 \). Now the lemma follows from (3.34) and (3.40).

### 4. Proof of Lemma B

Let \( n \leq 6 \) and \( \gamma > 1 \). Let \( y(t, \gamma), S_1(\gamma), \) and \( S_0(\gamma) \) be as in (2.1), (2.2), (3.23), respectively. For the sequel we use \( C, C_1, C_2, \) etc., to denote positive constants independent of \( \gamma \), but which may be different in different inequalities. We have the following

**Lemma 4.1.** For \( \gamma \) large,

\[
S_0(\gamma) = O(\gamma), \tag{4.1}
\]

\[
y(t, \gamma) \leq 1 + Ct/\gamma \quad \text{for } t \geq S_0(\gamma), \tag{4.2}
\]

\[
y(\gamma^2, \gamma) \geq C\gamma, \tag{4.3}
\]

\[
C_1/\gamma \leq y'(2S_0(\gamma), \gamma) \leq C_2/\gamma. \tag{4.4}
\]
For $t \in (2S_0(\gamma), \gamma^2)$
\[ 1 + \frac{C_1(t - S_0(\gamma))}{\gamma} \leq y(t, \gamma) \leq 1 + \frac{C_2(t - S_0(\gamma))}{\gamma}. \]  
(4.5)

**Proof.** By Lemma (3.5), $S_0(\gamma)$ exists and from (3.18),
\[ Z_1(S_0(\gamma), \gamma) \leq y(S_0(\gamma), \gamma) = 1. \]  
(4.6)

This implies that
\[ S_0(\gamma)^{k-2} \leq \frac{\gamma^{p-1}}{(k-1)(\gamma^k - 1)}. \]  
(4.7)

Since $p = 2k - 3$, it follows from (4.7) that $S_0(\gamma) = O(\gamma)$ as $\gamma \to \infty$. This proves (4.1).

For large $\gamma$ we have, \( \frac{\gamma^p - \gamma}{(k-1)(\gamma^k - 1)} \leq C\gamma^{2(k-2)} \) and hence from (3.19),
\[ y(t, \gamma) \leq 1 + \frac{C_0 t}{(k-2)\gamma} \leq 1 + \frac{Ct}{\gamma}. \]  
for all $t \geq S_0(\gamma)$. This proves (4.2).

Again from (3.18), we have
\[ y(\gamma^2, \gamma) \geq Z_1(\gamma^2, \gamma) \geq C\gamma \]  
for $\gamma$ large. This proves (4.3).

From the concavity of $y$ in $[S_0(\gamma), 2S_0(\gamma)]$ and from (4.2) we have for large $\gamma$
\[ y'(2S_0(\gamma), \gamma) \geq \frac{y(2S_0(\gamma), \gamma) - 1}{2S_0(\gamma)} \leq \frac{C_2S_0(\gamma)}{\gamma S_0(\gamma)} = \frac{C_2}{\gamma}. \]  
(4.8)

Again, from the concavity of $y$ in $[2S_0(\gamma), \gamma^2]$ and from (4.1)--(4.3), we have for large $\gamma$ that
\[ y'(2S_0(\gamma), \gamma) \leq \frac{y(\gamma^2, \gamma) - y(2S_0(\gamma), \gamma)}{\gamma^2 - 2S_0(\gamma)} \leq \frac{C_1 + O(1)}{\gamma^2 + O(\gamma)} \geq \frac{C_1}{\gamma}. \]  
This together with (4.8) proves (4.4).

Let $t \in [2S_0(\gamma), \gamma^2]$. From (4.2), we then have
\[ y(t, \gamma) \leq 1 + \frac{Ct}{\gamma} = 1 + \frac{C(t - S_0(\gamma))}{\gamma} \leq 1 + \frac{C_2(t - S_0(\gamma))}{\gamma}. \]  
(4.9)

From the concavity of $y$ in $[S_0(\gamma), \gamma^2]$ and from (4.1)--(4.3), it follows that
\[ \frac{y(t, \gamma) - y(S_0(\gamma), \gamma)}{t - S_0(\gamma)} \geq \frac{y(\gamma^2, \gamma) - y(S_0(\gamma), \gamma)}{\gamma^2 - S_0(\gamma)} \geq \frac{C\gamma + O(1)}{\gamma^2 + O(\gamma)} \geq \frac{C_1}{\gamma}. \]
for $\gamma$ large and for $t \in [S_0(\gamma), \gamma^2]$. Hence

$$y(t, \gamma) \geq 1 + \frac{C_1(t - S_0(\gamma))}{\gamma}.$$ 

This together with (4.9) proves (4.5) and hence the lemma.

