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Critical Sobolev exponent problem in $\mathbb{R}^n (n \geqslant 4)$ with Neumann boundary condition

ADIMURTHI and S L YADAVA
TIFR Centre, P. B. 1234, Bangalore 560012, India

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Abstract. In this paper we study the existence and non existence of positive solution for the critical Sobolev exponent problem

$$-\Delta u = u^{(n+2)/(n-2)} + \lambda\alpha(x)u \quad \text{in}\,\Omega$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on}\,\partial B,$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geqslant 4)$.

Keywords. Critical exponent; flatness condition; Neumann boundary.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geqslant 3)$ be a bounded domain with smooth boundary. Let $\alpha \in C^\infty(\bar{\Omega})$ and consider the following problem:

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} + \lambda\alpha(x)u \quad \text{in } \Omega \\ u &> 0 \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$. Problem (1) with Drichlet boundary condition instead of Neumann, has been studied by Brezis-Nirenberg [5] in detail. For the Neumann boundary condition, namely, for problem (1), Brezis [3] raised the following question:

"Under what conditions on α and Ω, problem (1) admits a solution?".

When $n = 2$, and the nonlinearity $u^{(n+2)/(n-2)}$ is replaced by $u^2 \exp(bu^2)$, $b > 0$, problem (1) has been studied by authors in [1]. More precisely, in [1] it has been shown that, under suitable assumptions on α, there exists a $\lambda(\alpha) > 0$ such that

$$\begin{aligned} -\Delta u &= u^2 \exp(bu^2) + \lambda\alpha(x)u \quad \text{in } \Omega \\ u &> 0 \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

admits a solution if and only if $\lambda \in (0, \lambda(\alpha))$.

In this paper we have made an attempt to answer the question of Brezis (see also Cherrier [9] for some partial results for the problem similar to (1)). Obviously if $\lambda\alpha \geqslant 0$, then (1) does not admit any solution. Here we consider problem (1) when α changes sign in Ω. If $\int_\Omega \alpha(x)\,dx = 0$, then (1) does not admit any solution (see remark 2 in §4).

Let α change sign in Ω and $\int_\Omega \alpha(x)\,dx < 0$. Let $\lambda(\alpha) > 0$ be the unique real number such that

$$\begin{aligned} -\Delta\varphi &= \lambda(\alpha)\alpha(x)\varphi \quad \text{in } \Omega \\ \varphi &> 0 \\ \frac{\partial\varphi}{\partial\nu} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3}$$

admits a solution. For the existence of such $\lambda(\alpha)$, we refer Brown–Lin [6] and Senn–Hess [12]. Now by exploiting the techniques used in Brezis–Nirenberg [5] and Adimurthi–Yadava [1], we prove the following

Theorem. *Let $n \geqslant 4$ and assume that*

(i) *α changes sign in Ω and $\int_\Omega \alpha(x)\,dx < 0$. Let $\lambda(\alpha)$ be given by* (3).

(ii) *There exists a $x_0 \in \partial\Omega$ such $\alpha(x_0) > 0$ and $\partial\Omega$ is flat of order $k > 3$ at x_0.*

Then problem (1) *admits a solution $u \in C^2(\bar\Omega)$ if and only if $\lambda \in (0, \lambda(\alpha))$.*

For the meaning of flatness of $\partial\Omega$ of order k at x_0, see definition (2.1). Here we remark that the flatness condition in the theorem is not satisfied for the ball.

2. Preliminaries

Let $H^1(\Omega)$ denote the usual Sobolev space. For $u \in H^1(\Omega)$ and $1 \leqslant p \leqslant 2n/(n-2)$, let

$$|\nabla u|^2_{2,\Omega} = \int_\Omega |\nabla u|^2\,dx$$

$$|u|_{p,\Omega} = \left(\int_\Omega |u|^p\,dx\right)^{1/p}.$$

To prove the theorem, we need the following

PROPOSITION 2.

Let $\alpha \in C^\infty(\bar\Omega)$ be such that α changes sign in Ω and $\int_\Omega \alpha(x)\,dx < 0$. Let $\lambda(\alpha)$ be given by (3). *Then we have*

(i) *for all $\lambda \in (0, \lambda(\alpha))$,*

$$\left\{\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega \alpha(x)u^2\,dx\right\}^{1/2}$$

defines an equivalent norm on $H^1(\Omega)$.

