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Formula for a solution of $u_t + H(u, Du) = g$

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Abstract. We study the continuous as well as the discontinuous solutions of Hamilton–Jacobi equation $u_t + H(u, Du) = g$ in $\mathbb{R}^n \times \mathbb{R}_+$ with $u(x, 0) = u_0(x)$. The Hamiltonian $H(s, p)$ is assumed to be convex and positively homogeneous of degree one in p for each s in $\mathbb{R}$. If H is non increasing in s, in general, this problem need not admit a continuous viscosity solution. Even in this case we obtain a formula for discontinuous viscosity solutions.

Keywords. Hamilton–Jacobi equation; dynamic programming principle; viscosity sub and super solutions.

1. Introduction

Let $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $u_0 \in W^{1,\infty}(\mathbb{R}^n), g \in W^{1,\infty}(\mathbb{R}^n \times \mathbb{R}_+)$ and consider the Hamilton–Jacobi equation

$$\begin{aligned} &u_t + H(u, Du) = g \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ &u(x, 0) = u_0(x) \qquad \text{for } x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$. This problem has been studied extensively using the method of viscosity solutions developed by Crandall, Evans and Lions [7, 10]. An excellent reference for this is the lecture notes of Evans [7]. It has been shown ([10], Ch. 9, Remark 9.1) that apart from the usual hypothesis on H, if there exist a $\gamma \in \mathbb{R}$ such that for all $(s, p) \in \mathbb{R} \times \mathbb{R}^n$,

$$\frac{\partial H}{\partial s}(s, p) \geq \gamma, \tag{1.2}$$

then (1.1) admits a viscosity solution $u \in W^{1,\infty}(\mathbb{R}^n \times \mathbb{R}_+)$. The question is to obtain a formula for the solution of (1.1).

Under the conditions (1.2), $p \mapsto H(s, p)$ being convex and positively homogeneous of degree one and $g \equiv 0$, Barron, Jenssen and Liu [5] have obtained an explicit formula for the viscosity solution. In general for $g \not\equiv 0$, one cannot expect an explicit formula in the sense of Hopf and Lax, but one can hope to get an infinite dimensional representation in terms of a control problem. This has been carried out by Barron and Ishii [3] (representation formula) and Barron and Liu [4] (existence of a minimizer).

If H does not satisfy (1.2), in general (1.1) need not admit a continuous viscosity solution [1,10]. In this paper we study this problem under the hypothesis, $s \mapsto H(s, p)$ is non-increasing and $p \mapsto H(s, p)$ is convex and positively homogeneous of degree one. As far as our knowledge goes this problem has not been tackled in the literature. But in [1], the authors considered this problem with $g = 0$ and obtained explicit formula for solutions. Here we extend this result for $g \not\equiv 0$ (see Theorem 2.2). The main ingredients in the proof of this are to prove semicontinuity property for the constraints (Corollary 4.1) and the dynamic programming principle (Lemmas 4.7 and 4.8). The same idea allows us to study problem (1.1) when H satisfies (1.2) (see Theorem 2.1) and of course this result can be obtained also from the results of [3] and [4] with proper modifications.

2. Main results

Let $x \in \mathbb{R}^n, t > 0, 0 \le s < t, M > 0$ and define

$$C(x,t,s) = \{\xi \in W^{1,\infty}([s,t],\mathbb{R}^n);\ \xi(t) = x\}, \tag{2.1}$$

$$C_M(x,t,s) = \{\xi \in C(x,t,s);\ |\dot{\xi}|_\infty \le M\}, \tag{2.2}$$

$$C(x,t) = C(x,t,0), C_M(x,t) = C_M(x,t,0),$$

where $\dot{\xi}(\theta) = \mathrm{d}\xi/\mathrm{d}\theta(\theta)$. Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ be functions and $g \in W^{1,\infty}(\mathbb{R}^n \times \mathbb{R}_+)$. Let $0 \le s \le \theta \le t$, $\xi \in C(x,t,s)$, define

$$|u|_t = \sup\{|u(y,\theta)|;\ y \in \mathbb{R}^n, \theta \in [0,t]\},$$

$$\int_s^\theta g(\xi) = \int_s^\theta g(\xi(\lambda),\lambda)\,\mathrm{d}\lambda, \tag{2.3}$$

$$\rho_+(\xi,t,s,h,g) = \operatorname*{ess\,sup}_{\theta\in[s,t]} \left\{h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi)\right\}, \tag{2.4}$$

$$\rho_-(\xi,t,s,h,g) = \operatorname*{ess\,inf}_{\theta\in[s,t]} \left\{h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi)\right\}, \tag{2.5}$$

$$\rho_\pm(\xi,t,h,g) = \rho_\pm(\xi,t,0,h,g). \tag{2.6}$$

Then we have the following results.

Theorem 2.1. *Let $u_0 \in W^{1,\infty}(\mathbb{R}^n)$, $g \in W^{1,\infty}(\mathbb{R}^n \times \mathbb{R}_+)$. Assume that H satisfies,*

(H_1) *$s \mapsto H(s,p)$ is non decreasing for all $p \in \mathbb{R}^n$.*
(H_2) *$p \mapsto H(s,p)$ is convex, positively homogeneous of degree one for each $s \in \mathbb{R}$.*

Let h denote the quasi convex dual of H defined by

$$h(q) = \inf\{\gamma : H(\gamma,p) \ge \langle p,q\rangle\ \forall\,|p| = 1\}. \tag{2.7}$$

For $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, define

$$u(x,t) = \inf_{\xi\in C(x,t)} \left\{u_0(\xi(0)) \vee \rho_+(\xi,t,h,g) + \int_0^t g(\xi)\right\}. \tag{2.8}$$

Then for each $T > 0$, $u \in W^{1,\infty}(\mathbb{R}^n \times [0,T])$ and is a viscosity solution of (1.1). Furthermore infimum is achieved in (2.8).

Theorem 2.2. *Let $u_0 \in W^{1,\infty}(\mathbb{R}^n), g \in W^{1,\infty}(\mathbb{R}^n \times \mathbb{R}_+)$. Assume that H satisfies*

(H_3) *$s \mapsto H(s,p)$ is non-increasing for all $p \in \mathbb{R}^n$,*
(H_4) *$p \mapsto H(s,p)$ is convex, positively homogeneous of degree one, for each $s \in \mathbb{R}$.*

Let h denote the quasi concave dual of H defined by

$$h(q) = \sup\{\gamma : H(\gamma, p) \geq \langle p, q\rangle \ \forall\ |p| = 1\}. \tag{2.9}$$

For $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, define

$$\underline{u}(x,t) = \inf_{\xi \in C(x,t)} \left\{ u_0(\xi(0)) + \int_0^t g(\xi);\ u_0(\xi(0)) \leq \rho_-(\xi, t, h, g) \right\}, \tag{2.10}$$

$$\overline{u}(x,t) = \inf_{\xi \in C(x,t)} \left\{ u_0(\xi(0)) + \int_0^t g(\xi);\ u_0(\xi(0)) < \rho_-(\xi, t, h, g) \right\}. \tag{2.11}$$

Then $\underline{u}$ is a lower semicontinuous viscosity solution of (1.1) and $\overline{u}$ is an upper semicontinuous viscosity solution of (1.1). Also infimum is achieved in (2.10). Furthermore if $g(x,t) = g_1(x,t) + g_2(t)$, $t \to tg_1$ is non-increasing in t, $g_2(t) \leq 0$ and $H(u,p) > 0$ for all $p \neq 0, u \in \mathbb{R}$, then $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$. In this case the two solutions coincide.

