Critical Sobolev exponent problem in $\mathbb{R}^n (n \geq 4)$ with Neumann boundary condition

ADIMURTHI and S L YADAVA
TIFR Centre, P. B. 1234, Bangalore 560 012, India

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Abstract. In this paper we study the existence and non existence of positive solution for the critical Sobolev exponent problem

$$-\Delta u = u^{(p+2)/(n-2)} + \lambda(x)u \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial v} = 0 \quad \text{on } \partial \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n (n \geq 4)$.

Keywords. Critical exponent; flatness condition; Neumann boundary.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. Let $\alpha \in C^0(\bar{\Omega})$ and consider the following problem:

$$-\Delta u = u^{(p+2)/(n-2)} + \lambda(x)u \quad \text{in } \Omega$$

$$u > 0$$

$$\frac{\partial u}{\partial v} = 0 \quad \text{on } \partial \Omega,$$

(1)

where $\lambda \in \mathbb{R}$. Problem (1) with Dirichlet boundary condition instead of Neumann, has been studied by Brezis-Nirenberg [5] in detail. For the Neumann boundary condition, namely, for problem (1), Brezis [3] raised the following question:

"Under what conditions on $\alpha$ and $\Omega$, problem (1) admits a solution?".

When $n = 2$, and the nonlinearity $u^{(p+2)/(n-2)}$ is replaced by $u^2 \exp(bu^2)$, $b > 0$, problem (1) has been studied by authors in [1]. More precisely, in [1] it has been shown that, under suitable assumptions on $\alpha$, there exists a $\lambda(\alpha) > 0$ such that

$$-\Delta u = u^2 \exp(bu^2) + \lambda(x)u \quad \text{in } \Omega$$

$$u > 0$$

$$\frac{\partial u}{\partial v} = 0 \quad \text{on } \partial \Omega$$

(2)

admits a solution if and only if $\lambda \in (0, \lambda(\alpha))$. 275
In this paper we have made an attempt to answer the question of Brezis (see also Cherrier [9] for some partial results for the problem similar to (1)). Obviously if \( \lambda \sigma > 0 \), then (1) does not admit any solution. Here we consider problem (1) when \( \sigma \) changes sign in \( \Omega \). If \( \int_{\Omega} \sigma(x) \, dx = 0 \), then (1) does not admit any solution (see remark 2 in §4).

Let \( \sigma \) change sign in \( \Omega \) and \( \int_{\Omega} \sigma(x) \, dx < 0 \). Let \( \lambda(\sigma) > 0 \) be the unique real number such that

\[
- \Delta \varphi = \lambda(\sigma) \sigma(x) \varphi \quad \text{in } \Omega
\]
\[
\varphi > 0
\]
\[
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial \Omega
\]

admits a solution. For the existence of such \( \lambda(\sigma) \), we refer Brown–Lin [6] and Senn–Hess [12]. Now by exploiting the techniques used in Brezis–Nirenberg [5] and Adimurthi–Yadava [1], we prove the following

**Theorem.** Let \( n \geq 4 \) and assume that

(i) \( \sigma \) changes sign in \( \Omega \) and \( \int_{\Omega} \sigma(x) \, dx < 0 \). Let \( \lambda(\sigma) \) be given by (3).

(ii) There exists a \( x_0 \in \partial \Omega \) such that \( \sigma(x_0) > 0 \) and \( \partial \Omega \) is flat of order \( k \geq 3 \) at \( x_0 \).

Then problem (1) admits a solution \( u \in C^2(\overline{\Omega}) \) if and only if \( \lambda \in (0, \lambda(\sigma)) \).

For the meaning of flatness of \( \partial \Omega \) of order \( k \) at \( x_0 \), see definition (2.1). Here we remark that the flatness condition in the theorem is not satisfied for the ball.