(ii) *Let* $u \in H^2(\Omega) \cap C(\bar{\Omega})$ *be such that*

$$\lambda \notin (0, \lambda(\alpha))$$

$$\Delta u + \lambda \alpha u \neq 0$$

$$\Delta u + \lambda \alpha u \leqslant 0, \ \frac{\partial u}{\partial \nu} = 0,$$

then u cannot be positive.

For the proof of above proposition we refer Brown–Lin [6] (Theorems 3.10 and 3.11) and Senn–Hess [12] (Proposition 6).

Flatness condition

Let $x_0 \in \partial\Omega$. After a translation and rotation, we assume that $x_0 = 0$ and there exist $R > 0$ and $\rho: B(0, R) \cap \{x_n = 0\} \to \mathbb{R}$ a smooth function such that

$$\rho(0) = 0, \nabla\rho(0) = 0$$

$$\Omega \cap B(0, R) = \{x \in B(0, R); x_n > \rho(x')\}$$

$$\partial\Omega \cap B(0, R) = \{x \in B(0, R); x_n = \rho(x')\}$$

where $x' = (x_1 \ldots x_{n-1}, 0)$.

DEFINITION 2.1.

We say that $\partial\Omega$ is flat of order k at 0 if $\rho(x') = 0(|x'|^k)$ as $|x'| \to 0$.

3. Proof of the theorem

Let $\Omega \subset \mathbb{R}^n (n \geqslant 3)$ be a bounded domain with smooth boundary. Let Γ_0, Γ_1 be disjoint submanifolds of $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and let

$$H^1(\Gamma_0) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}. \tag{4}$$

Let $a \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma_1)$ be such that

$$\left\{ \int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega a u^2 \mathrm{d}x + \int_{\Gamma_1} b u^2 \mathrm{d}x \right\}^{1/2} \tag{5}$$

defines an equivalent norm on $H^1(\Gamma_0)$. We denote this norm by $\|u\|$. For $u \in H^1(\Gamma_0)$ and $p = (n+2)/(n-2)$, define

$$J(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} \mathrm{d}x \tag{6}$$

$$Q(u) = \frac{\|u\|^2}{\left(\int_\Omega |u|^{p+1} \mathrm{d}x\right)^{2/p+1}} \tag{7}$$

$$S(\Gamma_0, a, b) = \inf\{Q(u): u \in H^1(\Gamma_0) \backslash \{0\}\}. \tag{8}$$

Let

$$S = \inf\left\{ \int_\Omega |\nabla u|^2 \mathrm{d}x; u \in H_0^1(\Omega), \int_\Omega |u|^{2n/(n-2)} \mathrm{d}x = 1 \right\} \tag{9}$$

be the best Sobolev constant. Then by Cherrier [7, 8] we have

Lemma 3.1. *For every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $u \in H^1(\Omega)$,*

$$\left(\int_\Omega |u|^{p+1}\,dx\right)^{2/(p+1)} \leqslant \left(\frac{2^{2/n}}{S} + \varepsilon\right)\int_\Omega |\nabla u|^2\,dx + C(\varepsilon)\int_\Omega u^2\,dx.$$

For the proof of this lemma we also refer Aubin [2] (Theorem 2.30).

Now by using the above mentioned Cherrier's result and Brezis–Lieb lemma [4], we have

Lemma 3.2.

(i) $S(\Gamma_0, a, b) > 0$
(ii) *Assume $S(\Gamma_0, a, b) < S/(2^{2/n})$, then there exists a $v \geqslant 0$ such that $S(\Gamma_0, a, b) = Q(v)$. Further if we define $u_0 = S(\Gamma_0, a, b)^{(n-2)/4}\, v$, then u_0 satisfies*

$$\begin{aligned} -\Delta u_0 &= u_0^{(n+2)/(n-2)} + a(x)u_0 \ \text{ in } \Omega \\ u_0 &> 0 \\ u_0 &= 0 \text{ on } \Gamma_0,\ \frac{\partial u_0}{\partial \nu} + bu_0 = 0 \ \text{ on } \Gamma_1 \end{aligned} \tag{10}$$

and $J(u_0) < (S^{n/2})/2n$.