Remark 2.3. For $g \equiv 0$, using Jenssen's inequality as in Lemma 3.3 of [5], Theorem 2.1 reduces to Theorem 3.1 of [5]. Also Theorem 2.2 reduces to Theorem 2.1 of [1].

3. Preliminaries

In this section we recall the definitions and known results from [9, 6, 5] and [4] without proofs.

DEFINITION 3.1

Let $\Omega \subset \mathbb{R}^n$ be a domain and V be a locally bounded function. For $x \in \overline{\Omega}$ define

$$V^*(x) = \limsup_{r \to 0} \{V(z) : |z - x| \leq r\},$$
$$V_*(x) = \liminf_{r \to 0} \{V(z) : |z - x| \leq r\}.$$

Then V^* is an upper semicontinuous and V_* is a lower semicontinuous functions and $V_* \leq V \leq V^*$.

As in [6] and [9], we have the following:

DEFINITION 3.2

Let U be a locally bounded function in $\mathbb{R}^n \times \mathbb{R}_+$.

1. U is said to be a subsolution of (1.1) if for any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+, \varphi \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ such that (x_0, t_0) is a local maximum for $U^* - \varphi$ with $U^*(x_0, t_0) = \varphi(x_0, t_0)$, then at (x_0, t_0), $\varphi_t + H(\varphi, D\varphi) \leq g$ and $U^*(x, 0) \leq u_0(x)$.
2. U is said to be a super solution of (1.1) if for any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+, \varphi \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ such that (x_0, t_0) is a local minimum for $U_* - \varphi$ with $U_*(x_0, t_0) = \varphi(x_0, t_0)$, then at $(x_0, t_0), \varphi_t + H(\varphi, D\varphi) \geq g$ and $U_*(x, 0) \geq u_0(x)$.
3. U is said to be a viscosity solution of (1.1) if U is both a sub and a super solution.

Now recall some properties of quasi convex (concave) dual of H.

Let H satisfy (H_1) and (H_2). Then

(A_1) h is a lower semicontinuous quasi convex function i.e, for any $q_1, q_2 \in \mathbb{R}^n$, $t \in [0, 1]$, $h(tq_1 + (1 - t)q_2) \leq \max\{h(q_1), h(q_2)\}$,

(A_2) $\inf h = -\infty, \lim_{|q|\to\infty} h(q) = \infty$,
(A_3) $H(s,p) = \sup\{\langle p,q\rangle; h(q) \le s\}$.

Proofs of (A_1) to (A_3) follow from lemmas (2.1) and (2.2) of [5].

Let H satisfy (H_3) and (H_4) of Theorem 2.2 and let h be the quasi concave dual of H. Then

(A_4) h is an upper semicontinuous quasi concave function i.e, for $t \in [0,1]$, $q_1, q_2 \in \mathbb{R}^n$, $h(tq_1 + (1-t)q_2) \ge \min\{h(q_1), h(q_2)\}$,
(A_5) $\sup h = +\infty, \lim_{|q|\to\infty} h(q) = -\infty$,
(A_6) $H(s,p) = \sup\{\langle p,q\rangle : s \le h(q)\}$.

(A_4) to (A_6) follow from (A_1) to (A_3) applied to the Hamiltonian $\tilde{H}(s,p) = H(-s,p)$.

4. Proof of theorems

Before going to the proof of the theorems, we need the following lemma for proving the existence of a minimizer and the semicontinuity of $\underline{u}$ and $\overline{u}$.

Let $0 \le s < t$ and $1 \le p \le \infty$. Let $b, b_\kappa : L^p([s,t]) \times [s,t] \to \mathbb{R}$ be continuous functions. Assume that b, b_κ satisfies the following hypothesis: For $\xi, \xi_\kappa \in L^p([s,t]), \theta$, $\theta_\kappa \in [s,t]$ such that $\xi_\kappa \to \xi$ strongly in L^p and $\theta_\kappa \to \theta$ as $\kappa \to \infty$, then

$$\lim_{\kappa\to\infty} b_\kappa(\xi_\kappa, \theta_\kappa) = b(\xi, \theta).$$

Lemma 4.1. *Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function and b, b_κ satisfying the above hypothesis and $\eta, \eta_\kappa \in L^p([s,t], \mathbb{R}^n)$ such that $\eta_\kappa \rightharpoonup \eta$ weakly. Then*

(a) *Assume that h is a lower semicontinuous quasi convex function. Then*

$$\varliminf_{\kappa\to\infty} \operatorname*{ess\,sup}_{\theta\in[s,t]} \{h(\eta_\kappa(\theta)) + b_\kappa(\xi_\kappa, \theta)\} \ge \operatorname*{ess\,sup}_{\theta\in[s,t]} \{h(\eta(\theta)) + b(\xi, \theta)\} \tag{4.1}$$

(b) *Assume that h is an upper semicontinuous quasi concave function. Then*

$$\varlimsup_{\kappa\to\infty} \operatorname*{ess\,inf}_{\theta\in[s,t]} \{h(\eta_\kappa(\theta)) + b_\kappa(\xi_\kappa, \theta)\} \le \operatorname*{ess\,inf}_{\theta\in[s,t]} \{h(\eta(\theta)) + b(\xi, \theta)\} \tag{4.2}$$

Proof. Observe that (a) follows from (b) by changing h to $-h, b_\kappa$ to $-b_\kappa$. Hence it is enough to prove (b). Let

$$C = \varlimsup_{\kappa\to\infty} \operatorname*{ess\,inf}_{\theta\in[s,t]} \{h(\eta_\kappa(\theta)) + b_\kappa(\xi_\kappa, \theta)\}.$$

If $C = -\infty$, then there is nothing to prove. Let $m > 0$ and choose $\kappa(m)$ such that $\kappa(m) \to \infty$ as $m \to \infty$ and for any $\kappa \ge \kappa(m)$

$$C - \frac{1}{m} \le \operatorname*{ess\,inf}_{\theta\in[s,t]} \{h(\eta_\kappa(\theta)) + b_\kappa(\xi_\kappa, \theta)\}.$$

Since $\eta_\kappa \rightharpoonup \eta$ as $\kappa \to \infty$, hence there exist $0 \le \alpha_{\kappa l} \le 1, \Sigma_{\kappa\ge\kappa(m)}\alpha_{\kappa l} = 1, \alpha_{\kappa l \neq 0}$ for all but a finitely many κ such that $f_l = \Sigma_{\kappa\ge\kappa(m)}\alpha_{\kappa l}\eta_\kappa \to \eta$ strongly in L^p as $l \to \infty$, ([11], theorem 3.13). Hence extracting a subsequence still denoted by f_l and a null set $N \subset [s,t]$

such that for all $\theta \notin N$, $f_l(\theta) \to \eta(\theta)$. Let $\theta \notin N$ and choose a $\kappa_l \geq \kappa(m)$ such that

$$\min_{\alpha_{\kappa l} \neq 0} \{h(\eta_\kappa(\theta))\} = h(\eta_{\kappa_l}(\theta)).$$