**Lemma 4.2.** $\lim_{\gamma \to \infty} S_0(\gamma) > 0$.

**Proof.** Integrating (2.1) and using (4.4) and (4.5) we obtain for $\gamma$ large that

$$\frac{C_2}{\gamma} \geq \int_{S_0(\gamma)}^{\gamma} t^{-k}(y^p - y) \, dt \geq (p - 1) \int_{S_0(\gamma)}^{\gamma} t^{-k}(y - 1) \, dt \geq \frac{C}{\gamma} \int_{S_0(\gamma)}^{\gamma} t^{-k}(t - S_0(\gamma)) \, dt \geq \frac{C S_0(\gamma)^{2-k}}{\gamma}.$$ 

which implies that $\lim_{\gamma \to \infty} S_0(\gamma) \geq C_4 > 0$, since $\gamma^2/S_0(\gamma) \to \infty$ as $\gamma \to \infty$ and $k > 2$. This proves the lemma.

**Lemma 4.3.** For $\gamma$ large,

$$C_1 \gamma \leq S_0(\gamma) \leq C_2 \gamma,$$  

and

$$C_1/\gamma \leq y'(S_0(\gamma), \gamma) \leq C_2/\gamma.$$  

**Proof.** Let $v = y - 1$ and $f_1(s) = (s+1)^p - (s+1)$. Then $v$ satisfies

$$-v'' = t^{-k} f_1(v),$$

$$v(\infty) = \gamma - 1, \quad v'(\infty) = 0$$

and $S_0(\gamma)$ is the first zero of $v$. Let $F_1(s)$ be the primitive of $f_1$ and let

$$H(t) = \frac{1}{2} tv^2 - \frac{1}{2} vv' + t^{1-k} F_1(v).$$

Then from (3.6) we have

$$-H'(t) = \frac{t^{-k}}{2} h(v + 1),$$

where

$$h(s) = s^p - \frac{p-1}{2} s^2 - s + \left( \frac{p-1}{2} \right).$$

Since $n \leq 6$, we have $p \geq 2$, and therefore we obtain that $h$ is convex for $s \geq 1$ and satisfies

$$h(s) \geq C(s - 1)^p$$

for $\gamma$ large and for $t \in [S_0(\gamma), \gamma^2]$. Hence

$$y(t, \gamma) \geq 1 + \frac{C_1(t - S_0(\gamma))}{\gamma}.$$ 

This together with (4.9) proves (4.5) and hence the lemma.

**Lemma 4.2.** $\lim_{\gamma \to \infty} S_0(\gamma) > 0$.

**Proof.** Integrating (2.1) and using (4.4) and (4.5) we obtain for $\gamma$ large that

$$\frac{C_2}{\gamma} \geq \int_{S_0(\gamma)}^{\gamma} t^{-k}(y^p - y) \, dt \geq (p - 1) \int_{S_0(\gamma)}^{\gamma} t^{-k}(y - 1) \, dt \geq \frac{C}{\gamma} \int_{S_0(\gamma)}^{\gamma} t^{-k}(t - S_0(\gamma)) \, dt \geq \frac{C S_0(\gamma)^{2-k}}{\gamma},$$

which implies that $\lim_{\gamma \to \infty} S_0(\gamma) \geq C_4 > 0$, since $\gamma^2/S_0(\gamma) \to \infty$ as $\gamma \to \infty$ and $k > 2$. This proves the lemma.