Proof. (i) By Sobolev imbedding theorem, there exists a constant $C > 0$ such that for all $u \in H^1(\Gamma_0)$,

$$\left(\int_\Omega |u|^{p+1}\,dx\right)^{2/(p+1)} \leqslant C\|u\|^2.$$

Now (i) follows from the definition of $S(\Gamma_0, a, b)$. (ii) Let $\{u_k\}$ be a minimizing sequence in (8) with $\int_\Omega |u_k|^{p+1}\,dx = 1$. Let for a subsequence, $u_k \to v$ weakly in $H^1(\Gamma_0)$ and almost everywhere.

Claim 1. $v \neq 0$.

Suppose $v \equiv 0$. Then by Rellich lemma and lemma (3.1) we have

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega |\nabla u_k|^2\,dx &= \lim_{k\to\infty} \|u_k\|^2 \\ &= S(\Gamma_0, a, b) \\ &\leqslant S(\Gamma_0, a, b)\left(\frac{2^{2/n}}{S} + \varepsilon\right)\lim_{k\to\infty}\int_\Omega |\nabla u_k|^2\,dx \end{aligned}$$

for every $\varepsilon > 0$. Hence we have,

$$1 \leqslant S(\Gamma_0, a, b)\left(\frac{2^{2/n}}{S} + \varepsilon\right).$$

This contradicts $S(\Gamma_0, a, b) < S/2^{2/n}$. Hence $v \neq 0$.

Claim 2. $Q(v) = S(\Gamma_0, a, b)$.

Let $v_k = u_k - v$. Then $v_k \to 0$ weakly and almost everywhere. Now by Rellich lemma

$$\|u_k\|^2 = \|v\|^2 + \|v_k\|^2 + \mathbf{o}(1)$$
$$= \|v\|^2 + |\nabla v_k|_{2,\Omega}^2 + \mathbf{o}(1)$$

which gives

$$S(\Gamma_0, a, b) = \|v\|^2 + |\nabla v_k|_{2,\Omega}^2 + \mathbf{o}(1). \tag{11}$$

By Brezis–Lieb lemma [4] and lemma 3.1, we have

$$1 = |u_k|_{p+1,\Omega}^2 = |v|_{p+1,\Omega}^2 + |v_k|_{p+1,\Omega}^2 + \mathbf{o}(1)$$
$$\leqslant |v|_{p+1,\Omega}^2 + \left(\frac{2^{2/n}}{S} + \varepsilon\right)|\nabla v_k|_{2,\Omega}^2 + \mathbf{o}(1)$$

for every $\varepsilon > 0$. Hence

$$S(\Gamma_0, a, b) \leqslant S(\Gamma_0, a, b)|v|_{p+1,\Omega}^2 + S(\Gamma_0, a, b)\left(\frac{2^{2/n}}{S} + \varepsilon\right)|\nabla v_k|_{2,\Omega}^2 + \mathbf{o}(1)$$
$$\leqslant S(\Gamma_0, a, b)|v|_{p+1,\Omega}^2 + |\nabla v_k|_{2,\Omega}^2 + \mathbf{o}(1). \tag{12}$$

Now from (11) and (12), we get

$$\frac{\|v\|^2}{|v|_{p+1,\Omega}^2} \leqslant S(\Gamma_0, a, b).$$

Hence v is a minimizer in (8).

Since $Q(v) = Q(|v|)$, we may assume $v \geqslant 0$. Finally if we take $u_0 = S(\Gamma_0, a, b)^{(n-2)/4}$ v, then it is easy to check that u_0 satisfies (10) and $J(u_0) < S^{n/2}/2n$. This completes the proof of the lemma.

Lemma 3.3. *Let* $\alpha \in C(\bar{\Omega})$. *Assume that there exists some* $x_0 \in \partial\Omega$ *such that* $\alpha(x_0) > 0$ *and* $\partial\Omega$ *is flat of order* $k > 3$ *at* x_0. *Then for every* $\lambda > 0$.

$$S(\lambda\alpha) < \frac{S}{2^{2/n}} \tag{13}$$

where $S(\lambda\alpha) = S(\phi, \lambda\alpha, 0)$.

Proof. Without loss of generality we may assume $x_0 = 0$. Hence $0 \in \partial\Omega$, $\alpha(0) > 0$ and $\partial\Omega$ is flat at 0 of order k which is strictly greater than 3. Let $\rho: B(0, R) \cap \{x : x_n = 0\} \to \mathbb{R}$ be the function appeared in the definition of flatness.