Then by quasi concavity we have

$$\begin{aligned} h(f_l(\theta)) + b_{\kappa_l}(\xi_{\kappa_l}, \theta) &\geq \min_{\alpha_{\kappa_l} \neq 0} \{h(\eta_\kappa(\theta)) + b_{\kappa_l}(\xi_{\kappa_l}, \theta)\} \\ &= h(\eta_{\kappa_l}(\theta)) + b_{\kappa_l}(\xi_{\kappa_l}, \theta) \geq C - \frac{1}{m}. \end{aligned}$$

Since h is upper semicontinuous, now letting $l \to \infty$ and $m \to \infty$, we obtain for all $\theta \notin N$

$$C \leq h(\eta(\theta)) + b(\xi, \theta)$$

and this proves (4.2).

As a consequence of this lemma we have the following result. Let $0 \leq s_\kappa < t_\kappa, 0 \leq s < t$ such that $(s_\kappa, t_\kappa) \to (s, t)$ as $\kappa \to \infty$. Let $\{\xi_\kappa\} \in W^{1,\infty}([s_\kappa, t_\kappa], \mathbb{R}^n)$ be a bounded sequence and $g \in C^0(\mathbb{R}^n \times \mathbb{R}_+) \cap L^\infty$. Let $\alpha_\kappa : [s_\kappa, t_\kappa] \to [s, t]$ be defined by $\alpha_\kappa(\theta) = \frac{t-s}{t_\kappa - s_\kappa}\theta + \frac{t_\kappa s - t s_\kappa}{t_\kappa - s_\kappa}$, $\tilde{\xi}_\kappa(\theta) = \xi_\kappa(\alpha_\kappa^{-1}(\theta))$ and $b_\kappa(\tilde{\xi}_\kappa, \theta) = \frac{s_\kappa - t_\kappa}{t-s}\int_s^\theta g(\tilde{\xi}_\kappa(\lambda), \alpha_\kappa^{-1}(\lambda))\mathrm{d}\lambda$.

COROLLARY 4.1

Assume that as $\kappa \to \infty$, $\tilde{\xi}_\kappa \to \xi$ *in* C^0*-topology for some* $\xi \in W^{1,\infty}([s,t], \mathbb{R}^n)$. *Let* $h : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *be a function. Then*

(a) *Assume that h is a lower semicontinuous and quasi convex function, then*

$$\varliminf_{\kappa \to \infty} \operatorname*{ess\,sup}_{\theta \in [s_\kappa, t_\kappa]} \left\{h(\dot{\xi}_\kappa(\theta)) - \int_{s_\kappa}^\theta g(\xi_\kappa)\right\} \geq \operatorname*{ess\,sup}_{\theta \in [s,t]} \left\{h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi)\right\}. \tag{4.3}$$

(b) *Assume that h is an upper semicontinuous and quasi concave function, then*

$$\varlimsup_{\kappa \to \infty} \operatorname*{ess\,inf}_{\theta \in [s_\kappa, t_\kappa]} \left\{h(\dot{\xi}_\kappa(\theta)) - \int_{s_\kappa}^\theta g(\xi_\kappa)\right\} \leq \operatorname*{ess\,inf}_{\theta \in [s,t]} \left\{h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi)\right\}. \tag{4.4}$$

Proof. (a) follows from (b) by changing h to $-h$ and g to $-g$. Hence it is enough to prove (b). By change of variables we have

$$\begin{aligned} &\operatorname*{ess\,inf}_{\theta \in [s_\kappa, t_\kappa]} \left\{h(\dot{\xi}_\kappa(\theta)) - \int_{s_\kappa}^\theta g(\xi_\kappa)\right\} \\ &\quad = \operatorname*{ess\,inf}_{\theta \in [s,t]} \left\{h\left(\left(\frac{t-s}{t_\kappa - s_\kappa}\right)\dot{\tilde{\xi}}_\kappa(\theta)\right) + b_\kappa(\tilde{\xi}_\kappa, \theta)\right\}. \end{aligned}$$

Since $\{\dot{\tilde{\xi}}_\kappa\}$ is a bounded sequence in $L^2([s,t], \mathbb{R}^n)$ and $\tilde{\xi}_\kappa \to \xi$ strongly in L^2, hence if $\eta_\kappa(\theta) = \frac{t-s}{t_\kappa - s_\kappa}\dot{\tilde{\xi}}_\kappa(\theta)$, then $\eta_\kappa \rightharpoonup \dot{\xi}$ weakly in L^2. Now (4.4) follows from (4.2). This proves the corollary.

Theorem 2.1.

In order to prove Theorem 2.1, we will first establish the dynamic programming principle. In order to do this we need some information on the bounds of the cost function.

Lemma 4.2. *Let W be a function on* $\mathbb{R}^n \times \mathbb{R}_+$. *Assume that for every* $T > 0$, $|W|_T = \sup\{|W(x,t)|; (x,t) \in \mathbb{R}^n \times [0,T]\} < \infty$. *For* $0 \le s < t \le T$ *and* $x \in \mathbb{R}^n$, *define*

$$V(x,t) = \inf_{\xi \in C(x,t,s)} \left\{ W(\xi(s),s) \vee \rho_+(\xi,t,s,h,g) + \int_s^t g(\xi) \right\}. \tag{4.5}$$

Then there exist a constant $M(T) > 0$ *such that*

$$|V(x,t)| \le |W|_T + T|g|_\infty, \tag{4.6}$$

$$V(x,t) = \inf_{\xi \in C_{M(T)}(x,t,s)} \left\{ W(\xi(s),s) \vee \rho_+(\xi,t,s,h,g) + \int_s^t g(\xi) \right\}. \tag{4.7}$$

Furthermore if W is a continuous function, then there exist a $\xi \in C_{M(T)}(x,t,s)$ *such that*

$$V(x,t) = W(\xi(s),s) \vee \rho_+(\xi,t,s,h,g) + \int_s^t g(\xi). \tag{4.8}$$

Proof. Let $M_1(T) = |W|_T + T|g|_\infty$. Since $\inf h = -\infty$, there exist a $q \in \mathbb{R}^n$ such that $h(q) + T|g|_\infty < -|W|_T$. Let $\xi(\theta) = x + q(\theta - t) \in C(x,t,s)$ and hence $\rho_+(\xi,t,s,h,g) \le h(q) + T|g|_\infty < -|W|_T$. Therefore $V(x,t) \le |W|_T + T|g|_\infty \le M_1(T)$. Also for any $\xi \in C(x,t,s)$,

$$W(\xi(s),s) \vee \rho_+(\xi,t,s,h,g) + \int_s^t g(\xi) \ge W(\xi(s),s) + \int_s^t g(\xi) \ge -|W|_T - T|g|_\infty.$$

Hence $|V(x,t)| \le M_1(T)$. This proves (4.6).