2. Preliminaries

Let \( H^1(\Omega) \) denote the usual Sobolev space. For \( u \in H^1(\Omega) \) and \( 1 \leq p \leq 2n/(n-2) \), let

\[
|\nabla u|^2_{L^2(\Omega)} = \int_{\Omega} |\nabla u|^2 \, dx
\]
\[
|u|^p_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p \, dx \right)^{1/p}
\]

To prove the theorem, we need the following

**PROPOSITION 2.**

Let \( u \in C^2(\overline{\Omega}) \) be such that \( \sigma \) changes sign in \( \Omega \) and \( \int_{\Omega} \sigma(x) \, dx < 0 \). Let \( \lambda(\sigma) \) be given by (3). Then we have

(i) for all \( \lambda \in (0, \lambda(\sigma)) \),

\[
\left\{ \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} \sigma(x) u^2 \, dx \right\}^{1/2}
\]

defines an equivalent norm on \( H^1(\Omega) \).
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(ii) Let $u \in H^2(\Omega) \cap C(\bar{\Omega})$ be such that
\[
\lambda \neq 0, \lambda(\alpha)
\]
\[
\Delta u + \lambda xu \neq 0
\]
\[
\Delta u + \lambda xu \leq 0, \quad \frac{\partial u}{\partial v} = 0,
\]
then $u$ cannot be positive.

For the proof of above proposition we refer Brown–Lin [6] (Theorems 3.10 and 3.11) and Senn–Hess [12] (Proposition 6).

Flatness condition

Let $x_0 \in \partial \Omega$. After a translation and rotation, we assume that $x_0 = 0$ and there exist $R > 0$ and $\rho : B(0, R) \cap \{x_n = 0\} \rightarrow \mathbb{R}$ a smooth function such that
\[
\rho(0) = 0, \nabla \rho(0) = 0
\]
\[
\Omega \cap B(0, R) = \{x \in B(0, R); x_n > \rho(x')\}
\]
\[
\partial \Omega \cap B(0, R) = \{x \in B(0, R); x_n = \rho(x')\}
\]
where $x' = (x_1, \ldots, x_{n-1}, 0)$.

DEFINITION 2.1.

We say that $\partial \Omega$ is flat of order $k$ at 0 if $\rho(x') = 0(|x'|^k)$ as $|x'| \to 0$.

3. Proof of the theorem

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with smooth boundary. Let $\Gamma_0, \Gamma_1$ be disjoint submanifolds of $\partial \Omega$ such that $\partial \Omega = \Gamma_0 \cup \Gamma_1$ and let
\[
H^1(\Gamma_0) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}.
\]
(4)

Let $a \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma_1)$ be such that
\[
\left\{ \int_{\Omega} |Vu|^2 \, dx - \int_{\Omega} au^2 \, dx + \int_{\Gamma_1} bu^2 \, dx \right\}^{1/2}
\]
defines an equivalent norm on $H^1(\Gamma_0)$. We denote this norm by $\|u\|$. For $u \in H^1(\Gamma_0)$ and $p = (n+2)/(n-2)$, define
\[
J(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p + 1} \int_{\Omega} |u|^{p+1} \, dx
\]
(6)

\[
Q(u) = \left( \int_{\Omega} |u|^{p+1} \, dx \right)^{2/p+1}
\]
(7)

\[
S(\Gamma_0, a, b) = \inf \{Q(u); u \in H^1(\Gamma_0)\} \setminus \{0\}.
\]
(8)

Let
\[
S = \inf \left\{ \int_{\Omega} |Vu|^2 \, dx; u \in H^1(\Omega), \int_{\Omega} |u|^{2n/(n-2)} \, dx = 1 \right\}
\]
(9)

be the best Sobolev constant. Then by Cherrier [7, 8] we have
Lemma 3.1. For every \( \varepsilon > 0 \), there exists \( C(\varepsilon) > 0 \) such that for all \( u \in H^1(\Omega) \),
\[
\left( \int_\Omega |u|^{p-1} \, dx \right)^{2/(p+1)} \leq \left( \frac{2^{2^p}}{S} + \varepsilon \right) \int_\Omega |\nabla u|^2 \, dx + C(\varepsilon) \int_\Omega u^2 \, dx.
\]
For the proof of this lemma we also refer Aubin [2] (Theorem 2.30).

Now by using the above mentioned Cherrier's result and Brezis–Lieb lemma [4], we have

Lemma 3.2.
(i) \( S(\Gamma_0, a, b) > 0 \)
(ii) Assume \( S(\Gamma_0, a, b) < 2^{2^p} \), then there exists a \( \nu > 0 \) such that \( S(\Gamma_0, a, b) = \mathcal{Q}(\nu) \).

