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Existence and Nonexistence of Positive Radial Solutions of Neumann Problems with Critical Sobolev Exponents

ADIMURTHI & S. L. YADAVA

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\alpha \in C^\infty(\Omega)$. For $\lambda > 0$, $p > 1$, $n \geqq 3$ we consider the following problem

$$\begin{aligned} -\Delta u &= u^p + \lambda\alpha(x)\, u \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

When $p < \dfrac{n+2}{n-2}$ and $\alpha(x) = -1$, this problem has been discussed extensively in the works of NI [12], LIN & NI [10] and LIN, NI & TAKAGI [11]. They have proved that there exist positive constants λ_0 and λ_1, with $\lambda_0 \leqq \lambda_1$, such that (1.1) admits a non-constant solution for $\lambda \geqq \lambda_1$ and does not admit any non-constant solution for $\lambda < \lambda_0$. In view of their results, it was conjectured by LIN & NI [10] that a similar result holds even for $p \geqq \dfrac{n+2}{n-2}$.

When $p = \dfrac{n+2}{n-2}$, BREZIS [7] posed the question of finding conditions on α and Ω for which (1.1) admits a solution. Clearly when $\alpha(x) \geqq 0$, (1.1) does not admit any solution. Therefore we have to consider two cases: (i) $\alpha(x)$ changes sign, (ii) $\alpha(x) \leqq 0$.

In case (i) some partial results have been obtained in [3] by using the variational methods of BREZIS & NIRENBERG [8]. To describe the results of [3], we further assume that $\int_\Omega \alpha(x)\, dx < 0$, that there exists an $x_0 \in \partial\Omega$ such that $\alpha(x_0) > 0$, and that $\partial\Omega$ is flat at x_0 of order at least four. Under these assumptions, it was shown that for $n \geqq 4$ there exists a $\lambda^* > 0$ such that (1.1) admits a solution if and only if $\lambda \in (0, \lambda^*)$.

In case (ii) the standard variational arguments do not seem to work. On the other hand, in this situation it is easy to construct an example (see Remark 2 at the end of Section 4) such that for any Ω we can find a negative function $\alpha(x)$ for which (1.1) admits a solution. In view of this and the results of LIN, NI & TAKAGI [11], we shall consider the very restricted case of problem (1.1) when $\lambda\alpha(x) \equiv -1$, Ω is a ball and the solution is radial.

Let $B(R)$ denote the ball of radius R with center at the origin and let $\mu_1(R)$ be the first non-zero eigenvalue of the radial problem

$$\begin{aligned} -\Delta\varphi &= \mu\varphi \quad \text{in } B(R), \\ \frac{\partial\varphi}{\partial\nu} &= 0 \quad \text{on } \partial B(R). \end{aligned} \tag{1.2}$$

We consider the problem

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} - u \quad \text{in } B(R), \\ u > 0, &\quad u \text{ is radial in } B(R), \\ \frac{\partial u}{\partial\nu} &= 0 \quad \text{on } \partial B(R) \end{aligned} \tag{1.3}$$

and prove the following

Theorem. *Let* $p = (n+2)/(n-2)$. *The following conclusions hold*:
(a) *If* $n \geqq 3$ *and* $p - 1 > \mu_1(R)$, *then* (1.3) *admits a solution which is radially increasing.*
(b) *If* $n \in \{4, 5, 6\}$ *and* $p - 1 < \mu_1(R)$, *then* (1.3) *admits a solution which is radially decreasing.*
(c) *If* $n = 3$, *then there exists an* $R^* > 0$ *such that for* $0 < R < R^*$, (1.3) *does not admit any nonconstant solution.*

Here we remark that part (a) of the theorem has been proved by NI [12] and LIN & NI [10], and that part (b) gives a counter-example to a part of the conjecture of LIN & NI [10].

Since we are looking for radial solutions, (1.3) reduces to studying the first turning point $R_1(\gamma)$ of $v(r,\gamma)$, where v satisfies

$$\begin{aligned} -v'' - \frac{n-1}{r}v' &= v^{\frac{n+2}{n-2}} - v, \\ v'(0) = 0, &\quad v(0) = \gamma > 0 \end{aligned} \tag{1.4}$$

and $R_1(\gamma)$ is defined by

$$R_1(\gamma) = \sup\{r;\, v'(s,\gamma) \neq 0 \quad \forall\, s \in (0, r)\}. \tag{1.5}$$

Because of the continuity of $\gamma \to R_1(\gamma)$, we shall be able to deduce the theorem from knowledge of the behavior of $R_1(\gamma)$ as $\gamma \to 0$, 1 and ∞. Information about the behavior of $R_1(\gamma)$ as $\gamma \to 0$, 1 is available in the literature. Therefore the main difficulty lies in understanding its behavior at ∞. We illustrate this for $n = 6$.

Let $n = 6$, $\gamma > 1$, $\eta = v(R_1(\gamma), \gamma)$ and $w = v - \eta$. Then w satisfies

$$-\Delta w = w^2 + (2\eta - 1)\, w + \eta(\eta - 1) \quad \text{in } B(R_1(\gamma)),$$
$$w > 0 \quad \text{in } B(R_1(\gamma)),$$
$$w = \frac{\partial w}{\partial v} = 0, \quad \text{on } \partial B(R_1(\gamma)).$$

Hence by Pohožaev's identity we have

$$2(2\eta - 1) \int_{B(R_1(\gamma))} w^2\, dx + 8\eta(\eta - 1) \int_{B(R_1(\gamma))} w\, dx = 0.$$

This implies that $\eta > 1/2$ and hence $v(r, \gamma) > 1/2$ for all $r \in (0, R_1(\gamma))$. Now the asymptotic analysis of ATKINSON & PELETIER [5] suggests that we can find positive constants δ, C_1, C_2, C_3 and γ_0 such that, for $\gamma > \gamma_0$ and $R(\gamma) = C_1 \gamma^{-1/6}$,

$$R(\gamma) < R_1(\gamma), \tag{1.6}$$

$$1 - v(R(\gamma), \gamma) \geqq \delta, \tag{1.7}$$

$$C_1/\gamma^{1/6} \leqq |v'(R(\gamma), \gamma)| \leqq C_2/\gamma^{1/6}. \tag{1.8}$$

Integrating (1.4) from $R(\gamma)$ to $R_1(\gamma)$ and using (1.6)–(1.8), we obtain for $C = C_1^5 C_2$ that

$$C/\gamma \geqq -R(\gamma)^5\, v'(R(\gamma), \gamma) = \int_{R(\gamma)}^{R_1(\gamma)} r^5 v(1 - v)\, dr \geqq \delta/12(R_1(\gamma)^6 - C_1/\gamma).$$

Hence

$$R_1(\gamma)^6 \leqq \left(\frac{12C}{\delta} + C_1\right)/\gamma \to 0 \quad \text{as } \gamma \to \infty. \tag{1.9}$$

When $n \leqq 5$ it may not be true that $v(R_1(\gamma), \gamma)$ is bounded away from zero as $\gamma \to \infty$, whereas estimates similar to (1.6)–(1.8) still hold. Therefore in this case we have to adopt a different procedure to study $R_1(\gamma)$ as $\gamma \to \infty$.

The paper is divided into two parts. In the first part (Section 3), we study the behavior of $R_1(\gamma)$ as $\gamma \to 0, 1$. In the second part (Section 4), following the techniques developed in ATKINSON & PELETIER [5], we obtain estimates similar to (1.6)–(1.8). Using these (see Section 2) we obtain the proof of the theorem.

In a forthcoming paper we shall study problem (1.3) when $-\Delta$ is replaced by the p-Laplacian for $p \leqq n$.

While revising this paper, we learned of a recent result of BUDD, KNAAP & PELETIER [9], which discusses the question of existence and non-existence of solutions of (1.3) when $u^{(n+2)/(n-2)} - u$ is replaced by $u^{(n+2)/(n-2)} - u^q$ for $1 < q < 4/(n-2)$. This problem, for $q = 4/(n-2)$, has also been treated by ADIMURTHI, KNAAP & YADAVA [4].