Since $\lim_{|p|\to\infty} h(p) = \infty$, we can choose a $M(T) > 0$ such that whenever $|p| \ge M(T)$ then $h(p) \ge 3(|W|_T + T|g|_\infty)$. Let $\xi \in C(x,t,s)$ such that $|\dot{\xi}|_\infty \ge M(T)$. Then we have

$$\begin{aligned} \operatorname*{ess\,sup}_{\theta \in [s,t]} \left\{ h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi) \right\} &\ge \operatorname*{ess\,sup}_{\theta \in [s,t]} \{h(\dot{\xi}(\theta))\} - T|g|_\infty \\ &\ge 3(|W|_T + T|g|_\infty) - T|g|_\infty \\ &\ge 2|W|_T . \end{aligned}$$

Hence from (4.6) we have

$$\begin{aligned} W(\xi(s),s) \vee \rho_+(\xi,t,s,h,g) + \int_s^t g(\xi) &= \rho_+(\xi,t,s,h,g) + \int_s^t g(\xi) \\ &\ge 3(|W|_T + T|g|_\infty) - 2T|g|_\infty \\ &> |V(x,t)|. \end{aligned}$$

This proves (4.7).

Let W be a continuous function. Since V is bounded, from (4.7) we can choose a sequence $\xi_\kappa \in C_{M(T)}(x,t,s)$ such that

$$V(x,t) = \lim_{\kappa \to \infty} \left\{ W(\xi_\kappa(s),s) \vee \rho_+(\xi_\kappa,t,s,h,g) + \int_s^t g(\xi_\kappa) \right\}.$$

Since $|\dot{\xi}_\kappa| \le M(T)$ and $\xi_\kappa(t) = x$, hence by Arzela–Ascoli, for a subsequence still denoted by ξ_κ such that $\xi_\kappa \to \xi$ uniformly in $[s,t]$. Since $|\dot{\xi}_\kappa| \le M(T)$ implies that $\xi \in C_{M(T)}(x,t,s)$.

Again by going to a subsequence we can assume that $\xi_\kappa \rightharpoonup \xi$ weakly in $W^{1,2}([s,t],\mathbb{R}^n)$. Hence by (4.3) and continuity of W and g it follows that

$$\begin{aligned} V(x,t) &= \lim_{\kappa\to\infty}\left\{W(\xi_\kappa(s),s)\vee\rho_+(\xi_\kappa,t,s,h,g)+\int_s^t g(\xi_\kappa)\right\} \\ &\geq W(\xi(s),s)\vee\rho_+(\xi,t,s,h,g)+\int_s^t g(\xi). \end{aligned}$$

Since $\xi \in C(x,t,s)$ and therefore by definition of V and the above inequality implies (4.8). This proves the lemma.

Lemma 4.3 (*Dynamic programming principle*). *Let u be as in* (2.8) *and* $0 \leq s < t$ *and* $x \in \mathbb{R}^n$, *then*

$$u(x,t) = \inf_{\xi\in C(x,t,s)}\left\{u(\xi(s),s)\vee\rho_+(\xi,t,s,h,g)+\int_s^t g(\xi)\right\}. \tag{4.9}$$

Proof. Let $v(x,t)$ denote the right hand side of (4.9). Let $\xi \in C(x,t)$. Then $\xi_1 = \xi|_{[0,s]} \in C(\xi(s),s)$ and $\xi_2 = \xi|_{[s,t]} \in C(x,t,s)$. Hence

$$\begin{aligned} v(x,t) &\leq u(\xi_2(s),s)\vee\rho_+(\xi_2,t,s,h,g)+\int_s^t g(\xi_2) \\ &\leq (u_0(\xi_1(0))\vee\rho_+(\xi_1,s,h,g)+\int_0^s g(\xi_1))\vee\rho_+(\xi_2,t,s,h,g)+\int_s^t g(\xi_2) \\ &= u_0(\xi(0))\vee\rho_+(\xi_1,s,h,g)\vee(\rho_+(\xi_2,t,s,h,g)-\int_0^s g(\xi_1))+\int_0^t g(\xi) \\ &= u_0(\xi(0))\vee\rho_+(\xi,t,h,g)+\int_0^t g(\xi). \end{aligned}$$

By taking infimum over ξ, this implies that $v(x,t) \leq u(x,t)$.

Since $u_0 \in W^{1,\infty}$, by Lemma (4.2), for any $T > 0$, $|u|_T < \infty$ and hence $|v|_T < \infty$. Hence for $\epsilon > 0$, choose $\xi_2 \in C(x,t,s)$ and $\xi_1 \in C(\xi_2(s),s)$ such that

$$\begin{aligned} v(x,t) &\geq u(\xi_2(s),s)\vee\rho_+(\xi_2,t,s,h,g)+\int_s^t g(\xi_2)-\epsilon, \\ u(\xi_2(s),s) &\geq u_0(\xi_1(0))\vee\rho_+(\xi_1,s,h,g)+\int_0^s g(\xi_1)-\epsilon. \end{aligned}$$

Let $\xi \in C(x,t)$ be defined by $\xi|_{[0,s]} = \xi_1, \xi|_{[s,t]} = \xi_2$. Then

$$\begin{aligned} v(x,t) &\geq \left(u_0(\xi(0))\vee\rho_+(\xi,s,h,g)+\int_0^s g(\xi)-\epsilon\right)\vee\rho_+(\xi,t,s,h,g) \\ &\qquad +\int_s^t g(\xi)-\epsilon \\ &= (u_0(\xi(0))\vee\rho_+(\xi,s,h,g)-\epsilon)\vee\left(\rho_+(\xi,t,s,h,g)-\int_0^s g(\xi)\right) \\ &\qquad +\int_0^t g(\xi)-\epsilon \end{aligned}$$

$$
\begin{aligned}
&\geq u_0(\xi(0)) \vee \rho_+(\xi, s, h, g) \vee \left(\rho_+(\xi, t, s, h, g) - \int_0^s g(\xi)\right) + \int_0^t g(\xi) - 2\epsilon \\
&= u_0(\xi(0)) \vee \operatorname*{ess\,sup}_{\theta\in[0,s]} \left\{ h(\dot{\xi}(\theta)) - \int_0^\theta g(\xi) \right\} \\
&\quad \vee \left(\operatorname*{ess\,sup}_{\theta\in[s,t]} \left\{ h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi) \right\} - \int_0^s g(\xi) \right) + \int_0^t g(\xi) - 2\epsilon \\
&= u_0(\xi(0)) \vee \operatorname*{ess\,sup}_{\theta\in[0,s]} \left\{ h(\dot{\xi}(\theta)) - \int_0^\theta g(\xi) \right\} \\
&\quad \vee \operatorname*{ess\,sup}_{\theta\in[s,t]} \left\{ h(\dot{\xi}) - \int_0^\theta g(\xi) \right\} + \int_0^t g(\xi) - 2\epsilon \\
&= u_0(\xi(0)) \vee \operatorname*{ess\,sup}_{\theta\in[0,t]} \left\{ h(\dot{\xi}(\theta)) - \int_0^\theta g(\xi) \right\} + \int_0^t g(\xi) - 2\epsilon \\
&\geq u(x,t) - 3\epsilon.
\end{aligned}
$$

Letting $\epsilon \to 0$ to obtain $v(x,t) \geq u(x,t)$ and hence $v = u$. This proves the lemma.