** Lemma 4.3.** For $\gamma$ large,

$$C_1 \gamma \leq S_0(\gamma) \leq C_2 \gamma,$$  

and

$$C_1/\gamma \leq y'(S_0(\gamma), \gamma) \leq C_2/\gamma.$$  

**Proof.** Let $v = y - 1$ and $f_1(s) = (s+1)^p - (s+1)$. Then $v$ satisfies

$$-v'' = t^{-k} f_1(v),$$

$$v(\infty) = \gamma - 1, \quad v'(\infty) = 0$$

and $S_0(\gamma)$ is the first zero of $v$. Let $F_1(s)$ be the primitive of $f_1$ and let

$$H(t) = \frac{1}{2} tv^2 - \frac{1}{2} vv' + t^{1-k} F_1(v).$$

Then from (3.6) we have

$$-H'(t) = \frac{t^{-k}}{2} h(v + 1),$$

where

$$h(s) = s^p - \frac{p-1}{2} s^2 - s + \left( \frac{p-1}{2} \right).$$

Since $n \leq 6$, we have $p \geq 2$, and therefore we obtain that $h$ is convex for $s \geq 1$ and satisfies

$$h(s) \geq C(s - 1)^p$$
for \( s \geq 1 \). Integrating (4.13) and using (4.14) and (4.5) we obtain

\[
H(2S_0(\gamma)) = \int_{2S_0(\gamma)}^{\infty} t^{-k} h(v + 1) \, dt
\]

\[
\geq C \int_{2S_0(\gamma)}^{\gamma^2} t^{-k} \frac{(t - S_0(\gamma))^p}{\gamma} \, dt
\]

\[
= \frac{CS_0(\gamma)^{p-k+1}}{\gamma^p} \left[ \frac{\gamma^{2S_0(\gamma)}}{\gamma^{2(2S_0(\gamma))}} \right] \int_{2S_0(\gamma)}^{\gamma^2} t^{-k+p} \, dt = C/\gamma, \quad (4.15)
\]

since \( p = 2k - 3 \). On the other hand, we have from (4.1) and (4.4), that

\[
H(2S_0(\gamma)) \leq S_0(\gamma) v'(S_0(\gamma))^2 + 2^{1-k} S_0(\gamma)^{1-k} F_1(v(2S_0(\gamma)))
\]

\[
\leq C_1 \left\{ \frac{S_0(\gamma)}{\gamma^2} + S_0(\gamma)^{1-k} F_1 \left( C_2 \frac{S_0(\gamma)}{\gamma} \right) \right\}. \quad (4.16)
\]

Now we assert that

\[
\lim_{\gamma \to \infty} \frac{S_0(\gamma)}{\gamma} > 0. \quad (4.17)
\]

Suppose (4.17) is not true. Then for a subsequence \( \gamma \to \infty \), we can find \( C_3 > 0 \) such that

\[
F_1 \left( C_2 \frac{S_0(\gamma)}{\gamma} \right) \leq C_3 \left( \frac{S_0(\gamma)}{\gamma} \right)^2. \quad (4.18)
\]

From (4.15), (4.16) and (4.18) we have

\[
C/\gamma \leq H(2S_0(\gamma)) \leq C_4 \left\{ \frac{S_0(\gamma)}{\gamma^2} + S_0(\gamma)^{1-k} \left( \frac{S_0(\gamma)}{\gamma} \right)^2 \right\}
\]

\[
\leq \frac{C_4}{\gamma} \left( \frac{S_0(\gamma)}{\gamma} \right) \left[ 1 + \frac{1}{S_0(\gamma)^{k-2}} \right].
\]

This, together with Lemma (4.2), implies that

\[
0 < C_5 \leq \left( \frac{S_0(\gamma)}{\gamma} \right) \to 0
\]

as \( \gamma \to \infty \), which is a contradiction. This proves (4.17). Now (4.10) follows from (4.1) and (4.17).

From the concavity of \( y \) and (4.4), we have

\[
y'(S_0(\gamma), \gamma) \geq y'(2S_0(\gamma), \gamma) \geq \frac{C_1}{\gamma}. \quad (4.19)
\]
For $\gamma$ large it follows from (4.2) and (4.1) that $y(t, \gamma) \leq C$ for $t \in [S_0(\gamma), 2S_0(\gamma)]$. Hence from (4.4) and (4.10) we have

$$y'(S_0(\gamma), \gamma) = y'(2S_0(\gamma), \gamma) + \int_{S_0(\gamma)}^{2S_0(\gamma)} t^{-k}(y^p - y) \, dt \leq \frac{C}{\gamma} + O\left(\frac{1}{\gamma^{k-1}}\right) \leq \frac{C_2}{\gamma}.$$  

This, together with (4.19), proves (4.11) and the lemma.