For $u \in H^1(\Omega)$, we have

$$Q(u) = \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega \alpha(x) u^2 \, dx}{|u|_{p+1,\Omega}^2}. \tag{14}$$

Let $\varphi \in C_c^\infty(B(0,R/2))$ be such that φ is radial and $\varphi \equiv 1$ on $B(0,R/4)$ and for $\varepsilon > 0$,

$$U_\varepsilon(x) = \frac{\varphi(x)}{(\varepsilon + |x|^2)^{(n-2)/2}},$$

then we claim that there exists a constant $C = C(n) > 0$ such that as $\varepsilon \to 0$,

$$Q(U_\varepsilon) \leqslant \begin{cases} \dfrac{S}{2^{2/n}} - \lambda C\varepsilon + 0(\varepsilon^{(k-1)/2}) & \text{if } n \geqslant 5 \\ \dfrac{S}{2^{2/n}} - \lambda C\varepsilon|\log\varepsilon| + 0(\varepsilon) & \text{if } n = 4 \end{cases} \tag{15}$$

and this implies the lemma.

Proof of (15) follows in several steps. For simplicity we can assume that $\rho \geqslant 0$. For non-positive ρ's the estimate (15) follows exactly as in the case of positive ρ. Define

$$\sum = \{x \in B(0,R/2);\ 0 < x_n < \rho(x')\}.$$

Step 1. Let $l \geqslant 0$, then

$$\int_\Sigma \frac{\mathrm{d}x}{(\varepsilon + |x|^2)^l} = \begin{cases} |0(1) + 0(\varepsilon^{(n+k-1-2l)/2}) & \text{if } n+k-1-2l \neq 0 \\ 0\left(\log\dfrac{1}{\varepsilon}\right) & \text{if } n+k-1-2l = 0. \end{cases} \tag{16}$$

Since $\partial\Omega$ is flat of order k at 0, there exists a constant $c > 0$ such that $\rho(x') \leqslant c|x'|^k$. Therefore we have

$$\begin{aligned} \int_\Sigma \frac{\mathrm{d}x}{(\varepsilon + |x|^2)^l} &\leqslant \omega_{n-2} \int_0^R r^{n-2}\left(\int_0^{cr^k} \frac{\mathrm{d}x_n}{(\varepsilon + r^2 + x_n^2)^l}\right)\mathrm{d}r \\ &= \omega_{n-2} \int_0^R \frac{r^{n-2}}{(\varepsilon + r^2)^{l-1/2}}\left(\int_0^{cr^k/(\varepsilon + r^2)^{1/2}} \frac{1}{(1+t^2)^l}\mathrm{d}t\right)\mathrm{d}r \\ &\leqslant c\,\omega_{n-2} \int_0^R \frac{r^{n-2+k}}{(\varepsilon + r^2)^l}\mathrm{d}r \\ &= c\,\omega_{n-2}\,\varepsilon^{(n+k-1-2l)/2} \int_0^{R/\varepsilon^{1/2}} \frac{r^{n-2+k}}{(1+r^2)^l}\mathrm{d}r. \end{aligned} \tag{17}$$

Now

$$\begin{aligned} \int_0^{R/\varepsilon^{1/2}} \frac{r^{n-2+k}}{(1+r^2)^l}\mathrm{d}r &= \int_0^1 \frac{r^{n-2+k}}{(1+r^2)^l}\mathrm{d}r + \int_1^{R/\varepsilon^{1/2}} \frac{r^{n-2+k}}{(1+r^2)^l}\mathrm{d}r \\ &= 0(1) + 0\left(\int_1^{R/\varepsilon^{1/2}} r^{n-2+k-2l}\mathrm{d}r\right) \\ &= 0(1) + \begin{cases} 0(1) + 0(\varepsilon^{(-n-k+1+2l)/2}) & \text{if } n+k-1-2l \neq 0 \\ 0\left(\log\dfrac{1}{\varepsilon}\right) & \text{if } n+k-1-2l = 0 \end{cases} \end{aligned} \tag{18}$$

Hence from (18) and (17) we have (16).