Lemma 4.4. *Let u be as in* (2.8). *Then for every $T > 0$, $u \in W^{1,\infty}(\mathbb{R}^n \times [0,T])$ and* $\lim_{t\to 0} u(x,t) = u_0(x)$.

Proof. Let $T > 0$, $0 < t \leq T$, $x_1, x_2 \in \mathbb{R}^n$. Since $u_0 \in W^{1,\infty}(\mathbb{R}^n)$, hence by Lemma (4.2), there exist a constant $M(T) > 0$, $\xi_1 \in C_{M(T)}(x_1, t)$ such that $|u|_T < \infty$ and $u(x_1,t) = u_0(\xi_1(0)) \vee \rho_+(\xi_1, t, h, g) + \int_0^t g(\xi_1)$. Now define $\xi_2(\theta) = \xi_1(\theta) + x_2 - x_1$, then $\xi_2 \in C(x_2, t)$ with $\dot{\xi}_1 = \dot{\xi}_2$. Let M denote the maximum of the Lipschitz constants for u_0 and g. Then

$$
\begin{aligned}
h(\dot{\xi}_2(\theta)) - \int_0^\theta g(\xi_2) &= h(\dot{\xi}_1(\theta)) - \int_0^\theta g(\xi_1) + \int_0^\theta g(\xi_1) - \int_0^\theta g(\xi_2) \\
&\leq h(\dot{\xi}_1(\theta)) - \int_0^\theta g(\xi_1) + MT|\xi_1 - \xi_2|_\infty.
\end{aligned}
$$

Hence

$$\rho_+(\xi_2, t, h, g) \leq \rho_+(\xi_1, t, h, g) + MT|x_2 - x_1|.$$

Therefore

$$
\begin{aligned}
u(x_2,t) &\leq u_0(\xi_2(0)) \vee \rho_+(\xi_2, t, h, g) + \int_0^t g(\xi_2) \\
&\leq (u_0(\xi_1(0)) + u_0(\xi_2(0)) - u_0(\xi_1(0))) \vee (\rho_+(\xi_1, t, h, g) \\
&\quad + MT|x_2 - x_1|) + \int_0^t g(\xi_1) + \int_0^t (g(\xi_2) - g(\xi_1)) \\
&\leq (u_0(\xi_1(0)) + M|x_2 - x_1|) \vee (\rho_+(\xi_1, t, h, g) + MT|x_2 - x_1|) \\
&\quad + \int_0^t g(\xi_1) + MT|x_2 - x_1|
\end{aligned}
$$

$$\begin{aligned}&\leq (u_0(\xi_1(0)) + M(1+T)|x_2 - x_1|) \vee (\rho_+(\xi_1, t, h, g)\\ &\quad + M(1+T)|x_2 - x_1|) + \int_0^t g(\xi_1) + MT|x_2 - x_1|\\ &\leq u_0(\xi_1(0)) \vee \rho_+(\xi_1, t, h, g) + \int_0^t g(\xi_1) + 2M(1+T)|x_2 - x_1|\\ &= u(x_1, t) + 2M(1+T)|x_2 - x_1|.\end{aligned}$$

Since x_1 and x_2 are arbitrary, the above implies that

$$|u(x_1, t) - u(x_2, t)| \leq 2M(1+T)|x_2 - x_1|. \tag{4.10}$$

Let $0 \leq s < t \leq T$ and $x \in \mathbb{R}^n$. Since $\inf h = -\infty$ and hence there exist a $q \in \mathbb{R}^n$ such that $h(q) + T|g|_T < -|u|_T$. Let $\xi(\theta) = x + q(\theta - t) \in C(x, t, s)$. Then $|x - \xi(s)| = |q||t - s|$ and $\rho_+(\xi, t, s, h, g) \leq h(q) + T|g|_T < -|u|_T \leq u(\xi(s), s)$. Hence from (4.9) and (4.10)

$$\begin{aligned}u(x,t) - u(x,s) &\leq u(\xi(s), s) \vee \rho_+(\xi, t, s, h, g) + \int_s^t g(\xi) - u(x,s)\\ &\leq u(\xi(s), s) - u(x,s) + |g|_T|t - s|\\ &\leq 2M(1+T)|\xi(s) - x||g|_T|t - s|\\ &\leq (2M(1+T)|q| + |g|_T)|t - s|.\end{aligned} \tag{4.11}$$

Since $|u|_T < \infty$ and hence from (4.9), (4.7) and (4.10) there exist an $M(T) > 0$ such that

$$\begin{aligned}u(x,t) &= \inf_{\xi \in C_{M(T)}(x,t,s)} \left\{ u(\xi(s), s) \vee \rho_+(\xi, t, s, h, g) + \int_s^t g(\xi) \right\}\\ &\geq \inf_{\xi \in C_{M(T)}(x,t,s)} \left\{ u(\xi(s), s) + \int_s^t g(\xi) \right\}\\ &\geq \inf_{\xi \in C_{M(T)}(x,t,s)} \{ u(\xi(s), s) - u(x,s) - M|t - s| \} + u(x,s)\\ &\geq \inf_{\xi \in C_{M(T)}(x,t,s)} \{ -2M(1+T)|\xi(s) - x| \} - M|t - s| + u(x,s)\\ &\geq -2M(1+T)M(T)|s - t| - M|t - s| + u(x,s)\\ &\geq -M_1(T)|s - t| + u(x,s),\end{aligned}$$

where $M_1(T) = 2M(1+T)M(T) + M$. Combining this with (4.11) implies $|u(x,t) - u(x,s)| \leq M_1(T)|t - s|$. By taking $s = 0$, we obtain $\lim_{t \to 0} u(x,t) = u_0(x)$ and hence the lemma.