Recently, ADIMURTHI & MANCINI [1] have tackeled this problem in an arbitrary domain using variational techniques. We learned from Prof. J. SERRIN that X. J. WANG [13] has also found related results.

2. Proof of the Theorem

In order to prove the theorem, we make use of the standard substitutions,

$$t = \left(\frac{n-2}{r}\right)^{n-2}, \quad k = \frac{2(n-1)}{n-2}, \quad p = \frac{n+2}{n-2} = 2k-3, \quad y(t,\gamma) = v(r,\gamma),$$

introduced in [5]. Then from (1.4), y satisfies the Emden-Fowler equation

$$\begin{gathered} -y'' = t^{-k}(y^p - y), \\ y(\infty) = \gamma > 0, \quad y'(\infty) = 0. \end{gathered} \tag{2.1}$$

Let $S_1(\gamma)$ be the first turning point of $y(t, \gamma)$, defined by

$$S_1(\gamma) = \inf\{t; y'(s,\gamma) \neq 0 \quad \forall\, s \in (t,\infty)\}. \tag{2.2}$$

Let φ be the solution of

$$-\varphi'' = t^{-k}\varphi \quad \text{in } (0,\infty),$$

$$\varphi(\infty) = 1, \quad \varphi'(\infty) = 0 \tag{2.3}$$

and let τ_0 and τ_1 respectively be the first zero and first turning point of φ, i.e.,

$$\begin{aligned} \tau_0 &= \inf\{t; \varphi(s) > 0 \quad \text{for } s > t\}, \\ \tau_1 &= \inf\{t; \varphi'(s) > 0 \quad \text{for } s > t\}. \end{aligned} \tag{2.4}$$

Then we have

Lemma A. *Let* $\gamma \neq 0, 1$. *Then*
(i) $S_1(\gamma)$ *exists and* $y(S_1(\gamma), \gamma) > 0$.
(ii) *If* $\gamma \in (0, 1)$, *then* y *is decreasing, with*

$$\lim_{\gamma \to 0} S_1(\gamma) = 0, \tag{2.5}$$

$$\lim_{\gamma \to 1} S_1(\gamma) = (p-1)^{1/(k-2)}\,\tau_1. \tag{2.6}$$

(iii) *If* $\gamma > 1$, *then* y *is increasing, with*

$$\lim_{\gamma \to 1} S_1(\gamma) = (p-1)^{1/(k-2)}\,\tau_1. \tag{2.7}$$

This result is contained in the works of NI [12] and LIN & NI [10]. For the sake of completeness, we present the proof in Section 3.

Lemma B. *Let* $\gamma \in (1, \infty)$. *Then*
(i) *For* $t \geqq S_1(\gamma)$,

$$y(t,\gamma) \geqq Z_1(t,\gamma), \tag{2.8}$$

where

$$Z_1(t, \gamma) = \frac{\gamma t}{\{t^{k-2} + (\gamma^{p-1} - 1)/(k-1)\}^{1/(k-2)}}.$$

(ii) *If* $3 \leqq n \leqq 6$, *there exist positive constants* δ, C_1, C_2, C_3, C_4 *and* γ_0 *such that, for all* $\gamma \geqq \gamma_0$ *and* $S(\gamma) = C_1 \gamma^{1/(k-1)}$,

$$S_1(\gamma) < S(\gamma), \tag{2.9}$$

$$1 - y(S(\gamma), \gamma) \geqq \delta, \tag{2.10}$$

$$C_3/\gamma \leqq y'(S(\gamma), \gamma) \leqq C_2/\gamma, \tag{2.11}$$

$$\varliminf_{\gamma \to \infty} S_1(\gamma) \geqq C_4. \tag{2.12}$$

Assuming the validity of Lemmas A and B, we first complete the proof of the theorem. Since Lemma A gives the behavior of $S_1(\gamma)$ as $\gamma \to 0, 1$, to prove the theorem we must study its behavior at ∞. For this we need three further lemmas.

Lemma 2.1. *Let* Z_1 *be as defined in* (2.8). *Then*

$$-Z_1'' = \left(\frac{\gamma^p - \gamma}{\gamma^p}\right) t^{-k} Z_1^p \text{ in } (0, \infty), \tag{2.13}$$

$$\lim_{t \to \infty} Z_1 = \gamma, \tag{2.14}$$

$$\gamma - Z_1(t, \gamma) + t Z_1'(t, \gamma) = \left(\frac{\gamma^p - \gamma}{\gamma^p}\right) \int_t^\infty Z_1^p s^{-k+1}\, ds, \tag{2.15}$$

$$t Z_1'(t, \gamma) - Z_1(t, \gamma) = \frac{-\gamma t^{k-1}}{\{t^{k-2} + (\gamma^{p-1} - 1)/(k-1)\}^{(k-1)/(k-2)}}. \tag{2.16}$$

This lemma follows easily from the definition of Z_1.

Lemma 2.2. *If* $n = 3$ $(k = 4)$, *then*

$$\varlimsup_{\gamma \to \infty} S_1(\gamma) < \infty.$$

Proof. Let $\beta(t) = t \cosh \dfrac{1}{t}$. It is easy to verify that β satisfies

$$\beta'' = t^{-4} \beta \quad \text{in } (0, \infty), \tag{2.17}$$

$$\lim_{t \to 0} \beta(t) = \infty, \quad \beta(t) = t + C(t), \tag{2.18}$$

where $C(t) \geqq 0$. Let T_0 be such that $\beta'(T_0) = 0$. Then the lemma follows if we can show that

$$\varlimsup_{\gamma \to \infty} S_1(\gamma) \leqq T_0. \tag{2.19}$$

Let $W = (y\beta' - \beta y')$. Then $W(\infty) = \gamma$ and $W'(t) = t^{-4} y^5 \beta$. Integrating W' from $S_1(\gamma)$ to ∞ and using (2.8), (2.18), (2.15) and (2.16), we obtain

$$\begin{aligned} y(S_1(\gamma), \gamma)\, \beta'(S_1(\gamma)) &= \gamma - \int_{S_1(\gamma)}^{\infty} t^{-4} y^5 \beta \, dt \\ &\leqq \gamma - \int_{S_1(\gamma)}^{\infty} t^{-3} Z_1 \, dt \leqq \gamma - \frac{\gamma^5}{(\gamma^5 - \gamma)} [\gamma - Z_1 + S_1(\gamma) Z_1'] \\ &= -\frac{\gamma^2}{\gamma^5 - \gamma} + \left(\frac{\gamma^5}{(\gamma^5 - \gamma)}\right) \frac{\gamma S_1(\gamma)^3}{\{S_1(\gamma)^2 + \frac{1}{3}(\gamma^4 - 1)\}^{3/2}}. \end{aligned} \tag{2.20}$$

From (2.9) it follows that $S_1(\gamma) = 0(\gamma^{1/3})$ as $\gamma \to \infty$; hence we have

$$\left(\frac{\gamma^5}{(\gamma^5 - \gamma)}\right) \frac{\gamma S_1(\gamma)^3}{\{S_1(\gamma)^2 + \frac{1}{3}(\gamma^4 - 1)\}^{3/2}} = o\left(\frac{1}{\gamma^4}\right)$$

as $\gamma \to \infty$. This together with (2.20) and (i) of Lemma A implies that $\beta'(S_1(\gamma)) < 0$ for γ large, and so $S_1(\gamma) \leqq T_0$. This proves (2.19) and hence the lemma.

