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# *Nonexistence of Positive Radial Solutions of a Quasilinear Neumann Problem with a Critical Sobolev Exponent*

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## 1. Introduction

Let $B(1)$ denote the open unit ball in $\mathbb{R}^n$. We consider the following problem for $\lambda > 0$:

$$\begin{aligned} -\Delta u = u^{\frac{n+2}{n-2}} - \lambda u \quad & \text{in } B(1), \\ u \text{ is positive and radial} \quad & \text{in } B(1), \\ \frac{\partial u}{\partial \nu} = 0 \quad & \text{on } \partial B(1). \end{aligned} \tag{1.1}$$

Note that problem (1.1) always admits the constant solution $u_0 = \lambda^{(n-2)/4}$. In Adimurthi & Yadava [3] and Budd, Knaap & Peletier [6] it has been proved that there exists a constant $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, (1.1) admits a nonconstant solution for $n = 4, 5, 6$ and does not for $n = 3$.

In this paper we study problem (1.1) when $n \geqq 7$. In this case the method used in [3] and [6] breaks down due to the fact that $(n+2)/(n-2) < 2$, which leads to a drastic change in the behaviour of the associated Pohozhaev's functionals. In contrast to the case $n = 4, 5, 6$, we have

**Theorem 1.1.** *Suppose $n \geqq 7$. Then there exists a constant $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem (1.1) does not admit any nonconstant solution.*

The existence and nonexistence of nonconstant solutions of (1.1) is related to a conjecture of Lin & Ni [7] which we now briefly explain. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. For $1 < p < \infty$ and $\lambda > 0$, consider the problem

$$\begin{aligned} -\Delta u = u^p - \lambda u \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad & \text{on } \partial\Omega\,. \end{aligned} \tag{1.2}$$

In the subcritical case, i.e., when $1 < p < (n+2)/(n-2)$, LIN, NI & TAKAGI [8] have shown that there exist constants $\lambda_0$ and $\lambda_1$ with $0 < \lambda_0 \leqq \lambda_1$ such that problem (1.2) admits a nonconstant solution if $\lambda > \lambda_1$ and does not if $\lambda < \lambda_0$. Furthermore, LIN & NI [7] have shown that for $p > (n+2)/(n-2)$ and $\Omega$ a ball, this result continues to hold in the class of radial solutions. In view of these and some other connected results, they made the

**Conjecture:** *Let* $1 < p < \infty$. *Then there exist constants* $\lambda_0$ *and* $\lambda_1$ *with* $0 < \lambda_0 \leqq \lambda_1$ *such that*
*(a) for* $\lambda \geqq \lambda_1$, *problem* (1.2) *admits a nonconstant solution,*
*(b) for* $0 < \lambda < \lambda_0$, *problem* (1.2) *does not admit any nonconstant solution.*

Recently some progress has been made regarding this conjecture for the critical case $p = (n+2)/(n-2)$. Part (a) of the conjecture has been proved affirmatively by ADIMURTHI & MANCINI [2] & WANG [11]. They obtained a nonconstant minimal energy solution of (1.2) for $\lambda$ large. As mentioned earlier, for $n = 4, 5$ and 6, there is a counterexample to part (b) of the conjecture. However, in the class of minimal-energy solutions, it has been shown by ADIMURTHI & YADAVA [4] that part (b) of the conjecture is also true. Theorem 1.1 shows that for $n \geqq 7$, part (b) of the conjecture is true in the class of radial solutions and hence the conjecture is completely understood in this class.

Theorem 1.1 can be deduced from the following more general result for the $m$-Laplacian.

**Theorem 1.2.** *Let*

$$2 \leqq m < n, \; p = \frac{(m-1)n + m}{n-m}, \; p - 1 < q < p, \; \lambda > 0.$$

*Then there exists* $\lambda_0 > 0$ *such that the problem*

$$\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{m-2}\nabla u\Big) &= u^p - \lambda u^q \quad && \text{in } B(1), \\ u \text{ is positive and radial} & && \text{in } B(1), \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial B(1) \end{aligned} \tag{1.3}$$

*does not admit any nonconstant solution for* $0 < \lambda < \lambda_0$.

Theorem 1.1 obviously follows from Theorem 1.2 by taking $m = 2, \; q = 1, \; n \geqq 7$.

We mention that when $m = 2$, problem (1.3) has been studied extensively by BUDD, KNAAP & PELETIER [6] for the values $q < p - 1$ and by ADIMURTHI, KNAAP & YADAVA [1] for $q = p - 1$. The case $p - 1 < q < p$ has been open and Theorem 1.2 covers this range.

## 2. Proof of Theorem 1.2

In order to prove the theorem, we consider the following initial-value problem. Suppose that $\gamma > 0, \gamma \neq 1$ and let $w = w(\,\cdot\,, \gamma)$ (see [10]) denote the unique solution of

$$-\left(|w'|^{m-2}w'\right)' - \tfrac{n-1}{r}|w'|^{m-2}w' = w^p - w^q, \qquad w(0) = \gamma\,,\ w'(0) = 0\,. \tag{2.1}$$

Let $R_1(\gamma)$ be the first turning point of $w = w(\,\cdot\,,\gamma)$ and let $R_0(\gamma) < R_1(\gamma)$ be given by

$$R_1(\gamma) = \sup\{s; w(r,\gamma) > 0,\ w'(r,\gamma) \neq 0 \text{ for } r \in (0,s)\}, \tag{2.2}$$

$$w(R_0(\gamma),\gamma) = 1. \tag{2.3}$$

We now recall some facts about the solution $w(r,\gamma)$ of problem (2.1).