Proof of Theorem 2.1. First we prove that u is a subsolution. Suppose not, then there exist a $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$, $\epsilon > 0$, a ball B around (x_0, t_0) and a C^1 function φ such that $\varphi(x_0, t_0) = u(x_0, t_0)$, $u - \varphi$ has maximum at (x_0, t_0) in B and $\varphi_t + H(\varphi, D\varphi) - g \geq 4\epsilon$ at (x_0, t_0). By continuity we can choose a $\delta > 0$ such that at (x_0, t_0), $\varphi_t + H(\varphi - 2\delta, D\varphi) - g \geq 3\epsilon$. Hence from (A_3) of § 3 there exist q such that at (x_0, t_0), $\varphi_t + \langle q, D\varphi \rangle - g \geq 2\epsilon$, $h(q) \leq \varphi(x_0, t_0) - 2\delta$. Now by continuity, there exist a ball $B_1 \subset B$ around (x_0, t_0) such that in B_1

$$h(q) \leq \varphi - \delta, \varphi_t + \langle q, D\varphi \rangle - g \geq \epsilon. \tag{4.12}$$

Let $s_0 < t_0$ be such that the curve $\xi(\theta) = x_0 + q(\theta - t_0)$ for $\theta \in [s_0, t_0]$ is in B_1 and $\sup\{|\int_{s_0}^{\theta} g(\xi)|;\ \theta \in [s_0, t_0]\} < \delta$ from (4.9) and (4.12) we have

$$\begin{aligned}
\varphi(x_0, t_0) = u(x_0, t_0) &\leq u(\xi(s_0), s_0) \vee \rho_+(\xi, t_0, s_0, h, g) + \int_{s_0}^{t_0} g(\xi) \\
&= u(\xi(s_0), s_0) \vee \left\{ h(q) - \inf_{\theta \in [s_0, t_0]} \int_{s_0}^{\theta} g(\xi) \right\} + \int_{s_0}^{t_0} g(\xi) \\
&\leq u(\xi(s_0), s_0) \vee \{h(q) + \delta\} + \int_{s_0}^{t_0} g(\xi) \qquad (4.13) \\
&\leq u(\xi(s_0), s_0) \vee \varphi(\xi(s_0), s_0) + \int_{s_0}^{t_0} g(\xi) \\
&\leq \varphi(\xi(s_0), s_0) + \int_{s_0}^{t_0} g(\xi).
\end{aligned}$$

Also from (4.12)

$$\begin{aligned}
\varphi(x_0, t_0) - \varphi(\xi(s_0), s_0) &= \int_{s_0}^{t_0} \frac{\mathrm{d}}{\mathrm{d}\theta} \varphi(\xi(\theta), \theta) \mathrm{d}\theta \\
&= \int_{s_0}^{t_0} (\varphi_t + \langle q, D\varphi \rangle \mathrm{d}\theta) \geq \int_{s_0}^{t_0} g(\xi) + \epsilon(t_0 - s_0)
\end{aligned}$$

which contradicts (4.13). This proves that u is a subsolution.

Next we prove that u is a supersolution. Suppose not, then there exists $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+, \epsilon > 0$, a ball B around (x_0, t_0) and a C^1-function φ such that $u(x_0, t_0) = \varphi(x_0, t_0)$, $u - \varphi \geq 0$ in B, $\varphi_t + H(\varphi, D\varphi) - g \leq -3\epsilon$ in B. Hence from (A_3) of § 3, for $(x, t) \in B, q \in \mathbb{R}^n$

$$\begin{cases} (\varphi_t + \langle q, D\varphi \rangle - g)(x, t) \leq -3\epsilon, \\ \quad \text{whenever } h(q) \leq \varphi(x, t). \end{cases} \qquad (4.14)$$

From Lemma (4.4), u is continuous and $|u|_T < \infty$, for any $T > 0$. Hence from (4.9) and (4.8) for every $s < t$, there exist a $\xi_s \in C_{M(T)}(x_0, t_0, s)$ such that

$$u(x_0, t_0) = u(\xi(s), s) \vee \rho_+(\xi_s, t_0, s, h, g) + \int_s^{t_0} g(\xi_s). \qquad (4.15)$$

Since $|\dot{\xi}_s| \leq M(T)$ and hence by choosing s_0 sufficiently close to t_0, $\xi_s \in B$ for all $s \in [s_0, t_0]$.

Claim. There exist $s_1 \in [s_0, t_0)$ such that for a.e. $\theta \in [s_1, t_0]$

$$\varphi_t(\xi_{s_1}(\theta), \theta) + \langle \dot{\xi}_{s_1}(\theta), D\varphi(\xi_{s_1}(\theta), \theta) \rangle - g(\xi_{s_1}(\theta), \theta) \leq -\epsilon.$$

Suppose not, then there exist a sequence $s_m \to t_0, \theta_m \in (s_m, t_0)$ with $\xi_m = \xi_{s_m}$,

$$(\varphi_t + \langle \dot{\xi}_m, D\varphi \rangle - g)(\xi_m(\theta_m), \theta_m) > -\epsilon.$$

Let for a subsequence, $\dot{\xi}_m(\theta_m) \to q$ as $m \to \infty$. Since $\theta_m \to t_0, \xi_m(\theta_m) \to x_0$, we obtain from the above inequality

$$(\varphi_t + \langle q, D\varphi \rangle - g)(x_0, t_0) \geq -\epsilon. \qquad (4.16)$$

On the other hand from (4.15) and lower semicontinuity of h we have

$$\begin{aligned}\varphi(x_0,t_0)=u(x_0,t_0) &\geq \lim_{m\to\infty}\left\{\rho_+(\xi_m,t_0,s_m,h,g)+\int_{s_m}^{t_0} g(\xi_m)\right\}\\ &\geq \lim_{m\to\infty}\left[\left\{h(\dot{\xi}_m(\theta_m))-\int_{s_m}^{\theta_m} g(\xi_m)\right\}+\int_{s_m}^{t_0} g(\xi_m)\right]\\ &\geq h(q).\end{aligned}$$

Hence from (4.14), $(\varphi_t+\langle q,D\varphi\rangle-g)(x_0,t_0)\leq -3\epsilon$, contradicting (4.16). This proves the claim. From the above claim we have

$$\begin{aligned}\varphi(x_0,t_0)-\varphi(\xi_{s_1}(s_1),s_1) &= \int_{s_1}^{t_0}\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\xi_{s_1}(\theta),\theta)\mathrm{d}\theta\\ &= \int_{s_1}^{t_0}(\varphi_t+\langle\dot{\xi}_{s_1},D\varphi\rangle(\xi_{s_1}(\theta),\theta)\mathrm{d}\theta \qquad (4.17)\\ &\leq \int_{s_1}^{t_0} g(\xi_{s_1})-\epsilon(t_0-s_1).\end{aligned}$$

From (4.15) we have

$$\begin{aligned}\varphi(x_0,t_0)=u(x_0,t_0) &\geq u(\xi_{s_1}(s_1),s_1)+\int_{s_1}^{t_0} g(\xi_{s_1})\\ &\geq \varphi(\xi_{s_1}(s_1),s_1)+\int_{s_1}^{t_0} g(\xi_{s_1}),\end{aligned}$$

which contradicts (4.17). This proves that u is a super solution. Furthermore from (4.8) infimum is achieved and this proves the theorem.

Theorem 2.2.

From now on we assume that H satisfies (H_3) and (H_4) of Theorem 2.2 and h be its quasi concave dual. Let ρ_- be defined as in (2.5).