Further if we define \( u_\nu = S(\Gamma_0, a, b)^{\nu^{-2^{2^p}}}/\nu \), then \( u_\nu \) satisfies
\[
- \Delta u_\nu = u_\nu^{(p+2)/(\alpha-2)} + a(x)u_\nu \quad \text{in } \Omega
\]
\( u_\nu > 0 \)
\( u_\nu = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial u_\nu}{\partial y} + bu_\nu = 0 \quad \text{on } \Gamma_1 \)

and \( J(u_\nu) < (S_{2^p}/2)\nu \).

Proof. (i) By Sobolev imbedding theorem, there exists a constant \( C > 0 \) such that for all \( u \in H^1(\Gamma_0) \),
\[
\left( \int_\Omega |u|^{p+1} \, dx \right)^{2/(p+1)} \leq C ||u||^2.
\]
Now (i) follows from the definition of \( S(\Gamma_0, a, b) \). (ii) Let \( \{u_k\} \) be a minimizing sequence in (8) with \( \int_\Omega |u_k|^{p+1} \, dx = 1 \). Let for a subsequence, \( u_k \to v \) weakly in \( H^1(\Gamma_0) \) and almost everywhere.

Claim 1. \( v \neq 0 \).

Suppose \( v = 0 \). Then by Rellich lemma and lemma (3.1) we have
\[
\lim_{k \to \infty} \int_\Omega |\nabla u_k|^2 \, dx = \lim_{k \to \infty} ||u_k||^2 = S(\Gamma_0, a, b)
\]
\[
\leq S(\Gamma_0, a, b) \left( \frac{2^{2^p}}{S} + \varepsilon \right) \lim_{k \to \infty} \int_\Omega |\nabla u_k|^2 \, dx
\]
for every \( \varepsilon > 0 \). Hence we have,
\[
1 \leq S(\Gamma_0, a, b) \left( \frac{2^{2^p}}{S} + \varepsilon \right).
\]
This contradicts \( S(\Gamma_0, a, b) < S/2^{2^p} \). Hence \( v \neq 0 \).
Claim 2. \( Q(v) = S(\Gamma_0, a, b) \).

Let \( v_k = u_k - v \). Then \( v_k \to 0 \) weakly and almost everywhere. Now by Rellich lemma
\[
\|u_k\|^2 = \|v\|^2 + \|v_k\|^2 + o(1)
= \|v\|^2 + |\nabla v_k|^2, + o(1)
\]
which gives
\[
S(\Gamma_0, a, b) = \|v\|^2 + |\nabla v_k|^2, + o(1).
\]

By Brezis–Lieb lemma \([4]\) and lemma 3.1, we have
\[
1 = |u_k|^2 + o(1) \\
\leq |u|^2 + \left( \frac{22\pi}{5} + \varepsilon \right) |\nabla v_k|^2, + o(1)
\]
for every \( \varepsilon > 0 \). Hence
\[
S(\Gamma_0, a, b) \leq S(\Gamma_0, a, b)|u|^2 + |\nabla v_k|^2, + o(1)
\]
From (11) and (12), we get
\[
\frac{\|v\|^2}{|u|^2} \leq S(\Gamma_0, a, b).
\]

Hence \( v \) is a minimizer in (8).

Since \( Q(v) = Q(|v|) \), we may assume \( v \geq 0 \). Finally if we take \( u_0 = S(\Gamma_0, a, b)^{2/3} \), then it is easy to check that \( u_0 \) satisfies (10) and \( J(u_0) < S^4/2n \). This completes the proof of the lemma.

Lemma 3.3. Let \( \alpha \in C(\Omega) \). Assume that there exists some \( x_0 \in \partial \Omega \) such that \( \alpha(x_0) > 0 \) and \( \partial \Omega \) is flat of order \( k > 3 \) at \( x_0 \). Then for every \( \lambda > 0 \),
\[
S(\lambda x) < \frac{S}{2^{2+1}}
\]
where \( S(\lambda x) = S(\phi, \lambda x, 0) \).