Step 2. Since $k > 3$, without loss of generality we can assume that $k = 3 + \delta, 0 < \delta < 1$. Then

$$\int_{\Sigma} |U_\varepsilon|^2 \, dx = 0(1) + 0(\varepsilon^{(k+3-n)/2}) \tag{19}$$

$$\int_{\Sigma} |\nabla U_\varepsilon|^2 \, dx = 0(1) + 0(\varepsilon^{(k+1-n)/2}) \tag{20}$$

$$\int_{\Sigma} |U_\varepsilon|^{2n/(n-2)} \, dx = 0(1) + 0(\varepsilon^{(k-1-n)/2}). \tag{21}$$

Since

$$\nabla U_\varepsilon = \frac{\nabla \varphi}{(\varepsilon + |x|^2)^{(n-2)/2}} - \frac{(n-2)x}{(\varepsilon + |x|^2)^{n/2}},$$

hence we have

$$|U_\varepsilon|^2 = 0\left(\frac{1}{(\varepsilon + |x|^2)^{n-2}}\right), \quad |\nabla U_\varepsilon|^2 = 0\left(\frac{1}{(\varepsilon + |x|^2)^{n-1}}\right)$$

and

$$|U_\varepsilon|^{2n/n-2} = 0\left(\frac{1}{(\varepsilon + |x|^2)^n}\right).$$

Therefore from step (1), by taking $l = n-2$, $n-1$ and n, we obtain (19), (20) and (21) respectively.

Step 3. From Brezis–Nirenberg [5] (see page 144, eqs (1.11), (1.12) and (1.13)), there exist positive constants k_1, k_2 and k_3 such that $k_1/k_2 = S$ and ·

$$|\nabla U_\varepsilon|^2_{2,B(0,R)} = \frac{k_1}{\varepsilon^{(n-2)/2}} + 0(1) \tag{22}$$

$$|U_\varepsilon|^2_{2n/(n-2),B(0,R)} = \frac{K_2}{\varepsilon^{(n-2)/2}} + 0(1) \tag{23}$$

$$|U_\varepsilon|^2_{2,B(0,R)} = \begin{cases} \dfrac{k_3}{\varepsilon^{(n-4)/2}} + 0(1) & \text{if } n \geqslant 5 \\ k_3 |\log \varepsilon| + 0(1) & \text{if } n = 4. \end{cases} \tag{24}$$

Now from (19) to (24) we have

$$\begin{aligned} |\nabla U_\varepsilon|^2_{2,\Omega} &= \frac{1}{2} |\nabla U_\varepsilon|^2_{2,B(0,R)} - |\nabla U_\varepsilon|^2_{2,\Sigma} \\ &= \frac{k_1}{2\varepsilon^{(n-2)/2}} [1 + 0(\varepsilon^{(n-2)/2}) + 0(\varepsilon^{(k-1)/2})] \end{aligned} \tag{25}$$

$$|U_\varepsilon|^2_{2,\Omega} = \frac{1}{2}|U_\varepsilon|^2_{2,B(0,R)} - |U_\varepsilon|^2_{2,\Sigma}$$

$$= \begin{cases} \dfrac{k_3}{2\varepsilon^{(n-4)/2}} + [1 + 0(\varepsilon^{(k-1)/2}) + 0(\varepsilon^{(n-4)/2})] & \text{if } n \geqslant 5 \\ \dfrac{k_3}{2}|\log \varepsilon| + 0(1) + 0(\varepsilon^{(k+3-n)/2}) & \text{if } n = 4 \end{cases} \tag{26}$$

$$|U_\varepsilon|^{2n/n-2}_{2n/(n-2),\Omega} = \frac{1}{2}|U_\varepsilon|^{2n/(n-2)}_{2n/(n-2),B(0,R)} - |U_\varepsilon|^{2n/(n-2)}_{2n/(n-2),\Sigma}$$

$$= \frac{1}{2}\left(\frac{k_2}{\varepsilon^{(n-2)/2}}\right)^{n/(n-2)}[1 + 0(\varepsilon^{n/2}) + 0(\varepsilon^{(k-1)/2})].$$

Hence

$$|U_\varepsilon|^2_{2n/(n-2),\Omega} = \frac{k_2}{2^{(n-2)/n}\varepsilon^{(n-2)/2}}[1 + 0(\varepsilon^{n/2}) + 0(\varepsilon^{(k-1)/2})]. \tag{27}$$