Lemma 4.5. *Let W be a function on $\mathbb{R}^n\times\mathbb{R}_+$. Assume that for every $T>0, |W|_T=\sup\{|W(x,t)|:(x,t)\in\mathbb{R}^n\times[0,T]\}<\infty$. Let $0\leq s<t\leq T$ and $x\in\mathbb{R}^n$. Define*

$$\underline{V}(x,t)=\inf_{\xi\in C(x,t,s)}\left\{W(\xi(s),s)+\int_s^t g(\xi);W(\xi(s),s)\leq\rho_-(\xi,t,s,h,g)\right\}, \quad (4.18)$$

$$\overline{V}(x,t)=\inf_{\xi\in C(x,t,s)}\left\{W(\xi(s),s)+\int_s^t g(\xi);W(\xi(s),s)<\rho_-(\xi,t,s,h,g)\right\}, \quad (4.19)$$

then there exist a constant $M(T)>0$ such that

$$|\underline{V}(x,t)|\vee|\overline{V}(x,t)|\leq|W|_T+T|g|_\infty, \quad (4.20)$$

$$\underline{V}(x,t)=\inf_{\xi\in C_{M(T)}(x,t,s)}\left\{W(\xi(s),s)+\int_s^t g(\xi);W(\xi(s),s)\leq\rho_-(\xi,t,s,h,g)\right\}, \quad (4.21)$$

$$\overline{V}(x,t)=\inf_{\xi\in C_{M(T)}(x,t,s)}\left\{W(\xi(s),s)+\int_s^t g(\xi);W(\xi(s),s)<\rho_-(\xi,t,s,h,g)\right\}. \quad (4.22)$$

Furthermore if W is lower semicontinuous function, then there exist a $\xi \in C_{M(T)}(x,t,s)$ such that

$$W(\xi(s),s) \le \rho_-(\xi,t,s,h,g), \quad \underline{V}(x,t) = W(\xi(s),s) + \int_s^t g(\xi). \tag{4.23}$$

If W is continuous, then there exist $\{\eta_\kappa\} \subset C_{M(T)}(x,t,s), \eta \in C_{M(T)}(x,t,s)$ such that $\eta_\kappa \to \eta$ in C^0 and

$$\overline{V}(x,t) = W(\eta(s),s) + \int_s^t g(\eta), \tag{4.24}$$

$$W(\eta_\kappa(s),s) < \rho_-(\eta_\kappa,t,s,h,g). \tag{4.25}$$

Proof. Since $-(|W|_T + T|g|_\infty) \le W(\xi(s),s) + \int_s^t g(\xi) \le (|W|_T + T|g|_\infty)$ and hence (4.20) follows. Since $\lim_{|p|\to\infty} h(p) = -\infty$, there exist $M(T) > 0$ such that if $|p| > M(T)$, then $h(p) < -2(|W|_T + T|g|_\infty)$. Let $\xi \in C(x,t,s)$ such that $|\dot{\xi}|_\infty > M(T)$. Then for θ in a set of positive measure in $[s,t]$

$$h(\dot{\xi}(\theta)) - \int_s^\theta g(\xi) \le -2(|W|_T + T|g|_\infty) + T|g|_\infty \le -2|W|_\infty,$$

and hence $\rho_-(\xi,t,s,h,g) < W(\xi(s),s)$. This proves (4.21) and (4.22).

Let $\{\xi_\kappa\}$ be a minimizing sequence in (4.18). By going to a subsequence we can assume that $\xi_\kappa \to \xi$ in C^0 and $\xi_\kappa \to \xi$ weakly in $W^{1,2}([s,t],\mathbb{R}^n)$. Hence from (A_4), (A_5) of § 3, from (4.4) and by lower semicontinuity of W we have

$$W(\xi(s),s) \le \varliminf_{\kappa\to\infty} W(\xi_\kappa(s),s) \le \lim_{\kappa\to\infty} \rho_-(\xi_\kappa,t,s,h,g) \le \rho_-(\xi,t,s,h,g).$$

Hence

$$W(\xi(s),s) + \int_0^t g(\xi) \ge \underline{V}(x,t) = \lim \left\{ W(\xi_\kappa(s),s) + \int_0^t g(\xi_\kappa) \right\}$$
$$\ge W(\xi(s),s) + \int_0^t g(\xi).$$

This proves (4.23). Any minimizing sequence $\{\eta_\kappa\} \subset C_{M(T)}(x,t,s)$ of $\overline{V}$, we can extract a subsequence and still denote it by $\{\eta_\kappa\}$ converging strongly to η in C^0-topology. Now from continuity of W, (4.24) and (4.25) follow.

Lemma 4.6. Let $g(x,t) = g_1(x,t) + g_2(t)$. Assume that $tg_1(x,t)$ is non-increasing in $t, g_2(t) \le 0$ and $H(u,p) > 0$ for all $u \in \mathbb{R}, |p| = 1$. Then $\underline{u}^* = \overline{u}, \quad \overline{u}_* = \underline{u}$.

Proof. The proof is divided into three steps.

Step 1. Let $\alpha > 1$ and $\{q_k\}$ be a bounded sequence. Then $\varliminf_{k\to\infty} h(\alpha q_k) < \varliminf_{k\to\infty} h(q_k)$.

Suppose not, then let for a subsequence still denoted by $\{q_k\}$ such that

$$q_k \to q_0, \quad \lim_{k\to\infty} h(\alpha q_k) = \lim_{k\to\infty} h(q_k) = \eta.$$

Choose $|p_k| = |\tilde{p}_k| = 1$ such that for all $|p| = 1$

(i) $H(h(q_k),p_k) = \langle q_k,p_k\rangle, \qquad H(h(q_k),p) \ge \langle q_k,p\rangle$

(ii) $H(h(\alpha q_k),\tilde{p}_k) = \alpha\langle q_k,\tilde{p}_k\rangle, \quad H(h(\alpha q_k),p) \ge \alpha\langle q_k,p\rangle.$

Again going to a subsequence, one can assume that $p_k \to p_0$ $\tilde{p}_k \to \tilde{p}_0$ as $k \to \infty$. Then by continuity of H, $\langle q_0, p_0\rangle = H(\eta, p_0) = \lim_{k\to\infty} H(h(\alpha q_k), p_0) \geq \alpha\langle q_0, p_0\rangle$. Since $H(\eta, p_0) > 0$, it follows that $\alpha \leq 1$ which is a contradiction. This proves step 1.

Step 2. Let $t_1 > t$, then $\underline{u}(x,t) \geq \overline{u}(x,t_1)$. Let $\xi \in C_M(x,t)$ such that

$$\underline{u}(x,t) = u_0(\xi(0)) + \int_0^t g(\xi); \quad u_0(\xi(0)) \leq \operatorname*{ess\,inf}_{\theta\in[0,t]} \left\{ h(\dot{\xi}(\theta)) - \int_0^\theta g(\xi) \right\}.$$

Let $\xi_1(\theta) = \xi(\frac{t\theta}{t_1})$ for $\theta \in [0, t_1]$. Then $\xi_1 \in C(x, t_1)$ and $\xi_1(0) = \xi(0)$. Choose a sequence $\theta_k \in [0, t_1]$ and from step (1) to obtain

$$\begin{aligned}
\operatorname*{ess\,inf}_{\theta\in[0,t_1]} \left\{ h(\dot{\xi}_1(\theta)) - \int_0^{(t/t_1)\theta} g(\xi) \right\} &= \lim_{k\to\infty} \left\{ h(\dot{\xi}_1(\theta_k)) - \int_0^{(t/t_1)\theta_k} g(\xi) \right\} \\
&> \lim_{k\to\infty} \left\{ h\left(\frac{t_1}{t}\dot{\xi}_1(\theta_k)\right) - \int_0^{(t/t_1)\theta_k} g(\xi) \right\} \\
&= \lim_{k\to\infty} \left\{ h\left(\dot{\xi}\left(\frac{t}{t_1}\theta_k\right)\right) - \int_0^{(t/t_1)\theta_k} g(\xi) \right\} \\
&\geq \operatorname*{ess\,inf}_{\theta\in[0,t]} \left\{ h(\dot{\xi}(\theta)) - \int_0^\theta g(\xi) \right\}.
\end{aligned}$$