($A_1$) For $0 < \gamma < 1$, $w(\,\cdot\,,\gamma)$ is an increasing function in $(0, R_1(\gamma))$. For $1 < \gamma < \infty$, $w(\,\cdot\,,\gamma)$ is a decreasing function in $(0, R_1(\gamma))$.

($A_2$) $R_1(\gamma)$ exists and there exists a positive constant $\mu$ such that

$$\lim_{\gamma\to 1} R_1(\gamma) = \mu, \quad \lim_{\gamma\to 0} R_1(\gamma) = \infty.$$

The proofs of ($A_1$) and ($A_2$) are similar to that of Lemma A in [3] and therefore we omit them here.

Now we make the change of variables

$$v = \frac{n-m}{m-1}, \qquad k = \frac{m(n-1)}{n-m}, \qquad t = \left(\frac{v}{r}\right)^v,$$

$$T_0(\gamma) = \left(\frac{v}{R_0(\gamma)}\right)^v, \quad T_1(\gamma) = \left(\frac{v}{R_1(\gamma)}\right)^v.$$

Let $v(t) = w(r,\gamma)$. Then equation (2.1) transforms to

$$\begin{gathered} -\left(|v'|^{m-2}v'\right)' = t^{-k}(v^p - v^q) \text{ in } (T_1(\gamma),\infty), \\ v > 0 \text{ in } (T_1(\gamma),\infty), \\ v(\infty) = \gamma\,,\ v'(\infty) = v'(T_1(\gamma)) = 0. \end{gathered} \tag{2.4}$$

Define $f(s) = (1+s)^p - (1+s)^q$ and suppose $\gamma > 1$. Let $y = v - 1$. Then $y$ satisfies

$$\begin{gathered} -\left(|y'|^{m-2}y'\right)' = t^{-k}f(y) \quad \text{in } (T_0(\gamma),\infty), \\ y > 0 \quad \text{in } (T_0(\gamma),\infty), \\ y(\infty) = \gamma - 1, \quad y'(\infty) = y(T_0(\gamma)) = 0. \end{gathered} \tag{2.5}$$

Now the proof of Theorem 1.2 relies on the

**Main Lemma.** *Suppose that $m \geqq 2$, $p = ((m-1)n + m)/(n-m)$, and $p - 1 < q < p$. Then there exist $s_0 > 0$ and $\tau = \tau(\gamma) > T_0(\gamma)$ such that*

$$s_0 \leqq \underline{\lim}_{\gamma\to\infty} y(\tau) \leqq \overline{\lim}_{\gamma\to\infty} y(\tau) < \infty, \tag{2.6}$$

$$y'(\tau) \leqq \left(\frac{m-1}{m}\right)\frac{y(\tau)}{\tau}. \tag{2.7}$$

Assuming the validity of the Main Lemma, we complete the proof of Theorem 1.2. The proof of the Main Lemma will be given in Section 3.

**Proof of Theorem 1.2.** We first establish that

$$\overline{\lim}_{\gamma\to\infty} T_0(\gamma) < \infty. \tag{2.8}$$

**Proof of (2.8).** *Let* $s_0$ *and* $\tau$ be as in the Main Lemma. We observe that $y' > 0$ in $(T_0(\gamma), \infty)$. Now integrating (2.5) twice over $(T_0(\gamma), \tau)$ we obtain

$$y(\tau) = \int\limits_{T_0(\gamma)}^{\tau} \left( y'(\tau)^{m-1} + \int\limits_{\theta}^{\tau} s^{-k} f(y(s))ds \right)^{1/(m-1)} d\theta. \tag{2.9}$$

Since $f(0) = 0$ and $f$ is increasing, from (2.6) we can find a constant $C > 0$ such that for any $s \in [T_0(\gamma), \tau]$,

$$f(y(s)) \leqq f(y(\tau)) \leqq Cy(\tau). \tag{2.10}$$

Therefore from (2.7), (2.9), (2.10) and from the fact that $m \geqq 2$ we have

$$\begin{aligned} y(\tau) &\leqq \int\limits_{T_0(\gamma)}^{\tau} \left( y'(\tau) + \left( \frac{Cy(\tau)}{(k-1)\theta^{k-1}} \right)^{1/(m-1)} \right) d\theta \\ &= y'(\tau)(\tau - T_0(\gamma)) + \frac{C_1 y(\tau)^{1/(m-1)}}{T_0(\gamma)^{(k-m)/(m-1)}} \\ &\leqq \left( \frac{m-1}{m} \right) y(\tau) + \frac{C_1 y(\tau)^{1/(m-1)}}{T_0(\gamma)^{(k-m)/(m-1)}}, \end{aligned}$$

where $(k-m)C_1 = (m-1)(C/(k-1))^{1/(m-1)}$. Therefore from (2.6) and this inequality we get

$$T_0(\gamma) \leqq \frac{(C_1 m)^{(m-1)/(k-m)}}{s_0^{(m-2)/(k-m)}}.$$

This proves (2.8)