**Proof of Lemma B.** Inequality (2.8) follows from (3.18).

Let $t_0 \in [S_1(\gamma), S_0(\gamma)]$ be such that

$$y'(S_0(\gamma), \gamma) = y'(t_0, \gamma) = \frac{y'(S_0(\gamma), \gamma)}{2}. \quad (4.20)$$

Then from (4.11), (4.20), and the restriction that $0 < y \leq 1$, we obtain for $\gamma$ large that

$$C_1/\gamma \leq \frac{y'(S_0(\gamma), \gamma)}{2} = \int_{t_0}^{S_0(\gamma)} t^{-k}(y^p - y) \, dt \leq \frac{C}{t_0^{k-1}},$$

that is,

$$t_0 \leq C_1 \gamma^{1/(k-1)}. \quad (4.21)$$

Let $S(\gamma) = C_1 \gamma^{1/(k-1)}$; then clearly from (4.20) we have

$$S_1(\gamma) \leq S(\gamma), \quad (4.22)$$

$$C_3/\gamma \leq \frac{y'(S_0(\gamma), \gamma)}{2} = y'(t_0, \gamma) \leq y'(S(\gamma), \gamma) \leq y'(S_0(\gamma), \gamma) \leq C_4/\gamma. \quad (4.23)$$

This, together with (4.21) and (4.22), proves (2.9) and (2.11).

Now from the convexity of $y$ in $[S_1(\gamma), S_0(\gamma)]$ and (4.23) we have

$$\frac{1 - y(S(\gamma), \gamma)}{S_0(\gamma) - S(\gamma)} \geq y'(S(\gamma), \gamma) \geq C_3/\gamma. \quad (4.24)$$

From (4.24) and (4.10), we have

$$1 - y(S(\gamma), \gamma) \geq C - O\left(\frac{1}{\gamma^{(k-2)/(k-1)}}\right).$$

Hence we can find a $\delta > 0$ such that for $\gamma$ large, $1 - y(S(\gamma), \gamma) \geq \delta$ and this proves (2.10).

Since $S(\gamma) = O\left(\gamma^{1/(k-1)}\right)$, from (3.18) we get

$$y(t, \gamma) \geq Z_1(t, \gamma) \geq Ct/\gamma. \quad (4.25)$$
for all \( t \in [S_1(\gamma), S(\gamma)] \). From (2.9), (2.10), (2.11) and (4.25) we have
\[
C/\gamma \geq \gamma'(S(\gamma), \gamma) = \int_{S(\gamma)}^{S_1(\gamma)} t^{-k} \gamma(1 - \gamma^{p-1}) \, dt \geq \frac{C_1}{\gamma} \int_{S(\gamma)}^{S_1(\gamma)} t^{-k+1} \, dt
\]
\[
= \frac{C_2}{\gamma} \left( \frac{1}{S_1(\gamma)^{k-2}} - \frac{1}{S(\gamma)^{k-2}} \right).
\]

This implies that
\[
\lim_{\gamma \to \infty} S_1(\gamma) > 0.
\]

This proves (2.12) and hence the lemma.

**Remark 1.** Let \( n \geq 3 \) and \( p > 1 \). Then there exists an \( R_0 > 0 \) such that for \( 0 < R < R_0 \), the problem
\[
-\Delta u = u^p - u \quad \text{in } B(R),
\]
\[
u(0) = \gamma > 0, \quad \nu'(0) = 0.
\]

\( u > 0, \) \( u \) is radial \( \text{in } B(R) \),

\[ (4.26) \]
\[
\frac{\partial u}{\partial v} = 0 \quad \text{in } \partial B(R)
\]

does not admit any solution \( u \) such that \( u' \) changes sign.

**Proof.** We consider two cases: \( 1 < p < \frac{n+2}{n-2} \) and \( p \geq \frac{n+2}{n-2} \).

Case 1. \( 1 < p < \frac{n+2}{n-2} \). In this situation, by a result of LIN, NI & TAGAKI [11] there exists an \( R_0 > 0 \) such that for \( 0 < R < R_0 \), problem (4.26) does not admit a nonconstant solution. This proves the remark.