Now choose $R > 0$ and $\alpha_0 > 0$ such that $\alpha(x) \geqslant \alpha_0$ for all x in $B(0,R)\cap\bar{\Omega}$, then

$$Q(U_\varepsilon) \leqslant \frac{|\nabla U_\varepsilon|^2_{2,\Omega} - \lambda\alpha_0|U_\varepsilon|^2_{2,\Omega}}{|U_\varepsilon|^2_{2n/(n-2),\Omega}}. \tag{28}$$

From (25) to (28) we have

$$Q(U_\varepsilon) \leqslant \begin{cases} \dfrac{S}{2^{2/n}} - \dfrac{\lambda\alpha_0 k_3}{2^{2/n}k_2}\varepsilon + 0(\varepsilon^{(k-1)/2}) & \text{if } n \geqslant 5 \\ \dfrac{S}{2^{2/n}} - \dfrac{\lambda\alpha_0 k_3}{2^{2/n}k_2}\varepsilon|\log\varepsilon| + 0(\varepsilon) & \text{if } n = 4. \end{cases} \tag{29}$$

Let $C = \lambda\alpha_0 k_3/2^{2/n}k_2$, then (15) follows from (29). This proves the claim and hence the lemma.

Proof of the Theorem. Let $\lambda \in (0, \lambda(\alpha)$. By (i) of proposition 2,

$$\left\{\int_\Omega |\nabla u|^2 - \lambda\int_\Omega \alpha(x)u^2\,\mathrm{d}x\right\}^{1/2}$$

defines an equivalent norm on $H^1(\Omega)$. From lemma 3.3, $S(\lambda\alpha) < (S/2^{2/n})$. Hence from (ii) of lemma 3.2, there exists a $u_0 \in H^1(\Omega)$, $u_0 \geqslant 0$ which solves (1). By Cherrier [8], $u_0 \in C^\infty(\bar{\Omega})$ and by maximum principle $u_0 > 0$ in $\bar{\Omega}$. On the other hand, if $\lambda \notin (0, \lambda(\alpha))$, then by (ii) of proposition 2, (1) does not admit any solution. This completes the proof of the theorem.

4. Concluding remarks

1. Similar construction as in Brezis [3] (See page 21 example (2)), it is possible to construct a $\alpha(x) < 0$ such that (1) admits a solution for $\lambda = 1$. We do not know how to deal (1) when $\lambda\alpha(x) \leqslant 0$.

2. By a result of Senn–Hess [2] (See page 462, proposition (6)), it follows that if $\int_\Omega \alpha(x)\,dx = 0$, then for any $\lambda \in \mathbb{R}$, (1) does not admit a solution.

3. Following the method of this paper and using a recent result of Escobar [10], it is possible to show that under suitable flatness assumptions at a boundary point,

$$
\begin{aligned}
-\Delta u &= \lambda \alpha(x) u \text{ in } \Omega \\
u &> 0 \\
\frac{\partial u}{\partial \nu} &= u^{n/n-2} \text{ on } \partial\Omega
\end{aligned}
$$

admits a solution for all $\lambda \in (0, \lambda(\alpha))$.

4. Let $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \phi$, Γ_0 and Γ_1 the smooth submanifolds of dimension $n-1$. Consider the following mixed boundary value problem

$$
\begin{aligned}
-\Delta u &= u^{(n+2)/(n-2)} + \alpha(x)u + \mu u \text{ in } \Omega \\
u &> 0 \\
u &= 0 \quad \text{on } \Gamma_0 \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_1.
\end{aligned}
\tag{30}
$$

Let x_0 be in the interior of Γ_1 and assume that $\partial\Omega$ is flat at x_0 of order k strictly greater than 3 and $\alpha(x_0) \geqslant 0$. Further assume that $-\Delta - \alpha$ is positive on $H^1(\Gamma_0)$. Let μ_1 be the first eigenvalue of $-\Delta - \alpha$ on $H^1(\Gamma_0)$. Then by similar method used in this paper it follows that for $n \geqslant 4$, $\mu \in (0, \mu_1)$, (30) admits a weak solution.

However, it should be noted that under a stronger assumption on α, viz $\alpha(x_0) > 0$, (30) admits a solution even for $\mu = 0$.

When $\alpha \equiv 0$, $\mu = 0$, Lions–Pacella–Tricarico [11] have proved that, under suitable assumption on Γ_0 and Γ_1, with $\Gamma_0 \neq \phi$, problem (30) admits a solution.

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