Lemma 2.3. *If* $n \in \{4, 5, 6\}$, *then*

$$\lim_{\gamma \to \infty} S_1(\gamma) = \infty. \tag{2.21}$$

Proof. Suppose (2.21) is not true. Then for a sequence of values $\gamma \to \infty$, we have

$$\lim_{\gamma \to \infty} S_1(\gamma) < \infty. \tag{2.22}$$

For the sequel we use C, C_1, C_2, etc., to denote positive constants independent of γ. Now from (2.8), (2.9) we have for $t \in (S_1(\gamma), S(\gamma))$,

$$y(t, \gamma) \geqq Z_1(t, \gamma) \geqq C \frac{t}{\gamma}. \tag{2.23}$$

Let

$$H(t) = \tfrac{1}{2} t y'^2 - \tfrac{1}{2} y y' + t^{1-k} \left(\frac{y^{p+1}}{p+1} - \frac{y^2}{2}\right).$$

Then $H(\infty) = 0$ and $H'(t) = \dfrac{p-1}{2} t^{-k} y^2$. Hence $H(t) \leqq 0$. Now integrating $H'(t)$ from $S_1(\gamma)$ to $S(\gamma)$ and using (2.23), we obtain

$$\begin{aligned} -H(S_1(\gamma)) &\geqq \frac{p-1}{2} \int_{S_1(\gamma)}^{S(\gamma)} y^2 t^{-k} \, dt \\ &\geqq \frac{C}{\gamma^2} \int_{S_1(\gamma)}^{S(\gamma)} t^{-k+2} \, dt = C \frac{\varrho(\gamma)}{\gamma^2}, \end{aligned} \tag{2.24}$$

where

$$\varrho(\gamma) = \begin{cases} \log \dfrac{S(\gamma)}{S_1(\gamma)} & \text{if } k = 3, \\ (S(\gamma)^{3-k} - S_1(\gamma)^{3-k}) & \text{if } k < 3. \end{cases}$$

From (2.10), (2.11) and (2.22) we have

$$\begin{aligned} C_2/\gamma &\geqq y'(S(\gamma), \gamma) = \int\limits_{S_1(\gamma)}^{S(\gamma)} y(1 - y^{p-1})\, t^{-k}\, dt \\ &\geqq \frac{\delta}{k-1}\, y(S_1(\gamma), \gamma) \left(\frac{1}{S_1(\gamma)^{k-1}} - \frac{1}{S(\gamma)^{k-1}}\right) \\ &\geqq C y(S_1(\gamma), \gamma). \end{aligned}$$

Hence

$$\begin{aligned} -H(S_1(\gamma)) &= S_1(\gamma)^{1-k}\, y(S_1(\gamma), \gamma)^2 \left(\tfrac{1}{2} - \frac{y(S_1(\gamma), \gamma)^{p-1}}{p+1}\right) \\ &\leqq C_3 \frac{S_1(\gamma)^{1-k}}{\gamma^2}. \end{aligned}$$

This combined with (2.24) gives

$$S_1(\gamma)^{k-1} \leqq C_4/\varrho(\gamma). \tag{2.25}$$

Since $S_1(\gamma)$ is bounded by assumption, it follows that $\varrho(\gamma) \to \infty$ as $\gamma \to \infty$. Therefore from (2.25), $S_1(\gamma) \to 0$ as $\gamma \to \infty$, contradicting (2.12). This proves the lemma.

Proof of the Theorem. For $\gamma \neq 0, 1$, let $R_1(\gamma)$ and $u(r, \gamma)$ be defined by

$$t = \left(\frac{n-2}{r}\right)^{n-2}, \quad S_1(\gamma) = \left(\frac{n-2}{R_1(\gamma)}\right)^{n-2},$$

$$u(r, \gamma) = y(t, \gamma).$$

Then u satisfies

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} - u \quad \text{in } B(R_1(\gamma)), \\ u &> 0 \quad \text{in } B(R_1(\gamma)), \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial B(R_1(\gamma)). \end{aligned}$$

Define $R_1 = [(n-2)/\tau_1]^{n-2}$. It is easy to see that $\mu_1(R_1(\gamma)) = (R_1/R_1(\gamma))^2$. Since $\gamma \to R_1(\gamma)$ is continuous, (a) follows from (2.5) and (2.6), (b) follows from (2.7) and (2.21), and (c) follows from Lemma 2.2. This proves the theorem.

3. Proof of Lemma A

Let $k > 2$ and let $f: \mathbb{R} \to \mathbb{R}$ be a C^1-function. For $\gamma > 0$, let $Y(t, \gamma)$ be the solution of

$$-Y'' = t^{-k} f(Y),$$
$$Y(\infty) = \gamma, \qquad Y'(\infty) = 0. \tag{3.1}$$

Let

$$F(s) = \int_0^s f(r)\,dr,$$

$$H(t) = \tfrac{1}{2} tY'^2 - \tfrac{1}{2} YY' + t^{1-k} F(Y), \tag{3.2}$$

$$H_1(t) = \tfrac{1}{2} tY'^2 - \tfrac{1}{2} YY' + \frac{t^{1-k}}{2(k-1)} Yf(Y). \tag{3.3}$$

It is then easy to see that Y satisfies

$$\lim_{t\to\infty} H(t) = \lim_{t\to\infty} H_1(t) = 0, \tag{3.4}$$

$$\lim_{t\to\infty} Y'(t,\gamma)\, t^{k-1} = \frac{f(\gamma)}{(k-1)}, \tag{3.5}$$

$$H'(t) = \tfrac{1}{2} t^{-k}[Yf(Y) - 2(k-1)\,F(Y)], \tag{3.6}$$

$$H_1'(t) = \frac{Y' t^{1-k}}{2(k-1)} [Yf'(Y) - (2k-3) f(Y)], \tag{3.7}$$

$$(Y' Y^{1-k}\, t^{k-1})' = -2(k-1)\, t^{k-2}\, Y^{-k}\, H_1(t). \tag{3.8}$$

From now on, we assume that $f(0) = f(1) = 0$ and $f'(1) > 0$. Furthermore, we assume that

$$(s-1) f(s) > 0 \qquad \text{for } s > 0 \text{ and } s \neq 1. \tag{3.9}$$

For $\gamma > 0$, $\gamma \neq 1$, put

$$S_0(\gamma, f) = \inf\{t;\ Y(s,\gamma) \neq 1,\ Y'(s,\gamma) \neq 0 \quad \forall\, s > t\}, \tag{3.10}$$

$$S_1(\gamma, f) = \inf\{t;\ Y(s,\gamma) > 0,\ Y'(s,\gamma) \neq 0 \quad \forall\, s > t\}. \tag{3.11}$$

We then have the following

Lemma 3.1. *For* $s \geqq 0$, *assume that* f *satisfies*

$$sf'(s) - (2k-3) f(s) \geqq 0. \tag{3.12}$$

Then

$$Y(t,\gamma) \geqq \eta_1(t,\gamma) \tag{3.13}$$

for $\gamma > 1$ *and* $t \geqq S_1(\gamma, f)$, *where*

$$\eta_1(t,\gamma) = \frac{\gamma t}{\left\{t^{k-2} + \dfrac{f(\gamma)}{(k-1)\gamma}\right\}^{1/(k-2)}}.$$

Proof. Let $t > S_1(\gamma, f)$. Since $\gamma > 1$, it follows from (3.9) that $Y'(t, \gamma) > 0$. Therefore from (3.12) and (3.7), $H_1'(t) \geqq 0$. Hence H_1 is increasing and from (3.4), $H_1(t) \leqq 0$. From (3.8), we have

$$(Y' Y^{1-k} t^{k-1})' \geqq 0.$$

Integrating this twice from t to ∞ and using (3.5), we obtain

$$\frac{1}{Y^{k-2}} - \frac{1}{\gamma^{k-2}} \leqq \frac{\gamma^{1-k} f(\gamma)}{(k-1)\, t^{k-2}},$$

which gives

$$Y(t, \gamma) \geqq \frac{\gamma t}{\left\{t^{k-2} + \dfrac{f(\gamma)}{(k-1)\gamma}\right\}^{1/(k-2)}}.$$

This proves the lemma.