Since $R_0(\gamma) < R_1(\gamma)$, from (2.8) it follows that

$$\underline{\lim}_{\gamma\to\infty} R_1(\gamma) > 0\,. \tag{2.11}$$

Let $u$ be a nonconstant solution of (1.3). Let

$$r = |x|,\ \eta(r) = \frac{1}{\lambda^{1/(p-1)}} u\left( \frac{x}{\lambda^{(p+1-m)/(m(p-q))}} \right),\ R_\lambda = \lambda^{(p+1-m)/(m(p-q))}\,.$$

Then $\eta$ satisfies

$$-\left(|\eta'|^{m-2}\eta'\right)' - \frac{n-1}{r}|\eta'|^{m-2}\eta' = \eta^p - \eta^q \text{ in } (0, R_\lambda), \qquad \eta > 0,\ \eta'(0) = \eta'(R_\lambda) = 0. \tag{2.12}$$

By the uniqueness of the solution of the initial-value problem (2.1), and from ($A_1$) and ($A_2$), there exists $\tilde{R} > 0$ such that for $R_\lambda < \tilde{R}$ and $\gamma = \eta(0) > 1$ we have

$$R_1(\gamma) = R_\lambda, \quad \eta(r) = w(r, \gamma). \tag{2.13}$$

Also from (2.11) and (2.13), we can find some $\bar{R}$ with $0 < \bar{R} < \tilde{R}$ such that $R_\lambda \geqq \bar{R}$ if $R_\lambda \leqq \tilde{R}$. This implies that there exists a $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem (1.3) has no nonconstant solution. This proves the theorem.

## 3. Proof of the Main Lemma

Let $\gamma > 1$ and let $v, y, T_0(\gamma), T_1(\gamma), f$ be as described in Section 2. For $s \geqq 0$ and $t \geqq T_0(\gamma)$, define

$$\begin{aligned} G(s) &= \frac{s^{p+1}}{p+1} - \frac{s^{q+1}}{q+1}, \\ h(s) &= sf'(s) - pf(s) \\ &= (p-q)(1+s)^q - p(1+s)^{p-1} + q(1+s)^{q-1}, \end{aligned} \tag{3.1}$$

$$H(t) = t(v')^m - v(v')^{m-1} + \frac{mG(v)}{(m-1)t^{k-1}}, \tag{3.2}$$

$$H_1(t) = t(y')^m - y(y')^{m-1} + \frac{yf(y)}{(k-1)t^{k-1}}. \tag{3.3}$$

By (2.4) and (2.5), it follows that

$$\lim_{t\to\infty} H(t) = \lim_{t\to\infty} H_1(t) = 0, \tag{3.4}$$

$$H'(t) = \frac{p-q}{q+1} t^{-k} v^{q+1}, \tag{3.5}$$

$$H_1'(t) = \frac{t^{1-k}y'}{(k-1)}(yf'(y) - pf(y)) = \frac{t^{1-k}y'}{(k-1)}h(y), \tag{3.6}$$

$$\left((y')^{m-1}y^{1-k}t^{k-1}\right)' = -(k-1)t^{k-2}y^{-k}H_1(t). \tag{3.7}$$

Since $p - 1 < q < p$, from (3.1), we see that $h(s) \to \infty$ as $s \to \infty$ and that $h$ is negative near zero. Hence there exists a $s_0 > 0$ such that

$$h(s) \geqq 0 \text{ for } s \geqq s_0,\ h(s) \leqq 0 \text{ for } 0 \leqq s \leqq s_0. \tag{3.8}$$

Now also require that $\gamma > \max\left\{s_0, ((p+1)/(q+1))^{1/(p-q)}\right\} + 1$ and choose $\tau_0 = \tau_0(\gamma) > T_0(\gamma)$ such that

$$y(\tau_0) = \max\left\{s_0, \left(\frac{p+1}{q+1}\right)^{1/(p-q)}\right\}. \tag{3.9}$$

Then we have

**Lemma 3.1.** *Let* $\tau \geqq \tau_0$ *and define*

$$\alpha(\tau)^{m-1} = \Big((y')^{m-1} y^{1-k} t^{k-1}\Big)(\tau), \tag{3.10}$$

$$\beta(\tau) = \Big(y^{-(k-m)/(m-1)} - t^{-(k-m)/(m-1)}\alpha(\tau)\Big)(\tau), \tag{3.11}$$

$$\zeta_\tau(t) = t\Big(\alpha(\tau) + \beta(\tau) t^{(k-m)/(m-1)}\Big)^{-(m-1)/(k-m)}. \tag{3.12}$$

*Then for all* $t \in [\tau_0, \tau]$,

$$\zeta_\tau(t) \leqq y(t). \tag{3.13}$$

*Moreover* $\alpha, \beta$ *and* $\zeta_\tau$ *satisfy*

$$-\Big((\zeta_\tau')^{m-1}\Big)' = (k-1)\alpha(\tau)^{(m-1)}\beta(\tau) t^{-k}\zeta_\tau^p,$$

$$\zeta_\tau(\tau) = y(\tau), \;\; \zeta_\tau'(\tau) = y'(\tau). \tag{3.14}$$

*Furthermore, if* $y(\tau) \geqq 2$, *then*

$$\begin{aligned}&(k-1)\alpha(\tau)^{m-1}\beta(\tau) - 1 \\ &\quad = O\left(\frac{\tau^{k-1} y'(\tau)^{m-1}}{y(\tau)^{p+1}} + \frac{1}{y(\tau)^{p-q}} + \frac{\tau^{k-1}}{y(\tau)^{p+1}} \int_\tau^\infty s^{-k} y(s)^{q+1} ds\right).\end{aligned} \tag{3.15}$$