Proof. Without loss of generality we may assume \( x_0 = 0 \). Hence \( 0 \in \partial \Omega \), \( \alpha(0) > 0 \) and \( \partial \Omega \) is flat at \( 0 \) of order \( k \) which is strictly greater than \( 3 \). Let \( \rho : \mathbb{B}(0, R) \cap \{ x : x = 0 \} \to \mathbb{R} \) be the function appeared in the definition of flatness.

For \( u \in H^1(\Omega) \), we have
\[
Q(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega \alpha(x) u^2 dx \\
\frac{1}{|u|^2 + 1,}.
\]
Let \( \varphi \in C_0^\infty(B(0,R/2)) \) be such that \( \varphi \) is radial and \( \varphi \equiv 1 \) on \( B(0,R/4) \) and for \( \varepsilon > 0 \),

\[
U_\varepsilon(x) = \frac{\varphi(x)}{\varepsilon + |x|^2}^{n-2/2} \]

then we claim that there exists a constant \( C = C(n) > 0 \) such that as \( \varepsilon \to 0 \),

\[
Q(U_\varepsilon) \leq \begin{cases} 
\frac{S}{2^{2n}} - \lambda C\varepsilon + O(\varepsilon^{n-1}\lambda^{1/2}) & \text{if } n \geq 5 \\
\frac{S}{2^{2n}} - \lambda C\varepsilon \log \varepsilon + O(\varepsilon) & \text{if } n = 4
\end{cases}
\]  

(15)

and this implies the lemma.

Proof of (15) follows in several steps. For simplicity we can assume that \( \rho \geq 0 \). For non-positive \( \rho \)’s the estimate (15) follows exactly as in the case of positive \( \rho \). Define

\[
\Sigma = \{ x \in B(0,R/2); 0 < x_n < \rho(x') \}.
\]

Step 1. Let \( l \geq 0 \), then

\[
\int_{\Sigma} \frac{dx}{\varepsilon + |x|^2} = \begin{cases} 
0(1) + O(\varepsilon^{n+k-1/2}) & \text{if } n + k - 1 - 2l \neq 0 \\
0 \left( \log \frac{1}{\varepsilon} \right) & \text{if } n + k - 1 - 2l = 0.
\end{cases}
\]  

(16)

Since \( \partial \Omega \) is flat of order \( k \) at 0, there exists a constant \( c > 0 \) such that \( \rho(x') \leq c|x'|^k \). Therefore we have

\[
\int_{\Sigma} \frac{dx}{\varepsilon + |x|^2} \leq \omega_{n-2} \int_0^R r^{n-2-k} \left( \int_0^R \frac{dx_n}{(\varepsilon + r^2 + x_n^2)^{n-2}} \right) dr
\]

\[
= \omega_{n-2} \int_0^R r^{n-2-k} \left( \int_0^R \frac{1}{(1 + r^2)^{n-2}} dr \right) dr
\]

\[
\leq c \omega_{n-2} \int_0^R r^{n-2-k} \left( \int_0^R \frac{1}{1 + r^2} dr \right) dr
\]

\[
= c \omega_{n-2} e^{(n+k-1/2)} \int_0^{R^{1/2}} \frac{r^{n-2-k}}{(1 + r^2)^{n/2}} dr.
\]  

(17)

Now

\[
\int_0^{R^{1/2}} \frac{r^{n-2-k}}{(1 + r^2)^{n/2}} dr = \int_0^1 \frac{r^{n-2+k}}{(1 + r^2)^{n/2}} dr + \int_1^{R^{1/2}} \frac{r^{n-2+k}}{(1 + r^2)^{n/2}} dr
\]

\[
= 0(1) + O(\varepsilon^{n-k+1+2l}) \text{ if } n + k - 1 - 2l \neq 0
\]

(18)

Hence from (18) and (17) we have (16).
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Step 2. Since $k > 3$, without loss of generality we can assume that $k = 3 + \delta$, $0 < \delta < 1$. Then

\[ \int_{\mathbb{R}} |U_\varepsilon|^2 \, dx = 0(1) + 0(\varepsilon^{k+3-n/2}) \quad (19) \]

\[ \int_{\mathbb{R}} |\nabla U_\varepsilon|^2 \, dx = 0(1) + 0(\varepsilon^{k+1-n/2}) \quad (20) \]

\[ \int_{\mathbb{R}} |U_\varepsilon|^{2n/(n-2)} \, dx = 0(1) + 0(\varepsilon^{k-1-n/2}) \quad (21) \]