Lemma 3.2. *For* $s \geqq 0$, *assume that* f *satisfies*

$$s f'(s+1) - (2k-3) f(s+1) \leqq 0. \tag{3.14}$$

Then

$$Y(t, \gamma) \leqq 1 + \eta_2(t, \gamma), \tag{3.15}$$

for $\gamma > 1$ *and* $t \geqq S_1(\gamma, f)$, *where*

$$\eta_2(t, \gamma) = \frac{(\gamma - 1)\, t}{\left\{t^{k-2} + \dfrac{f(\gamma)}{(k-1)(\gamma-1)}\right\}^{1/(k-2)}}.$$

Proof. Let $V = Y - 1$, $f_1(s) = f(s+1)$. Then V satisfies

$$-V'' = t^{-k} f_1(V),$$

$$V(\infty) = \gamma - 1, \qquad V'(\infty) = 0. \tag{3.16}$$

Since $\gamma > 1$, from (3.9), we get $Y(t, \gamma) \geqq 1$ and $Y'(t, \gamma) > 0$ for $t \geqq S_0(\gamma, f)$. Hence $V(t) \geqq 0$ and $V'(t) > 0$. Therefore for $t \geqq S_0(\gamma, f)$, we have from (3.16), (3.7) and (3.14) that $H_1'(t) \leqq 0$. So we deduce that $H_1(t) \geqq 0$ from (3.4) and that

$$(V' V^{1-k} t^{k-1})' \leqq 0$$

from (3.8). Integrating twice and using (3.5) we obtain for all $t \geqq S_0(\gamma, f)$ that

$$V(t, \gamma) \leqq \frac{(\gamma - 1)\, t}{\left\{t^{k-2} + \dfrac{f(\gamma)}{(k-1)(\gamma-1)}\right\}^{1/(k-2)}} = \eta_2(t, \gamma),$$

that is, for $t \geqq S_0(\gamma, f)$,

$$Y(t, \gamma) \leqq 1 + \eta_2(t, \gamma). \tag{3.17}$$

Since $Y(t,\gamma) \leqq 1$ for $t \in [S_1(\gamma,f), S_0(\gamma,f)]$, inequality (3.17) continues to hold for $t \geqq S_1(\gamma,f)$. This proves the lemma.

As an immediate consequence of these lemmas we have the following

Lemma 3.3. *Let* $\gamma > 1$ *and let* $y(t,\gamma)$ *satisfy* (2.1). *For* $t \geqq S_1(\gamma)$,

(i) $y(t,\gamma) \geqq Z_1(t,\gamma)$ *if* $n \geqq 3$. (3.18)

(ii) $y(t,\gamma) \leqq 1 + Z_2(t,\gamma)$ *if* $3 \leqq n \leqq 6$, (3.19)

where

$$Z_1(t,\gamma) = \frac{\gamma t}{\left\{t^{k-2} + \dfrac{\gamma^{2(k-2)} - 1}{(k-1)}\right\}^{1/(k-2)}},$$

$$Z_2(t,\gamma) = \frac{(\gamma - 1)\, t}{\left\{t^{k-2} + \dfrac{\gamma(\gamma^{2(k-2)} - 1)}{(k-1)(\gamma - 1)}\right\}^{1/(k-2)}}.$$

Proof. Let $p = 2k - 3$ and $f(s) = s^p - s$ for $s \geqq 0$. Extend f as a C^1-function to $\mathbb{R}$. Then clearly f satisfies (3.9), and for $s \geqq 0$,

$$sf'(s) - (2k-3)f(s) = 2(k-2)\, s \geqq 0.$$

Hence, (3.18) follows from Lemma 3.1.

For $s \geqq 1$, $n \leqq 6$, let $h(s) = -ps^{p-1} + (p-1)\,s + 1$. Since $n \leqq 6$ we have $p \geqq 2$. Therefore $h''(s) = -p(p-1)(p-2)\, s^{p-3} \leqq 0$ and hence h is concave. Since $h(1) = 0$ and $h'(1) = -(p-1)^2$, we have $h(s) \leqq -(p-2)^2 (s-1) \leqq 0$.

For $s \geqq 0$, we have

$$\begin{aligned} sf'(s+1) - (2k-3)f(s+1) &= -p(s+1)^{p-1} + (p-1)(s+1) + 1 \\ &= h(s+1) \leqq 0. \end{aligned}$$

Hence (3.19) follows from Lemma 3.2. This proves the lemma.

For $i = 1, 2$, and $\gamma_i > 0$ let $\varrho_i : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions. Let φ_i satisfy

$$\begin{gathered} -\varphi_i'' = t^{-k}\,\varrho_i(t)\,\varphi_i, \\ \varphi_i(\infty) = \gamma_i, \quad \varphi_i'(\infty) = 0. \end{gathered} \tag{3.20}$$

Denote by $T_{0,i}$ and $T_{1,i}$ respectively the first zero and first turning point of φ_i (see (2.4)). Then

Lemma 3.4. (i) *Assume that* $T_{0,1}$ *exists and also that* $\varrho_2(t) \geqq \varrho_1(t)$ *for* $t \geqq T_{0,2}$. *Then* $T_{0,2} > 0$ *and* $T_{0,2} \geqq T_{0,1}$.

(ii) *Assume that* $T_{0,1}$ *and* $T_{1,1}$ *exist and also that* $\varrho_2(t) \geqq \varrho_1(t)$ *for* $t \geqq T_{1,2}$. *Then* $T_{1,2} > 0$ *and* $T_{1,2} \geqq T_{1,1}$.

Proof. Let $W = \varphi_1'\varphi_2 - \varphi_1'\varphi_2$. Then $W(\infty) = 0$ and

$$W'(t) = t^{-k}(\varrho_2 - \varrho_1)\,\varphi_1\varphi_2. \tag{3.21}$$

Suppose that (i) is not true. Then $T_{0,2} < T_{0,1}$ and hence from (3.21), $W'(t) \geqq 0$ for all $t \geqq T_{0,1}$. Therefore $W(T_{0,1}) \leqq 0$. But $W(T_{0,1}) = \varphi_1'(T_{0,1})\varphi_2(T_{0,1}) > 0$, which is a contradiction. This proves (i).

Suppose that (ii) is not true. Then $T_{1,2} < T_{1,1}$. From (i) it follows that $T_{0,1} < T_{0,2}$. Using (3.21), we obtain $W'(t) \geqq 0$ for $t \in [T_{1,1}, T_{0,1}]$. Therefore we have

$$\begin{aligned} 0 < -\varphi_1(T_{1,1})\,\varphi_2'(T_{1,1}) &= W(T_{1,1}) \leqq W(T_{0,1}) \\ &= \varphi_1'(T_{0,1})\,\varphi_2(T_{0,1}) < 0, \end{aligned}$$

which is a contradiction. This proves (ii) and hence the lemma.