**Proof.** From the choice of $\tau_0$ and from (3.8) and (3.6) we get $H_1'(t) \geqq 0$ for $t \geqq \tau_0$. Therefore, from (3.7) and (3.4) we conclude that $\Big((y')^{m-1} y^{1-k} t^{k-1}\Big)$ is an increasing function for $t \geqq \tau_0$. Hence for $t \in [\tau_0, \tau]$,

$$(y')^{m-1} y^{1-k} t^{k-1} \leqq \Big((y')^{m-1} y^{1-k} t^{k-1}\Big)(\tau) = \alpha(\tau)^{m-1}.$$

This gives

$$\Big(y^{-(k-m)/(m-1)} - t^{-(k-m)/(m-1)}\alpha(\tau)\Big)' \geqq 0.$$

Hence for $t \in [\tau_0, \tau]$,

$$y(t)^{(m-k)/(m-1)} - y(t)^{(m-k)/(m-1)}\alpha(\tau) \leqq \Big(y^{(m-k)/(m-1)} - t^{(m-k)/(m-1)}\alpha(\tau)\Big)(\tau) = \beta(\tau),$$

which gives

$$y(t) \geqq t\Big(\alpha(\tau) + \beta(\tau) t^{(k-m)/((m-1))}\Big)^{-(m-1)/(k-m)} = \zeta_\tau(t).$$

This proves (3.13). A direct verification gives (3.14).

To simplify the notation, we let $y = y(\tau)$ and $y' = y'(\tau)$. Then from (3.10) and (3.11),

$$
\begin{aligned}
(k-1)\alpha(\tau)^{m-1}\beta(\tau)-1 &= (k-1)(y')^{m-1}\left(\frac{\tau}{y}\right)^{k-1}\left(\frac{1}{y^{(k-m)/(m-1)}}-\frac{\tau y'}{y^{(k-1)/(m-1)}}\right)-1 \\
&= \frac{(k-1)(y')^{m-1}\tau^{k-1}(y-\tau y')}{y^{m(k-1)/(m-1)}}-1 \qquad (3.16)\\
&= \frac{(k-1)\tau^{k-1}(y(y')^{m-1}-\tau(y')^{m})-y^{p+1}}{y^{p+1}}.
\end{aligned}
$$

Integrating (3.5) from $\tau$ to $\infty$ yields

$$
-\tau(y')^m+(1+y)(y')^{m-1}-\frac{mG(1+y)}{(m-1)\tau^{k-1}}=\frac{p-q}{q+1}\int_\tau^\infty s^{-k}(1+y(s))^{q+1}ds,
$$

that is,

$$
y(y')^{m-1}-\tau(y')^m=-(y')^{m-1}+\frac{mG(1+y)}{(m-1)\tau^{k-1}}+\frac{p-q}{q+1}\int_\tau^\infty s^{-k}(1+y(s))^{q+1}ds.
$$

Hence from (3.16) we have

$$
\begin{aligned}
(k-1)\alpha(\tau)^{m-1}\beta(\tau)-1 &= \frac{1}{y^{p+1}}\Bigg\{-(k-1)\tau^{k-1}(y')^{m-1} \\
&\quad +(p+1)\left(\frac{(1+y)^{p+1}}{p+1}-\frac{(1+y)^{q+1}}{q+1}\right) \\
&\quad -y^{p+1}+\left(\frac{p-q}{q+1}\right)(k-1)\tau^{k-1}\int_\tau^\infty s^{-k}(1+y(s))^{q+1}ds\Bigg\} \\
&= \frac{1}{y^{p+1}}\Bigg\{-(k-1)\tau^{k-1}(y')^{m-1}+y^{p+1}+0\big(y^{q+1}\big)-y^{p+1} \\
&\quad +\left(\frac{p-q}{q+1}\right)(k-1)\tau^{k-1}\int_\tau^\infty s^{-k}(1+y(s))^{q+1}ds\Bigg\} \\
&= O\left(\frac{\tau^{k-1}(y')^{m-1}}{y^{p+1}}+\frac{1}{y^{p-q}}+\frac{\tau^{k-1}}{y^{p+1}}\int_\tau^\infty s^{-k}y(s)^{q+1}ds\right).
\end{aligned}
$$

This proves (3.15) and hence the lemma.

**Lemma 3.2.** *There exists a positive constant $C$ such that for, $t \geqq \tau_0$,*

$$
y' \leqq y/t, \qquad (3.17)
$$

$$
y^{m/(m-1)} \leqq Ct. \qquad (3.18)
$$

**Proof.** Integrating (3.6) from $t$ to $\infty$, we obtain

$$t(y')^m - y(y')^{m-1} + \frac{yf(y)}{(k-1)t^{k-1}} + \frac{1}{(k-1)}\int_t^\infty s^{(1-k)} y' h(y)ds = 0. \qquad (3.19)$$

Since $h(y(t)) \geqq 0$ for $t \geqq \tau_0$, it follows from (3.19) that

$$t(y')^m - y(y')^{m-1} \leqq 0\,.$$

This proves (3.17).