Since

\[ \nabla U_\varepsilon = \frac{\nabla \varphi}{(e + |x|^2)^{(n-2)/2}} - \frac{(n-2)x}{(e + |x|^2)^n/2}, \]

hence we have

\[ |U_\varepsilon|^2 = 0\left(\frac{1}{(e + |x|^2)^{(n-2)/2}}\right), \quad |\nabla U_\varepsilon|^2 = 0\left(\frac{1}{(e + |x|^2)^{(n-1)}}\right) \]

and

\[ |U_\varepsilon|^{2n/(n-2)} = 0\left(\frac{1}{(e + |x|^2)^n}\right). \]

Therefore from step (1), by taking $l = n - 2$, $n - 1$ and $n$, we obtain (19), (20) and (21) respectively.

Step 3. From Brezis–Nirenberg [5] (see page 144, eqs (1.11), (1.12) and (1.13)), there exist positive constants $k_1$, $k_2$ and $k_3$ such that $k_1/k_2 = S$ and

\[ |\nabla U_\varepsilon|^2_{L^2, \omega(0,R)} = k_1 \varepsilon^{(n-2)/2} + 0(1) \quad (22) \]

\[ |U_\varepsilon|^{2n/(n-2)}_{L^{(n-2)/2}, \omega(0,R)} = k_2 \varepsilon^{(n-2)/2} + 0(1) \quad (23) \]

\[ |U_\varepsilon|^{k_3}_{L^{k_3}, \omega(0,R)} = \begin{cases} k_3 \varepsilon^{(n-4)/2} + 0(1) & \text{if } n \geq 5 \\ k_3 \log \varepsilon + 0(1) & \text{if } n = 4 \end{cases} \quad (24) \]

Now from (19) to (24) we have

\[ |\nabla U_\varepsilon|^2_{L^2, \omega} - |\nabla U_\varepsilon|^2_{L^2, \omega} = k_1 \varepsilon^{(n-2)/2} \left[1 + 0(\varepsilon^{(n-2)/2}) + 0(\varepsilon^{(n-1)/2})\right] \quad (25) \]
\[ |U_{\epsilon}|_{2,\Omega}^2 = \frac{1}{2} |U_{\epsilon}|_{2,\Omega(0,\epsilon)}^2 - |U_{\epsilon}|_{2,\Omega}^2, \]

\[
= \begin{cases} 
\frac{k_3}{2(\epsilon^{n-4})^{1/2}} + [1 + O(\epsilon^{n-1/2}) + O(\epsilon^{n-4})] & \text{if } n \geq 5 \\
\frac{k_3}{2} |\log \epsilon| + O(1) + O(\epsilon^{n-3-n/2}) & \text{if } n = 4 
\end{cases} \tag{26}
\]

\[
|U_{\epsilon}|_{2(n(n-2)),\Omega}^{2n(n-2)} - \frac{1}{2} |U_{\epsilon}|_{2(n(n-2)),\Omega}^{2n(n-2),\Omega(0,\epsilon)} - |U_{\epsilon}|_{2(n(n-2)),\Omega}^{2n(n-2)}\Omega(0,\epsilon) - |U_{\epsilon}|_{2(n(n-2)),\Omega}^{2n(n-2)}\Omega(0,\epsilon) \nonumber
\]

\[
\frac{k_3}{2(\epsilon^{n-2})^{1/2}} [1 + O(\epsilon^{n/2}) + O(\epsilon^{n-1}/2)].
\]

Hence

\[
|U_{\epsilon}|_{2(n(n-2)),\Omega}^2 = \frac{k_2}{2n(n-2)} \epsilon^{n-2} [1 + O(\epsilon^{n/2}) + O(\epsilon^{n-1}/2)] \tag{27}
\]

Now choose \( R > 0 \) and \( \alpha_0 > 0 \) such that \( \alpha(x) \geq \alpha_0 \) for all \( x \) in \( B(0, R) \cap \Omega \), then

\[
Q(U_{\epsilon}) \leq \frac{\|VU_{\epsilon}\|_{2,\Omega}^2 - \lambda \alpha_0 |U_{\epsilon}|_{2,\Omega}^2}{|U_{\epsilon}|_{2(n(n-2)),\Omega}^2} \tag{28}
\]