Let φ, τ_0, τ_1 be as in (2.3) and (2.4). For $a > 0$, denote $\varphi(t, a) = \varphi(at)$ and let $\tau_{0,a}$ and $\tau_{1,a}$ be the first zero and first turning point of $\varphi(\cdot, a)$. Then we have

$$\begin{gathered} \tau_{0,a} = \frac{\tau_0}{a}, \quad \tau_{1,a} = \frac{\tau_1}{a}, \\ -\varphi''(\cdot, a) = a^{2-k}\, t^{-k}\, \varphi(\cdot, a), \\ \varphi(\infty, a) = 1, \quad \varphi'(\infty, a) = 0. \end{gathered} \tag{3.22}$$

Let $y(t, \gamma)$ and $S_1(\gamma)$ be as in (2.1) and (2.2). Define

$$S_0(\gamma) = \inf\{t, y(s, \gamma) \neq 1, y'(s, \gamma) \neq 0 \quad \forall\, s > t\}. \tag{3.23}$$

We then have

Lemma 3.5. *If* $\gamma \neq 0, 1$, *then* $S_0(\gamma)$ *exists and*

$$\lim_{\gamma \to 0} S_0(\gamma) = 0.$$

Proof. First consider the case $\gamma > 1$. Let

$$\begin{gathered} \varphi_2(t) = y(t, \gamma) - 1, \\ \varrho_2(t) = \frac{(\varphi_2 + 1)^p - (\varphi_2 + 1)}{\varphi_2}. \end{gathered}$$

Then φ_2 satisfies

$$\begin{gathered} -\varphi_2'' = t^{-k}\,\varrho_2\varphi_2, \\ \varphi_2(\infty) = \gamma - 1, \quad \varphi_2'(\infty) = 0. \end{gathered}$$

From (3.23) it follows that $S_0(\gamma)$ is the first zero of φ_2 and that $\varrho_2(t) \geqq (p - 1)$ for $t \geqq S_0(\gamma)$. Taking $\varrho_1 = (p - 1)$, $\varphi_1(t) = \varphi(t, (p - 1)^{-1/(k-2)})$ in (i) of Lemma 3.4, we conclude that $S_0(\gamma)$ exists and

$$(p - 1)^{1/(k-2)}\,\tau_0 \leqq S_0(\gamma). \tag{3.24}$$

Now consider the case $0 < \gamma < 1$. Let

$$\varphi_2 = 1 - y(t, \gamma),$$
$$\varrho_2(t) = \frac{(1 - \varphi_2) - (1 - \varphi_2)^p}{\varphi_2}.$$

Then φ_2 satisfies

$$-\varphi_2'' = t^{-k} \varrho_2(t) \varphi_2,$$
$$\varphi_2(\infty) = 1 - \gamma, \quad \varphi_2'(\infty) = 0,$$

with $S_0(\gamma)$ as its first zero. By taking $\varrho_1 = \min\{\varrho_2(t), t \geqq S_0(\gamma)\}$, $\varphi_1(t) = \varphi(t, \varrho_1^{1/(k-2)})$ in (i) of Lemma 3.4, we obtain the existence of $S_0(\gamma)$. Since $\varrho_2(t) \leqq (p-1)$ for $t \geqq S_0(\gamma)$, again from (i) of Lemma 3.4, we obtain

$$S_0(\gamma) \leqq (p-1)^{1/(k-2)} \tau_0. \tag{3.25}$$

Now suppose that $S_0(\gamma)$ does not tend to zero as γ approaches zero. Then by going to a subsequence and using (3.25), we have

$$\lim_{\gamma \to 0} S_0(\gamma) = S_0 > 0. \tag{3.26}$$

Since the boundedness of y implies that y' and y'' are uniformly bounded in $(S_0(\gamma), \infty)$, the Arzelà-Ascoli Theorem implies that there exists a subsequence such that $y(t, \gamma) \to y_0(t)$ uniformly on compact sets and that y_0 satisfies

$$-y_0'' = t^{-k}(y_0^p - y_0) \quad \text{in } (S_0, \infty), \tag{3.27}$$
$$y_0(\infty) = y_0'(\infty) = 0.$$

From the uniqueness of the solution of (3.27), $y_0 \equiv 0$. But $y_0(S_0) = 1$. This contradiction proves the lemma.

Proof of the Lemma A. From (2.1), it follows that y is increasing for $\gamma > 1$, and y is decreasing for $\gamma < 1$.

First consider the case $\gamma > 1$. Suppose $S_1(\gamma) = 0$. Then $y(t, \gamma) > 0$ for $t > 0$ by (3.18), and $y(t, \gamma)$ is an increasing function by (2.1). Since $y(S_0(\gamma), \gamma) = 1$, from Lemma 3.5 we can find a $C > 0$ such that for $t \in 0, (S_0(\gamma)/2)$,

$$1 - y^{p-1}(t, \gamma) \geqq C. \tag{3.28}$$

From (3.18), we can find a $C_1 > 0$ such that for $t \in (0, S_0(\gamma)/2)$,

$$y(t, \gamma) \geqq C_1 t. \tag{3.29}$$

Integrating (2.1) and using (3.28) and (3.29), we have

$$\infty > y'(S_0(\gamma)/2, \gamma) \geqq \int_0^{S_0(\gamma)/2} t^{-k} y(1 - y^{p-1}) \, dt$$
$$\geqq CC_1 \int_0^{S_0(\gamma)/2} t^{-k+1} \, dt = \infty,$$

which is a contradiction. Hence $S_1(\gamma) > 0$; from (3.18), we have $y(S_1(\gamma), \gamma > 0)$.

Let $v = y - 1$ and $f_1(s) = (s+1)^p - (s+1)$. Then $v(\infty) = \gamma - 1$, $v'(\infty) = 0$, and $S_0(\gamma)$ and $S_1(\gamma)$ respectively are the first zero and first turning points of v. Moreover, v satisfies

$$-v'' = t^{-k}\left(\frac{f_1(v)}{v}\right)v. \tag{3.30}$$

Now integrating (3.30) and using (3.24), we can find a $C > 0$ such that, for all $1 < \gamma \leqq 2$,

$$v'(S_0(\gamma)) = \int_{S_0(\gamma)}^{\infty} t^{-k}\left(\frac{f_1(v)}{v}\right)v\,dt \leqq C(\gamma - 1). \tag{3.31}$$

Since

$$\sup\left\{\frac{f_1(v)}{v}\,;\, t \geqq S_0(\gamma), \gamma \in (1, 2]\right\} < \infty,$$

as a consequence of (i) of Lemma 3.4, $S_0(\gamma)$ is bounded for $\gamma \in (0, 2]$. From this and from (3.31), we can find a $C_1 > 0$ such that for $1 < \gamma \leqq 2$,

$$|v(S_1(\gamma))| \leqq v'(S_0(\gamma))\,(S_0(\gamma) - S_1(\gamma)) \leqq C(\gamma - 1). \tag{3.32}$$

This inequality implies that for any $\varepsilon > 0$, we can find a $\delta > 0$ such that whenever $\gamma - 1 \leqq \delta$, $t \geqq S_1(\gamma)$,

$$(1-\varepsilon)(p-1) \leqq \frac{f_1(v(t))}{v(t)} \leqq (1+\varepsilon)(p-1). \tag{3.33}$$

From (3.22), (3.33) and (ii) of Lemma 3.4, we obtain

$$[(1-\varepsilon)(p-1)]^{1/(k-2)}\tau_1 \leqq S_1(\gamma) \leqq [(1+\varepsilon)(p-1)]^{1/(k-2)}\tau_1$$

for $\gamma \leqq 1 + \delta$. This inequality implies that

$$\lim_{\gamma \to 1} S_1(\gamma) = (p-1)^{1/(k-2)}\tau_1. \tag{3.34}$$

Now consider the case in which $0 < \gamma < 1$. Suppose $S_1(\gamma) = 0$. From Lemma 3.5, $S_0(\gamma)$ exists and $y(S_0(\gamma), \gamma) = 1$. Hence from (2.1),

$$-y'(t, \gamma) \leqq -y'(S_0(\gamma), \gamma) \tag{3.35}$$

for all $t \in (0, (S_0(\gamma))$. Also we can find a $C > 0$ such that for $t \in (0, S_0(\gamma)/2)$,

$$y^p(t, \gamma) - y(t, \gamma) \geqq C. \tag{3.36}$$

Integrating (2.1) and using (3.35) and (3.36), we have

$$\begin{aligned} -y'((S_0(\gamma), \gamma) \geqq -y'(t, \gamma) &\geqq \int_t^{S_0(\gamma)/2} s^{-k}(y^p - y)\,dt \\ &\geqq C\left[\frac{1}{t^{k-1}} - \left(\frac{2}{S_0(\gamma)}\right)^{k-1}\right] \to \infty \end{aligned}$$

as $t \to 0$, which is a contradiction. This implies that $S_1(\gamma)$ exists.