Let $t \geqq \tau_0$ be fixed, and for $X \geqq 0$ define

$$\rho(X) = X^m - yX^{m-1} + \frac{yf(y)}{(k-1)t^{k-m}} + \frac{t^{m-1}}{(k-1)}\int_t^\infty s^{(1-k)} y' h(y)ds.$$

Then $\rho(0) > 0, \rho(\infty) = \infty$, and from (3.19) we see that $\rho(ty'(t)) = 0$. Hence $\rho$ has a minimum at some point $X_0$, with $\rho(X_0) \leqq 0,\ \rho'(X_0) = 0$. This implies that

$$mX_0^{m-1} - (m-1)yX_0^{m-2} = 0,$$

and so

$$X_0 = \frac{m-1}{m}y.$$

Hence

$$0 \geqq \rho(X_0) = \left(\frac{m-1}{m}\right)^m y^m - \left(\frac{m-1}{m}\right)^{m-1} y^m + \frac{yf(y)}{(k-1)t^{k-m}} + \frac{t^{m-1}}{k-1}\int_t^\infty s^{(1-k)} y' h(y)ds.$$

Since $h(y(s)) \geqq 0$ for $s \geqq t \geqq \tau_0$, from this inequality follows

$$\frac{1}{m}\left(\frac{m-1}{m}\right)^{(m-1)} y^m \geqq \frac{yf(y)}{(k-1)t^{k-m}}.$$

Hence we can find a positive constant $C$ such that $y^{m/(m-1)} \leqq Ct$. This proves (3.18).

**Proof of the Main Lemma.** Let $\theta(t) = \dfrac{y^{m/(m-1)}}{t}$. Then,

$$\theta'(t) = \frac{y^{1/(m-1)}}{t^2}\left(\frac{m}{m-1}ty' - y\right), \qquad (3.20)$$

$$\begin{aligned}\theta''(t) &= \left(\frac{y^{1/(m-1)}}{t^2}\right)'\left(\frac{m}{m-1}ty' - y\right) + \frac{y^{1/(m-1)}}{(m-1)t^2}(mty'' + y') \qquad (3.21)\\ &= \left(\frac{y^{1/(m-1)}}{t^2}\right)'\left(\frac{m}{m-1}ty' - y\right) - \frac{y^{1/(m-1)}}{(m-1)t^2}\left(\frac{mt^{-k+1}f(y)}{(m-1)(y')^{m-2}} - y'\right).\end{aligned}$$

Now one of the following two cases holds:

Case (i): $\overline{\lim}_{\gamma\to\infty}\theta'(\tau_0) \leqq 0$.

Case (ii): There exists a sequence $\gamma_l \to \infty$ such that $\theta'(\tau_0) > 0$.

In case (i), from (3.20) and (3.9) we get $\tau_0 y'(\tau_0) \leqq ((m-1)/m)y(\tau_0)$ and $y(\tau_0) = s_0$. This proves the lemma.

In case (ii), since $\theta'(\tau_0) > 0$ and $\theta(\infty) = 0$, the function $\theta$ has a local maximum in $(\tau_0, \infty)$. Let $\tau = \tau(\gamma_l) > \tau_0$ be the first such point. Hence from (3.20) and (3.21), together with $\theta''(\tau) \leqq 0$, we have

$$\frac{(m-1)}{m}\frac{y}{t} \leqq y' \quad \text{for} \quad t \in [\tau_0, \tau], \tag{3.22}$$

$$\frac{(m-1)}{m}\frac{y(\tau)}{\tau} = y'(\tau), \tag{3.23}$$

$$\left(\frac{m-1}{m}\right)^{(m-1)}\frac{y(\tau)^{m-1}}{\tau^{m-1}} = y'(\tau)^{m-1} \leqq \frac{mf(y(\tau))}{(m-1)\tau^{k-1}}.$$

From the last inequality we can find a positive constant $C$ such that

$$y(\tau)^{m/(m-1)} \geqq C\tau. \tag{3.24}$$

Before going to the next lemma we introduce the following notation. For two functions $f$ and $g$ defined on a set $S$, we write $f \approx g$ if there exist positive constants $C_1$ and $C_2$ such that $C_1 g \leqq f \leqq C_2 g$ uniformly on $S$.

The Main Lemma now relies on

**Lemma 3.3.**

$$\overline{\lim}_{\gamma_l\to\infty} y(\tau) < \infty. \tag{3.25}$$

**Proof.** We argue by contradiction. Suppose the lemma were not true. Then for a subsequence, still denoted by $\gamma_l$, we would have

$$\lim_{\gamma_l\to\infty} y(\tau) = \infty. \tag{3.26}$$

Notice that from (3.18), $\tau \to \infty$ as $\gamma_l \to \infty$.

In the sequel, $C$ and $C_i$ denote positive constants independent of the sequence $\{\gamma_l\}$, which may however vary from one inequality to the next.

In view of (3.18) and (3.24) we can find positive constants $C_1$ and $C_2$ such that

$$C_1\tau \leqq y(\tau)^{m/(m-1)} \leqq C_2\tau.$$

Now from this inequality and (3.23) we obtain

$$\left(\frac{m-1}{m}\right)C_1^{(m-1)/m}\tau^{-1/m} \leqq y'(\tau) \leqq \left(\frac{m-1}{m}\right)C_2^{(m-1)/m}\tau^{-1/m}.$$

Therefore for all $\gamma_l$, the functions $y(\tau)$ and $y'(\tau)$ satisfy

$$y(\tau) \approx \tau^{1-1/m}, \tag{3.27}$$

$$y'(\tau) \approx \tau^{-1/m}. \tag{3.28}$$

We now divide the proof of the lemma into four steps.