From (25) to (28) we have

\[
Q(U_{\epsilon}) \leq \begin{cases} 
\frac{S}{2^{2n} - \frac{\lambda \alpha_0 k_3}{2^{2n} k_2}} \epsilon + O(\epsilon^{n-1/2}) & \text{if } n \geq 5 \\
\frac{S}{2^{2n} - \frac{\lambda \alpha_0 k_3}{2^{2n} k_2}} |\log \epsilon| + O(\epsilon) & \text{if } n = 4
\end{cases} \tag{29}
\]

Let \( C = \lambda \alpha_0 k_3/2^{2n} k_2 \), then (15) follows from (29). This proves the claim and hence the lemma.

**Proof of the Theorem.** Let \( \lambda \in (0, \lambda(\alpha)) \). By (i) of proposition 2,

\[
\left\{ \int_\Omega |Vu|^2 - \lambda \int_\Omega \alpha(x) \epsilon^2 \right\}^{1/2}
\]

defines an equivalent norm on \( H^1(\Omega) \). From lemma 3.3, \( S(\alpha) < (S/2^{2n}) \). Hence from (ii) of lemma 3.2, there exists a \( u_0 \in H^1(\Omega) \), \( u_0 \geq 0 \) which solves (1). By Cherrier [8], \( u_0 \in C^\infty(\overline{\Omega}) \) and by maximum principle \( u_0 > 0 \) in \( \overline{\Omega} \). On the other hand, if \( \lambda \not\in (0, \lambda(\alpha)) \), then by (ii) of proposition 2, (1) does not admit any solution. This completes the proof of the theorem.

4. Concluding remarks

1. Similar construction as in Brezis [3] (See page 21 example (2)), it is possible to construct a \( \alpha(x) < 0 \) such that (1) admits a solution for \( \lambda = 1 \). We do not know how to deal (1) when \( \lambda \alpha(x) \leq 0 \).
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2. By a result of Senn–Hess [2] (See page 462, proposition (6)), it follows that if $\int_{\Omega} x(x) dx = 0$, then for any $\lambda \in \mathbb{R}$, (1) does not admit a solution.

3. Following the method of this paper and using a recent result of Escobar [10], it is possible to show that under suitable flatness assumptions at a boundary point,

$$-\Delta u = \lambda x(x)u \text{ in } \Omega$$

$$u > 0$$

$$\frac{\partial u}{\partial v} = u^{n+2-2} \text{ on } \partial \Omega$$

admits a solution for all $\lambda \in (0, \lambda(\alpha))$.

4. Let $\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0$ and $\Gamma_1$ the smooth submanifolds of dimension $n - 1$. Consider the following mixed boundary value problem

$$-\Delta u = u^{(n+2)/(n-2)} + \alpha(x)u + \mu u \text{ in } \Omega$$

$$u > 0$$

$$u = 0 \text{ on } \Gamma_0$$

$$\frac{\partial u}{\partial v} = 0 \text{ on } \Gamma_1.$$  \hspace{1cm} (30)

Let $x_0$ be in the interior of $\Gamma_1$ and assume that $\partial \Omega$ is flat at $x_0$ of order $k$ strictly greater than 3 and $\alpha(x_0) \geq 0$. Further assume that $-\Delta - \alpha$ is positive on $H^2(\Gamma_0)$. Let $\mu_1$ be the first eigenvalue of $-\Delta - \alpha$ on $H^2(\Gamma_0)$. Then by similar method used in this paper it follows that for $n \geq 4$, $\mu \in (0, \mu_1)$, (30) admits a weak solution. However, it should be noted that under a stronger assumption on $\alpha$, viz $\alpha(x_0) > 0$, (30) admits a solution even for $\mu = 0$.

When $\alpha \equiv 0$, $\mu = 0$, Lions–Pacella–Tricarico [11] have proved that, under suitable assumption on $\Gamma_0$ and $\Gamma_1$, with $\Gamma_0 \neq \emptyset$, problem (30) admits a solution.

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References