Let $v = 1 - y$ and $f_1(s) = (1 - s) - (1 - s)^p$. Then $v(\infty) = 1 - \gamma$, $v'(\infty) = 0$, $S_0(\gamma)$ and $S_1(\gamma)$ are the first zero and first turning points of v. Moreover, v satisfies

$$-v'' = t^{-k}\left(\frac{f_1(v)}{v}\right)v. \tag{3.37}$$

Since

$$\inf\left\{\frac{f_1(v)}{v}\,;\, t \geqq S_0(\gamma), \tfrac{1}{2} \leqq \gamma < 1\right\} > 0,$$

by Lemma 3.4(i) and by (3.22) we have

$$\inf\{S_0(\gamma); \tfrac{1}{2} \leqq \gamma < 1\} > 0.$$

Therefore by integrating (3.37), we have for some constant $C > 0$,

$$v'(S_0(\gamma)) = \int_{S_0(\gamma)}^{\infty} t^{-k}\left(\frac{f_1(v)}{v}\right)v\,dt \leqq C(1 - \gamma). \tag{3.38}$$

From (3.25), (3.38) and the mean value theorem, we can find a $C_1 > 0$ such that

$$|v(S_1(\gamma))| \leqq |v'(S_0(\gamma))|\,(S_0(\gamma) - S_1(\gamma)) \leqq C_1(1 - \gamma).$$

This implies that for every $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$(1 - \varepsilon)(p - 1) \leqq \frac{f_1(v)}{v} \leqq (1 + \varepsilon)(p - 1) \tag{3.39}$$

whenever $1 - \gamma \leqq \delta$ and $t \geqq S_1(\gamma)$. From (3.22), (3.39), and Lemma 3.4(ii) we obtain

$$[(1 - \varepsilon)(p - 1)]^{1/(k-2)}\,\tau_1 \leqq S_1(\gamma) \leqq [(1 + \varepsilon)(p - 1)]^{1/(k-2)}\,\tau_1$$

for $1 - \gamma \leqq \delta$. This inequality implies that

$$\lim_{\gamma\to 1} S_1(\gamma) = (p - 1)^{1/(k-2)}\,\tau_1. \tag{3.40}$$

Since $S_1(\gamma) < S_0(\gamma)$, from Lemma 3.5 we have $\lim_{\gamma\to 0} S_1(\gamma) = 0$. Now the lemma follows from (3.34) and (3.40).

4. Proof of Lemma B

Let $n \leqq 6$ and $\gamma > 1$. Let $y(t, \gamma)$, $S_1(\gamma)$, and $S_0(\gamma)$ be as in (2.1), (2.2), (3.23), respectively. For the sequel we use C, C_1, C_2, etc., to denote positive constants independent of γ, but which may be different in different inequalities. We have the following

Lemma 4.1. *For γ large,*

$$S_0(\gamma) = O(\gamma), \tag{4.1}$$

$$y(t, \gamma) \leqq 1 + Ct/\gamma \quad \text{for } t \geqq S_0(\gamma), \tag{4.2}$$

$$y(\gamma^2, \gamma) \geqq C\gamma, \tag{4.3}$$

$$C_1/\gamma \leqq y'(2S_0(\gamma), \gamma) \leqq C_2/\gamma. \tag{4.4}$$

For $t \in (2S_0(\gamma), \gamma^2)$

$$1 + \frac{C_1(t - S_0(\gamma))}{\gamma} \leqq y(t, \gamma) \leqq 1 + \frac{C_2(t - S_0(\gamma))}{\gamma}. \tag{4.5}$$

Proof. By Lemma (3.5), $S_0(\gamma)$ exists and from (3.18),

$$Z_1(S_0(\gamma), \gamma) \leqq y(S_0(\gamma), \gamma) = 1. \tag{4.6}$$

This implies that

$$S_0(\gamma)^{k-2} \leqq \frac{\gamma^{p-1}}{(k-1)(\gamma^{k-2} - 1)}. \tag{4.7}$$

Since $p = 2k - 3$, it follows from (4.7) that $S_0(\gamma) = O(\gamma)$ as $\gamma \to \infty$. This proves (4.1).

For large γ we have, $\dfrac{\gamma^p - \gamma}{(k-1)(\gamma - 1)} \geqq C\gamma^{2(k-2)}$ and hence from (3.19),

$$y(t, \gamma) \leqq 1 + \frac{\gamma t}{\left\{t^{k-2} + \dfrac{\gamma^p - \gamma}{(k-1)(\gamma-1)}\right\}^{1/(k-2)}} \leqq 1 + \frac{Ct}{\gamma}.$$

for all $t \geqq S_0(\gamma)$. This proves (4.2).

Again from (3.18), we have

$$y(\gamma^2, \gamma) \geqq Z_1(\gamma^2, \gamma) \geqq C\gamma$$

for γ large. This proves (4.3).

From the concavity of y in $[S_0(\gamma), 2S_0(\gamma)]$ and from (4.2) we have for large γ that

$$y'(2S_0(\gamma), \gamma) \leqq \frac{y(2S_0(\gamma), \gamma) - 1}{S_0(\gamma)} \leqq \frac{C_2 S_0(\gamma)}{\gamma S_0(\gamma)} = \frac{C_2}{\gamma}. \tag{4.8}$$

Again, from the concavity of y in $[2S_0(\gamma), \gamma^2]$ and from (4.1)–(4.3), we have for large γ that

$$y'(2S_0(\gamma), \gamma) \geqq \frac{y(\gamma^2, \gamma) - y(2S_0(\gamma), \gamma)}{\gamma^2 - 2S_0(\gamma)} \geqq \frac{C\gamma + O(1)}{\gamma^2 + O(\gamma)} \geqq \frac{C_1}{\gamma}.$$

This together with (4.8)proves (4.4).

Let $t \in [2S_0(\gamma), \gamma^2]$. From (4.2), we then have

$$y(t, \gamma) \leqq 1 + \frac{Ct}{\gamma} = 1 + \frac{C(t - S_0(\gamma))}{\gamma} \frac{t}{(t - S_0(\gamma))} \leqq 1 + C_2 \frac{(t - S_0(\gamma))}{\gamma}. \tag{4.9}$$

From the concavity of y in $[S_0(\gamma), \gamma^2]$ and from (4.1)–(4.3), it follows that

$$\begin{aligned} \frac{y(t, \gamma) - y(S_0(\gamma), \gamma)}{t - S_0(\gamma)} &\geqq \frac{y(\gamma^2, \gamma) - y(S_0(\gamma), \gamma)}{\gamma^2 - S_0(\gamma)} \\ &\geqq \frac{C\gamma + O(1)}{\gamma^2 + O(\gamma)} \geqq \frac{C_1}{\gamma} \end{aligned}$$

for γ large and for $t \in [S_0(\gamma), \gamma^2]$. Hence

$$y(t,\gamma) \geqq 1 + \frac{C_1(t - S_0(\gamma))}{\gamma}.$$

This together with (4.9) proves (4.5) and hence the lemma.

Lemma 4.2. $\varliminf_{\gamma \to \infty} S_0(\gamma) > 0.$

Proof. Integrating (2.1) and using (4.4) and (4.5) we obtain for γ large that

$$\begin{aligned}
\frac{C_2}{\gamma} &\geqq y'(2S_0(\gamma), \gamma) = \int_{2S_0(\gamma)}^{\infty} t^{-k}(y^p - y)\, dt \\
&\geqq (p-1) \int_{2S_0(\gamma)}^{\gamma^2} t^{-k}(y-1)\, dt \geqq \frac{C}{\gamma} \int_{2S_0(\gamma)}^{\gamma^2} t^{-k}(t - S_0(\gamma))\, dt \\
&\geqq \frac{CS_0(\gamma)^{2-k}}{\gamma} \int_{2}^{\gamma^2/S_0(\gamma)} t^{-k}(t-1)\,dt \geqq \frac{CS_0(\gamma)^{2-k}}{\gamma},
\end{aligned}$$

which implies that $\varliminf_{\gamma \to \infty} S_0(\gamma) \geqq C_4 > 0$, since $\gamma^2/S_0(\gamma) \to \infty$ as $\gamma \to \infty$ and $k > 2$. This proves the lemma.