**Step 1.** Let $\tau_1 \in [\tau_0, \tau]$ be such that

$$y'(\tau_1) \leqq C\tau^{-1/m}. \tag{3.29}$$

Then

$$y(\tau_1) \geqq C\tau^{(m-1)(q-p+1)/m}. \tag{3.30}$$

**Proof of Step 1.** By the choice of $\tau_1$, we have $G(y(\tau_1)) \geqq 0$. Hence integrating (3.5) from $\tau_1$ to $\infty$ and using (3.4), (3.27) and (3.29), we have

$$\begin{aligned} C_1\tau^{-k+1+(m-1)(q+1)/m} &\leqq \left(\frac{p-q}{q+1}\right)\int_\tau^\infty t^{-k}y^{q+1}(t)dt \\ &\leqq \left(\frac{p-q}{q+1}\right)\int_{\tau_1}^\infty t^{-k}(1+y(t))^{q+1}dt \\ &\equiv -H(\tau_1) \leqq \frac{(1+y(\tau_1))}{\tau^{(m-1)/m}}. \end{aligned}$$

This implies (3.30), since

$$\frac{m-1}{m}(q+1) - k + 1 + \frac{m-1}{m} = \frac{m-1}{m}(q-p+1)$$

and $\tau \to \infty$ as $\gamma_l \to \infty$. This proves Step 1.

**Step 2.** For $\delta < 1$, define

$$\rho(\delta) = \frac{p\left(\delta - \frac{1}{m}\right) + \frac{m-1}{m}}{(k-1)}. \tag{3.31}$$

Then $\rho(\delta) < \delta$. Let $\delta_0 < 1$; if we now define a sequence $\{\delta_i\}_{i \geqq 0}$ by $\delta_{i+1} = \rho(\delta_i)$ for $i \geqq 0$, then

$$\lim_{i\to\infty} \delta_i = -\infty. \tag{3.32}$$

**Proof of Step 2.** Since $p+1 = nm/(n-m) > m$ and $\delta < 1$, from (3.31) we have

$$\rho(\delta) - \delta = \frac{p(\delta-1) + \frac{m-1}{m}(p+1)}{\frac{m-1}{m}(p+1)} - \delta = \frac{(1-\delta)(m-1-p)}{(m-1)(p+1)} < 0.$$

This proves that $\rho(\delta) < \delta$, and also implies that $\{\delta_i\}$ is a decreasing sequence. Let $\tilde{\delta} = \lim_{i\to\infty} \delta_i$. If $\tilde{\delta} > -\infty$, then

$$\tilde{\delta} = \lim_{i\to\infty} \delta_{i+1} = \rho(\tilde{\delta}) < \tilde{\delta},$$

which is a contradiction. This proves Step 2.

**Step 3.** Let $1/m < \delta < 1$. Assume that

$$y(\tau^{\delta}) \approx \tau^{\delta-1/m}, \tag{3.33}$$

$$y'(\tau^{\delta}) \approx \tau^{-1/m}. \tag{3.34}$$

Then $\rho(\delta) > 1/m$ and

$$\tau_0 < \tau^{\rho(\delta)} < \tau^{\delta}, \tag{3.35}$$

$$y'(\tau^{\rho(\delta)}) \approx \tau^{-1/m}, \tag{3.36}$$

$$y(\tau^{\rho(\delta)}) \approx \tau^{\rho(\delta)-1/m}, \tag{3.37}$$

$$y(\tau^{\rho(\delta)}) \geqq C\tau^{(m-1)(q-p+1)/m}. \tag{3.38}$$

**Proof of Step 3.** From Step 2, we have $\rho(\delta) < \delta < 1$. Let $\max\{\tau^{\rho(\delta)}, \tau_0\} \leqq t \leqq \tau^{\delta}$. Integrating (2.5) from $t$ to $\tau^{\delta}$ and using (3.33) and (3.34), we have

$$\begin{aligned} y'(t)^{m-1} &= y'(\tau^{\delta})^{m-1} + \int_t^{\tau^{\delta}} s^{-k} f(y(s))ds \\ &= O\left(\tau^{-(m-1)/m} + t^{1-k} y(\tau^{\delta})^p\right) \\ &= O\left(\tau^{-(m-1)/m} + \tau^{(1-k)\rho(\delta)}\tau^{(\delta-1/m)p}\right) = O\left(\tau^{-(m-1)/m}\right). \end{aligned} \tag{3.39}$$

Suppose that $\tau^{\rho(\delta)} \leqq \tau_0$. From (3.9), (3.39) and Step 1, we get

$$Cs_0 = Cy(\tau_0) = \lim_{\gamma_l\to\infty} Cy(\tau_0) \geqq \lim_{\gamma_l\to\infty} \tau^{(m-1)(q-p+1)/m} = \infty,$$

which is a contradiction. Hence $\tau^{\rho(\delta)} > \tau_0$. Thus from Step 1 and (3.39) we get $y\left(\tau^{\rho(\delta)}\right) \geqq C\tau^{(m-1)(q-p+1)/m}$. This proves (3.35) and (3.38). Furthermore, from (3.39) and (3.34), we have

$$C_1\tau^{-1/m} \leqq y'(\tau^{\delta}) \leqq y'\left(\tau^{\rho(\delta)}\right) \leqq C_2\tau^{-1/m}.$$

This proves (3.36). Next from (3.17), (3.22) and (3.36),

$$y\left(\tau^{\rho(\delta)}\right) \approx \tau^{\rho(\delta)} y'\left(\tau^{\rho(\delta)}\right) \approx \tau^{\rho(\delta)-1/m},$$

proving (3.37). Since $\tau \to \infty$ as $\gamma_l \to \infty$, and $q - p + 1 > 0$, we conclude from (3.37) and (3.38) that $\rho(\delta) > 1/m$. This proves Step 3.