Case 2. \( p \geq \frac{n+2}{n-2} \). Let \( v(r, \gamma) \) denote the solution of
\[
-\left( v'' + \frac{n-1}{r} v' \right) = v^p - v \quad \text{in } (0, \infty),
\]
\[
v(0) = \gamma > 0, \quad v'(0) = 0.
\]

Let \( R_1(\gamma) < R_2(\gamma) < \ldots \) be the turning points (i.e., \( v'(R_1(\gamma), \gamma) = 0 \)) of \( v(r, \gamma) \).

From the result of NI [12], we know that \( v(r, \gamma) > 0 \) for all \( \gamma > 0 \).

Now the remark follows from the following

**Assertion.** There exists a constant \( C > 0 \) such that
\[
\sup_{\gamma \in (0, \infty)} R_2(\gamma) \geq C.
\]
To prove this we adopt the method used in Atkinson, Brezis & Peletier [6] and in Adimurthi & Yadava [2]. Proceeding as in Lemma A, we obtain
\[ \lim_{\gamma \to 0} R_1(\gamma) = \infty, \quad \lim_{\gamma \to 1} R_1(\gamma) > 0. \]
Therefore it is sufficient to prove that
\[ \sup_{\gamma \in (1, \infty)} R_2(\gamma) \geq C. \] (4.28)
Let \( w(r, \gamma) = v(r, \gamma) - 1 \) and let \( T_1(\gamma) \) and \( T_2(\gamma) \) respectively be the first and second zeros of \( w(r, \gamma) \). Then
\[ T_1(\gamma) < R_1(\gamma) < T_2(\gamma) < R_2(\gamma). \]
Therefore, in order to prove (4.28), it is sufficient to show that
\[ \sup_{\gamma \in (1, \infty)} T_2(\gamma) \geq C. \] (4.29)
Since \( v(r, \gamma) > 0 \) for all \( \gamma > 1 \), we get
\[ \sup_{\gamma \in (1, \infty)} \{|w(r, \gamma)|; T_1(\gamma) < r < T_2(\gamma)\} \leq 1. \] (4.30)
Let \( Z(r) = \left( \frac{n - 2}{r} \right)^{\frac{n-2}{2}} \). Then \( Z \) satisfies
\[ Z'' + \left( \frac{n - 1}{r} \right) Z' + \frac{\gamma}{2} Z^{4/(n-2)} = 0 \quad \text{in } (0, \infty), \] (4.31)
\[ \lim_{r \to 0} Z(r) = \infty. \]
From (4.30) and (4.31) we can choose an \( r_0 > 0 \) such that for all \( \gamma > 1 \) and \( r \in (0, r_0) \cap [T_1(\gamma), T_2(\gamma)] \),
\[ \frac{(w + 1)^p - (w + 1)}{w} < \frac{1}{4} Z(r)^{4/(n-2)}. \]
Now by Sturm's comparison theorem, there exists a \( C > 0 \) such that (4.29) holds. This completes the proof of the remark.

**Remark 2.** Given any \( \Omega \), we can construct a negative function \( \alpha \in C^\infty(\Omega) \) such that the problem
\[ -\Delta u = u^p + \alpha(x) u \quad \text{in } \Omega, \]
\[ u > 0 \quad \text{in } \Omega, \] (4.32)
\[ \frac{\partial u}{\partial v} = 0 \quad \text{on } \partial \Omega \]
admits a solution.
The construction of $\alpha$ is similar to the construction given by Brezis [7] for the Dirichlet problem.

Let $a \in C^\infty(\Omega)$, be such that $a$ changes sign in $\Omega$ and $\int_\Omega a(x) \, dx < 0$. By the result of Hess & Senn [14] there exists a $\lambda_1(\Omega) > 0$ such that

$$-\Delta v = \lambda_1(\Omega) a(x) v \quad \text{in } \Omega,$$

$$v > 0 \quad \text{in } \Omega \quad \text{and}$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega$$

admits a solution. Define

$$\alpha(x) = \lambda_1(\Omega) a(x) - \mu^{p-1} v^{p-1}, \quad u = \mu v,$$

where $\mu$ is a positive real number. Obviously $u$ satisfies (4.32). By choosing $\mu$ large, we get $\alpha < 0$.

Acknowledgement. We thank Dr. Veerappa Gowda for assisting us in doing some numerical computation for this problem and also for several discussions.

References


Tata Institute of Fundamental Research
Post. Box 1234
Bangalore-560 012
INDIA

(Received January 2, 1991)