Lemma 4.3. *For γ large,*

$$C_1 \gamma \leqq S_0(\gamma) \leqq C_2 \gamma, \tag{4.10}$$

$$C_1/\gamma \leqq y'(S_0(\gamma), \gamma) \leqq C_2/\gamma. \tag{4.11}$$

Proof. Let $v = y - 1$ and $f_1(s) = (s+1)^p - (s+1)$. Then v satisfies

$$\begin{aligned}
-v'' &= t^{-k} f_1(v), \\
v(\infty) &= \gamma - 1, \quad v'(\infty) = 0
\end{aligned} \tag{4.12}$$

and $S_0(\gamma)$ is the first zero of v. Let $F_1(s)$ be the primitive of f_1 and let

$$H(t) = \tfrac{1}{2} tv'^2 - \tfrac{1}{2} vv' + t^{1-k} F_1(v).$$

Then from (3.6) we have

$$-H'(t) = \frac{t^{-k}}{2} h(v+1), \tag{4.13}$$

where

$$h(s) = s^p - \frac{p-1}{2} s^2 - s + \left(\frac{p-1}{2}\right).$$

Since $n \leqq 6$, we have $p \geqq 2$, and therefore we obtain that h is convex for $s \geqq 1$ and satisfies

$$h(s) \geqq C(s-1)^p \tag{4.14}$$

for $s \geqq 1$. Integrating (4.13) and using (4.14) and (4.5) we obtain

$$\begin{aligned} H(2S_0(\gamma)) &= \int_{2S_0(\gamma)}^{\infty} t^{-k}\, h(v+1)\, dt \\ &\geqq C \int_{S_0(\gamma)}^{\gamma^2} t^{-k} \frac{(t - S_0(\gamma))^p}{\gamma}\, dt \\ &= \frac{CS_0(\gamma)^{p-k+1}}{\gamma^p} \int_{\gamma^2/(2S_0(\gamma))}^{\gamma^2/S_0(\gamma)} t^{-k+p}\, dt = C/\gamma, \end{aligned} \tag{4.15}$$

since $p = 2k - 3$. On the other hand, we have from (4.1) and (4.4), that

$$\begin{aligned} H(2S_0(\gamma)) &\leqq S_0(\gamma)\, v'(S_0(\gamma))^2 + 2^{1-k}\, S_0(\gamma)^{1-k}\, F_1(v(2S_0(\gamma))) \\ &\leqq C_1 \left\{\frac{S_0(\gamma)}{\gamma^2} + S_0(\gamma)^{1-k}\, F_1\left(C_2 \frac{S_0(\gamma)}{\gamma}\right)\right\}. \end{aligned} \tag{4.16}$$

Now we assert that

$$\varliminf_{\gamma \to \infty} \frac{S_0(\gamma)}{\gamma} > 0. \tag{4.17}$$

Suppose (4.17) is not true. Then for a subsequence $\gamma \to \infty$, we can find $C_3 > 0$ such that

$$F_1\left(C_2 \frac{S_0(\gamma)}{\gamma}\right) \leqq C_3 \left(\frac{S_0(\gamma)}{\gamma}\right)^2. \tag{4.18}$$

From (4.15), (4.16) and (4.18) we have

$$\begin{aligned} C/\gamma \leqq H(2S_0(\gamma)) &\leqq C_4 \left\{\frac{S_0(\gamma)}{\gamma^2} + S_0(\gamma)^{1-k} \left(\frac{S_0(\gamma)}{\gamma}\right)^2\right\} \\ &\leqq \frac{C_4}{\gamma} \left(\frac{S_0(\gamma)}{\gamma}\right) \left\{1 + \frac{1}{S_0(\gamma)^{k-2}}\right\}. \end{aligned}$$

This, together with Lemma (4.2), implies that

$$0 < C_5 \leqq \left(\frac{S_0(\gamma)}{\gamma}\right) \to 0$$

as $\gamma \to \infty$, which is a contradiction. This proves (4.17). Now (4.10) follows from (4.1) and (4.17).

From the concavity of y and (4.4), we have

$$y'(S_0(\gamma), \gamma) \geqq y'(2S_0(\gamma), \gamma) \geqq \frac{C_1}{\gamma}. \tag{4.19}$$

For γ large it follows from (4.2) and (4.1) that $y(t, \gamma) \leqq C$ for $t \in [S_0(\gamma), 2S_0(\gamma)]$. Hence from (4.4) and (4.10) we have

$$y'(S_0(\gamma), \gamma) = y'(2S_0(\gamma), \gamma) + \int_{S_0(\gamma)}^{2S_0(\gamma)} t^{-k}(y^p - y)\, dt$$
$$\leqq \frac{C}{\gamma} + O\left(\frac{1}{\gamma^{k-1}}\right) \leqq \frac{C_2}{\gamma}.$$

This, together with (4.19), proves (4.11) and the lemma.

Proof of Lemma B. Inequality (2.8) follows from (3.18).

Let $t_0 \in [S_1(\gamma), S_0(\gamma)]$ be such that

$$y'(t_0, \gamma) = \frac{y'(S_0(\gamma), \gamma)}{2}. \tag{4.20}$$

Then from (4.11), (4.20), and the restriction that $0 < y \leqq 1$, we obtain for γ large that

$$C_1/\gamma \leqq \frac{y'(S_0(\gamma), \gamma)}{2} = \int_{t_0}^{S_0(\gamma)} t^{-k}(y - y^p)\, dt \leqq \frac{C}{t_0^{k-1}},$$

that is,

$$t_0 \leqq C_1 \gamma^{1/(k-1)}. \tag{4.21}$$

Let $S(\gamma) = C_1 \gamma^{1/(k-1)}$; then clearly from (4.20) we have

$$S_1(\gamma) \leqq S(\gamma), \tag{4.22}$$

$$C_3/\gamma \leqq \frac{y'(S_0(\gamma), \gamma)}{2} = y'(t_0, y)$$
$$\leqq y'(S(\gamma), \gamma) \leqq y'(S_0(\gamma), \gamma) \leqq C_4/\gamma. \tag{4.23}$$

This, together with (4.21) and (4.22), proves (2.9) and (2.11).

Now from the convexity of y in $[S_1(\gamma), S_0(\gamma)]$ and (4.23) we have

$$\frac{1 - y(S(\gamma), \gamma)}{S_0(\gamma) - S(\gamma)} \geqq y'(S(\gamma), \gamma) \geqq C_3/\gamma. \tag{4.24}$$

From (4.24) and (4.10), we have

$$1 - y(S(\gamma), \gamma) \geqq C - O\left(\frac{1}{\gamma^{(k-2)/(k-1)}}\right).$$

Hence we can find a $\delta > 0$ such that for γ large, $1 - y(S(\gamma), \gamma) \geqq \delta$ and this proves (2.10).

Since $S(\gamma) = O\left(\gamma^{\frac{1}{(k-1)}}\right)$, from (3.18) we get

$$y(t, \gamma) \geqq Z_1(t, \gamma) \geqq Ct/\gamma. \tag{4.25}$$

for all $t \in [S_1(\gamma), S(\gamma)]$. From (2.9), (2.10), (2.11) and (4.25) we have

$$C/\gamma \geqq y'(S(\gamma), \gamma) = \int_{S_1(\gamma)}^{S(\gamma)} t^{-k} y(1 - y^{p-1})\, dt \geqq \frac{C_1\, \delta}{\gamma} \int_{S_1(\gamma)}^{S(\gamma)} t^{-k+1}\, dt$$

$$\geqq \frac{C_2}{\gamma}\left(\frac{1}{S_1(\gamma)^{k-2}} - \frac{1}{S(\gamma)^{k-2}}\right).$$

This implies that

$$\varliminf_{\gamma \to \infty} S_1(\gamma) > 0.$$

This proves (2.12) and hence the lemma.