**Step 4.** We assert that

$$y\left(\tau^{(q-p+m)/m}\right) \approx \tau^{(q-p+m-1)/m}, \tag{3.40}$$

$$y'\left(\tau^{(q-p+m)/m}\right) \approx \tau^{-1/m}. \tag{3.41}$$

**Proof of Step 4.** Let $\alpha(\tau), \beta(\tau)$ and $\zeta_\tau$ be as in Lemma 3.1. Then from (3.27), (3.28) and (3.23) we have

$$\alpha(\tau)^{(m-1)} \approx \tau^{(k-m)/m}, \qquad \beta(\tau) \approx \tau^{-(k-m)/m}, \qquad \alpha(\tau)^{m-1}\beta(\tau) \approx 1. \tag{3.42}$$

Next by (3.15), (3.27), (3.28) and (3.18), we get

$$\begin{aligned}
&(k-1)\alpha(\tau)^{m-1}\beta(\tau) - 1 \\
&= O\left(\frac{\tau^{k-1}}{\tau^{(m-1)}\tau^{(m-1)(p+1)/m}} + \frac{1}{\tau^{(p-q)(m-1)/m}} + \frac{\tau^{k-1}}{\tau^{(m-1)(p+1)/m}}\int_\tau^\infty s^{-k+\frac{(m-1)(q+1)}{m}}ds\right) \\
&= O\left(\tau^{-(p-q)(m-1)/m}\right).
\end{aligned} \tag{3.43}$$

From (3.42), for $t \leqq \tau$ we have

$$\zeta_\tau(t) \approx \frac{t}{\tau^{1/m}}\left(1 + \left(\frac{t}{\tau}\right)^{(k-m)/(m-1)}\right)^{-(m-1)/(k-m)}. \tag{3.44}$$

Hence

$$\begin{aligned}
\int_t^\tau \left(\int_\theta^\tau s^{-k}\zeta_\tau(s)^q ds\right)^{1/(m-1)} d\theta &\leqq \frac{1}{\tau^{q/m(m-1)}}\int_t^\tau\left(\int_\theta^\tau s^{-k+q}ds\right)^{1/(m-1)} d\theta \\
&= O\left(\frac{\tau^{1+(-k+q+1)/(m-1)}}{\tau^{q/m(m-1)}}\right) = O\left(\tau^{(q-p+m-1)/m}\right).
\end{aligned} \tag{3.45}$$

Since $m-1 \geqq 1$, we have for $a \geqq b \geqq 0$, that

$$(a-b)^{1/(m-1)} \geqq a^{1/(m-1)} - b^{1/(m-1)}. \tag{3.46}$$

Let $\eta = (k-1)\alpha(\tau)^{m-1}\beta(\tau)$. For $t \in [\tau_0, \tau]$, by integrating equation (2.5) from $t$ to $\tau$ and using (3.13) and (3.14), we obtain

$$\begin{aligned}
y(t) &= y(\tau) - \int_t^\tau \left[y'(\tau)^{m-1} + \int_\theta^\tau s^{-k}f(y(s))ds\right]^{1/(m-1)} d\theta \\
&\leqq y(\tau) - \int_t^\tau \left[y'(\tau)^{m-1} + \int_\theta^\tau s^{-k}f(\zeta_\tau(s))ds\right]^{1/(m-1)} d\theta \\
&\leqq y(\tau) - \int_t^\tau \left[y'(\tau)^{m-1} + \int_\theta^\tau s^{-k}\zeta_\tau^p(s)ds + \int_\theta^\tau s^{-k}(f(\zeta_\tau) - \zeta_\tau^p)ds\right]^{1/(m-1)} d\theta \\
&= y(\tau) - \int_t^\tau \left[\left(1 - \frac{1}{\eta}\right)y'(\tau)^{m-1} + \frac{\zeta_\tau'(\theta)^{m-1}}{\eta} - \int_\theta^\tau s^{-k}\left|f(\zeta_\tau) - \zeta_\tau^p\right|ds\right]^{1/(m-1)} d\theta.
\end{aligned}$$

Now $f(\zeta_\tau) - \zeta_\tau^p \approx -\zeta_\tau^q$ for $t \geqq \tau_0$. Therefore using (3.46), (3.45) and (3.42), we have

$$
\begin{aligned}
y(t) &\leqq y(\tau) - \int_t^\tau \left[\left(1 - \frac{1}{\eta}\right) y'(\tau)^{m-1} + \frac{\zeta'_\tau(\theta)^{m-1}}{\eta}\right]^{1/(m-1)} d\theta \\
&\quad + O\left(\int_t^\tau \left[\int_\theta^\tau s^{-k}\zeta_\tau^q ds\right]^{1/(m-1)} d\theta\right) \\
&\leqq \begin{cases} y(\tau) + (\zeta_\tau(t) - \zeta_\tau(\tau))\eta^{-1/(m-1)} + O\left(\tau^{(q-p+m-1)/m}\right) & \text{if } \eta \geqq 1, \\ y(\tau) + \left|1 - \frac{1}{\eta}\right|^{1/(m-1)} y'(\tau)(\tau - t) + (\zeta_\tau(t) - \zeta_\tau(\tau))\eta^{-1/(m-1)} & \\ \qquad + O\left(\tau^{(q-p+m-1)/m}\right) & \text{if } \eta \leqq 1. \end{cases}
\end{aligned}
$$