Remark 1. *Let $n \geqq 3$ and $p > 1$. Then there exists an $R_0 > 0$ such that for $0 < R < R_0$, the problem*

$$\begin{aligned} -\Delta u = u^p - u \quad & \text{in } B(R), \\ u > 0,\ u \text{ is radial} \quad & \text{in } B(R), \\ \frac{\partial u}{\partial \nu} = 0 \quad & \text{in } \partial B(R) \end{aligned} \tag{4.26}$$

does not admit any solution u such that u' changes sign.

Proof. We consider two cases: $1 < p < \frac{n+2}{n-2}$ and $p \geqq \frac{n+2}{n-2}$.

Case 1. $1 < p < \frac{n+2}{n-2}$. In this situation, by a result of Lin, Ni & Tagaki [11] there exists an $R_0 > 0$ such that for $0 < R < R_0$, problem (4.26) does not admit a nonconstant solution. This proves the remark.

Case 2. $p \geqq \frac{n+2}{n-2}$. Let $v(r, \gamma)$ denote the solution of

$$-\left(v'' + \frac{n-1}{r} v'\right) = v^p - v \quad \text{in } (0, \infty),$$

$$v(0) = \gamma > 0, \quad v'(0) = 0.$$

Let $R_1(\gamma) < R_2(\gamma) < \dots$ be the turning points (i.e., $v'(R_i(\gamma), \gamma) = 0$) of $v(r, \gamma)$. From the result of Ni [12], we know that $v(r, \gamma) > 0$ for all $\gamma > 0$.

Now the remark follows from the following

Assertion. *There exists a constant $C > 0$ such that*

$$\sup_{\gamma \in (0, \infty)} R_2(\gamma) \geqq C. \tag{4.27}$$

To prove this we adopt the method used in ATKINSON, BREZIS & PELETIER [6] and in ADIMURTHI & YADAVA [2]. Proceeding as in Lemma A, we obtain

$$\lim_{\gamma\to 0} R_1(\gamma) = \infty, \quad \lim_{\gamma\to 1} R_1(\gamma) > 0.$$

Therefore it is sufficient to prove that

$$\sup_{\gamma\in(1,\infty)} R_2(\gamma) \geqq C. \tag{4.28}$$

Let $w(r,\gamma) = v(r,\gamma) - 1$ and let $T_1(\gamma)$ and $T_2(\gamma)$ respectively be the first and second zeros of $w(r,\gamma)$. Then

$$T_1(\gamma) < R_1(\gamma) < T_2(\gamma) < R_2(\gamma).$$

Therefore, in order to prove (4.28), it is sufficient to show that

$$\sup_{\gamma\in(1,\infty)} T_2(\gamma) \geqq C. \tag{4.29}$$

Since $v(r,\gamma) > 0$ for all $\gamma > 1$, we get

$$\sup_{\gamma\in(1,\infty)} \{|w(r,\gamma)|; T_1(\gamma) < r < T_2(\gamma)\} \leqq 1. \tag{4.30}$$

Let $Z(r) = \left(\frac{n-2}{r}\right)^{\frac{n-2}{2}}$. Then Z satisfies

$$Z'' + \left(\frac{n-1}{r}\right) Z' + \tfrac{1}{4} Z^{4/(n-2)} Z = 0 \quad \text{in } (0,\infty), \tag{4.31}$$

$$\lim_{r\to 0} Z(r) = \infty.$$

From (4.30) and (4.31) we can choose an $r_0 > 0$ such that for all $\gamma > 1$ and $r \in (0, r_0) \cap [T_1(\gamma), T_2(\gamma)]$,

$$\frac{(w+1)^p - (w+1)}{w} < \tfrac{1}{4} Z(r)^{4/(n-2}.$$

Now by Sturm's comparison theorem, there exists a $C > 0$ such that (4.29) holds. This completes the proof of the remark.

Remark 2. *Given any Ω, we can construct a negative function $\alpha \in C^\infty(\Omega)$ such that the problem*

$$\begin{aligned} -\Delta u &= u^p + \alpha(x)\, u \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.32}$$

admits a solution.

The construction of α is similar to the construction given by BREZIS [7] for the Dirichlet problem.

Let $a \in C^\infty(\Omega)$, be such that a changes sign in Ω and $\int_\Omega a(x)\,dx < 0$. By the result of HESS & SENN [14] there exists a $\lambda_1(\Omega) > 0$ such that

$$-\Delta v = \lambda_1(\Omega)\,a(x)\,v \quad \text{in } \Omega,$$

$$v > 0 \quad \text{in } \Omega \text{ and}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

admits a solution. Define

$$\alpha(x) = \lambda_1(\Omega)\,a(x) - \mu^{p-1}\,v^{p-1}, \quad u = \mu v,$$

where μ is a positive real number. Obviously u satisfies (4.32). By choosing μ large, we get $\alpha < 0$.

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References

1. ADIMURTHI & G. MANCINI, The Neumann problem for elliptic equations with critical non-linearity, preprint.
2. ADIMURTHI & S. L. YADAVA, Elementary proof of the non-existence of nodal solutions for the semilinear elliptic equations with Critical Sobolev exponent, *Nonlin. Anal., T.M.A.* **14** (1990), 785–787.
3. ADIMURTHI & S. L. YADAVA, Critical Sobolev exponent problem in $\mathbb{R}^n$ ($n \geqq 4$) with Neumann boundary condition, *Proc. Ind. Acad. of Sci.*, to appear.
4. ADIMURTHI, M. C. KNAAP & S. L. YADAVA, A note on a critical exponent problem with Neumann boundary conditions, preprint.
5. F. V. ATKINSON & L. A. PELETIER, Emden-Fowler equations involving critical exponents, *Nonlin. Anal. T.M.A.* **10** (1986), 755–776.
6. F. V. ATKINSON, H. BRREZIS & L. A. PELETIER, Solutions qui changent de signe d'equations elliptiques avec exposant de Sobolev critique, *C. R. Acad. Sci., Paris* **306** (1988), 711–714.
7. H. BREZIS, Nonlinear elliptic equations involving the Critical Sobolev exponent — survey and perspectives, *Directions in partial differential equations*, edited by M. G. CRANDALL et al. 1987, 17–36.
8. H. BREZIS & L. NIRENBERG, Positive solutions of nonlinear elliptic equations involving critical exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.
9. C. BUDD, M. C. KNAAP & L. A. PELETIER, Asymptotic behaviour of solutions of elliptic equations with critical exponents and Neumann boundary conditions, preprint.
10. C. S. LIN & W. M. NI, On the diffusion coefficient of a semilinear Neumann problem, *Springer Lecture Notes* **340**, 1986.

11. C. S. LIN, W. M. NI & I. TAKAGI, Large amplitude stationary solutions to a chemotaxis system, *J. Diff. Eqns.* **72** (1988), 1–27.
12. W. M. NI, On he positive radial solutions of some semilinear elliptic equations on $\mathbb{R}^n$, *Appl. Math. Optim.* **9** (1983), 373–380.
13. X. J. WANG, Neumann problem of semilinear elliptic equations involving critical Sobolev exponents, *J. Diff. Eqns.*, to appear.
14. P. HESS & S. SENN, On positive solutions of a linear elliptic eigenvalue problem with Neumann boundary conditions, *Math. Annalen* **258** (1982), 459–470.

Tata Institute of Fundamental Research
Post. Box 1234
Bangalore-560 012
INDIA

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