Since $\zeta_\tau(\tau) = y(\tau)$, we find from (3.42), (3.43), (3.27) and (3.28) that

$$
\begin{aligned}
y(t) &\leqq \left|1 - \eta^{-1/(m-1)}\right| y(\tau) + \left|1 - \frac{1}{\eta}\right|^{1/(m-1)} y'(\tau)\tau + \eta^{-1/(m-1)}\zeta_\tau(t) \\
&\quad + O\left(\tau^{(q-p+m-1)/m}\right) = O\left(\zeta_\tau(t) + \tau^{(q-p+m-1)/m}\right).
\end{aligned}
\tag{3.47}
$$

Also $\zeta_\tau\left(\tau^{(q-p+m)/m}\right) \approx \tau^{(q-p+m-1)/m} \to \infty$ as $\gamma_l \to \infty$, so that $\tau^{(q-p+m)/m} \geqq \tau_0$. Therefore from (3.13), (3.47) and (3.17) it follows that

$$
\begin{aligned}
C_1\tau^{(q-p+m-1)/m} &\leqq \zeta_\tau\left(\tau^{(q-p+m)/m}\right) \leqq y\left(\tau^{(q-p+m)/m}\right) \leqq C_2\tau^{(q-p+m-1)/m}, \\
C_1\tau^{-1/m} \leqq y'(\tau) &\leqq y'\left(\tau^{(q-p+m)/m}\right) \leqq y\left(\tau^{(q-p+m)/m}\right)\tau^{-(q-p+m)/m} \leqq C_2\tau^{-1/m}.
\end{aligned}
$$

This proves Step 4.

Now define the sequence $\{\delta_i\}$ by $\delta_0 = (q-p+m)/m$, $\delta_i = \rho(\delta_{i-1})$ for $i \geqq 1$. Since

$$
\begin{aligned}
\frac{q-p+m}{m} - 1 &= \frac{q-p}{m} < 0, \\
\frac{q-p+m}{m} - \frac{1}{m} &= \frac{q-p+m-1}{m} \geqq \frac{q-p+1}{m} > 0,
\end{aligned}
$$

we have $1/m < \delta_0 < 1$. Therefore from Steps 2, 3 and 4 we have $1/m < \delta_{i+1} < \delta_i < 1$. Hence $1/m \leqq \tilde{\delta} = \lim_{i\to\infty}\delta_i$, which contradicts Step 2. This proves the lemma and hence the Main Lemma.

*Remark 1.* In the case $n \geqq 7, m = 2, q = 1$, by using similar arguments, it is possible to show that for every positive integer $l$, there exists $\gamma_l > 0$ such that, for $\gamma \geqq \gamma_l$,

$$
y\left(\gamma^{2(2-p)^l}\right) \approx \gamma^{(2-p)^l}, \quad y'\left(\gamma^{2(2-p)^l}\right) \approx \gamma^{-(2-p)^l}.
$$

Since $0 < 2 - p < 1$, we see that $T_0(\gamma) = 0(\gamma^\varepsilon)$ for every $\varepsilon > 0$. Furthermore, if we define

$$Z(s) = \gamma^{-(2-p)^l} y\left(\gamma^{2(2-p)^l} s\right),$$

then $Z$ satisfies

$$-Z'' = t^{-k} f_{\gamma,l}(Z), \quad Z(1) \approx 1, \quad Z'(1) \approx 1,$$

where $f_{\gamma,l}(s) \to s^p$ as $\gamma \to \infty$. Hence $Z$ converges to a ground state solution $\zeta$ of the equation $-\zeta' = t^{-k}\zeta$. Such a point $t_l = \gamma^{2(2-p)^l}$ is called a blowup point for $y$. This implies that the solution $y$ exhibits an unbounded number of blow ups as $\gamma \to \infty$.

By a similar analysis, it can be shown for $\gamma$ large that

$$y\left(\gamma^{(3-p)(2-p)^l}\right) \approx \gamma^{(2-p)^{l+1}}, y'\left(\gamma^{(3-p)(2-p)^l}\right) \approx \gamma^{-(2-p)^l}.$$

Hence $y$ remains constant in $\left[\gamma^{2(2-p)^{l+1}}, \gamma^{(3-p)(2-p)^l}\right]$ while at the point $\gamma^{2(2-p)^{l+1}}$ the derivative $y'$ changes drastically. This phenomenon was noticed in [9] for similar problems in $n = 2$ with supercritical growth.

*Remark 2.* Consider the problem

$$\begin{array}{rcll} -\Delta u & = & (u+1)^{\frac{n+2}{n-2}} - (u+1) & \text{in } B(R)\,, \\ u & > & 0 & \text{in } B(R)\,, \\ u & = & 0 & \text{on } \partial B(R) \end{array}$$

where $B(R) \subset \mathbb{R}^n$ is a ball of radius $R$. From the analysis in [3, 5 and 6], it follows that for $n \in \{3, 4, 5, 6\}$, this above problem admits a solution for $R$ near zero, whereas for $n \geqq 7$, from (2.8), it follows that no solution exists for $R$ near zero. Thus the existence of a solution in the critical Sobolev exponent problem can be extremely sensitive to perturbations.

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