Hence from tg_1 non-increasing in t and $g_2(t) \leq 0$ we have that

$$\begin{aligned}
u_0(\xi_1(0)) &\leq \operatorname*{ess\,inf}_{\theta\in[0,t]} \left\{ h(\dot{\xi}(\theta)) - \int_0^\theta g(\xi) \right\} \\
&< \operatorname*{ess\,inf}_{\theta\in[0,t_1]} \left\{ h(\dot{\xi}_1(\theta)) - \int_0^{(t/t_1)\theta} g(\xi_1) \right\} \\
&= \operatorname*{ess\,inf}_{\theta\in[0,t_1]} \left\{ h(\dot{\xi}_1(\theta)) - \int_0^\theta g_1(\xi_1) + \int_0^\theta g_1(\xi_1) - \int_0^{(t/t_1)\theta} g_1(\xi) \right. \\
&\qquad \left. - \int_0^{(t/t_1)\theta} g_2(s)\mathrm{d}s + \int_0^\theta g_2(s)\mathrm{d}s - \int_0^\theta g_2(s)\mathrm{d}s \right\} \\
&\leq \operatorname*{ess\,inf}_{\theta\in[0,t_1]} \left\{ h(\dot{\xi}_1(\theta)) - \int_0^\theta g_1(\xi_1) - \int_0^\theta g_2(s)\mathrm{d}s \right\} \\
&= \operatorname*{ess\,inf}_{\theta\in[0,t_1]} \left\{ h(\dot{\xi}_1(\theta)) - \int_0^\theta g(\xi_1) \right\}.
\end{aligned}$$

Since $\int_0^\theta g_1(\xi_1) - \int_0^{(t/t_1)\theta} g_1(\xi) = \int_0^{(t/t_1)\theta} (\frac{t_1}{t} g_1(\xi(s), \frac{t_1}{t}s) - g_1(\xi(s), s))ds \leq 0$ and $g_2 \leq 0$, this implies that

$$\begin{aligned}
\underline{u}(x,t) &= u_0(\xi(0)) + \int_0^t g(\xi) \\
&= u_0(\xi_1(0)) + \int_0^{t_1} g(\xi_1) + \int_0^t g(\xi) - \int_0^{t_1} g(\xi_1) \\
&\geq \overline{u}(x, t_1).
\end{aligned}$$

Step 3. Let $B_r(x,t)$ be a ball centered at (x,t) with radius r. Let $t>0$ and let $t_k<t$ and $t_k\to t$. Then from step 2,

$$\begin{aligned}\underline{u}^*(x,t) &= \lim_{r\to 0}\sup_{B_r}\underline{u}(z)\\ &\geq \lim_{k\to\infty}\underline{u}(x,t_k)\geq \overline{u}(x,t).\end{aligned}$$

On the other hand $\underline{u}\leq\overline{u}$ and hence $\underline{u}^*(x,t)\leq\overline{u}(x,t)$, implies that $\underline{u}^*(x,t)=\overline{u}(x,t)$. Similarly $\overline{u}_*=\underline{u}$. This proves the Lemma.

In order to prove, the representation formula for a solution in the sense of viscosity, one has to establish a dynamic programming principle. This has been carried out for standard control problems and differential games in [2, 8] and [10]. We will next provide a proof of this fact for our problem.

Lemma 4.7 (*Dynamic programming principle*). *For every* $T>0$, *there exist* $M(T)>0$ *such that* $|\underline{u}|_T<\infty, \underline{u}$ *is lower semicontinuous and for* $0\leq s<t\leq T, x\in\mathbb{R}^n$,

$$\underline{u}(x,t)=\inf_{C_{M(t)}(x,t,s)}\left\{\underline{u}(\xi(s),s)+\int_s^t g(\xi):\underline{u}(\xi(s),s)\leq\rho_-(\xi,t,s,h,g)\right\}, \tag{4.26}$$

$$\underline{u}^*(x,t)\leq\inf_{C_{M(T)}(x,t,s)}\left\{\underline{u}^*(\xi(s),s)+\int_0^t g(\xi);\underline{u}^*(\xi(s),s)<\rho_-(\xi,t,s,h,g)\right\}. \tag{4.27}$$

Proof. Since $u_0\in W^{1,\infty}(\mathbb{R}^n)$ and hence by taking $s=0$ in (4.18), $|\underline{u}|_T<\infty$ follows from (4.20). Let $(x_m,t_m)\to(x,t)$ as $m\to\infty$. Since u_0 is continuous, from (4.23), for each m, there exist a $\xi_m\in C_{M(T)}(x_m,t_m)$ such that $\underline{u}(x_m,t_m)=u_0(\xi_m(0))+\int_0^{t_m}g(\xi_m)$ and $u_0(\xi_m(0))\leq\rho_-(\xi_m,t_m,h,g)$. From Arzela–Ascoli we can extract a subsequence still denoted by ξ_m such that $\xi_m\to\xi\in C_{M(T)}(x,t)$. Now from (4.4) we have

$$u_0(\xi(0))\leq\overline{\lim_{m\to\infty}}\,\rho_-(\xi_m,t_m,h,g)\leq\rho_-(\xi,t,h,g),$$

therefore,

$$\underline{\lim_{m\to\infty}}\,\underline{u}(x_m,t_m)=u_0(\xi(0))+\int_0^t g(\xi)\geq\underline{u}(x,t).$$

This proves $\underline{u}$ is lower semicontinuous. Let

$$v_1(x,t)=\inf_{C(x,t,s)}\left\{\underline{u}(\xi(s),s)+\int_s^t g(\xi);\underline{u}(\xi(s),s)\leq\rho_-(\xi,t,s,h,g)\right\}. \tag{4.28}$$

Since $|\underline{u}|_T<\infty$ and $\underline{u}$ is lower semicontinuous, hence from (4.23), there exist a $\xi_2\in C(x,t,s)$ such that $\underline{u}(\xi_2(s),s)\leq\rho_-(\xi_2,t,s,h,g)$ and $v_1(x,t)=\underline{u}(\xi_2(s),s)+\int_s^t g(\xi_2)$. Choose a $\xi_1\in C(\xi_2(s),s)$ such that $\underline{u}(\xi_2(s),s)=u_0(\xi_1(0))+\int_0^s g(\xi_1)$, $u_0(\xi_1(0))\leq\rho_-(\xi_1,s,h,g)$. Let $\eta\in C(x,t)$ defined by $\eta(\theta)=\xi_1(\theta)$ for $\theta\in[0,s]$ and $\eta(\theta)=\xi_2(\theta)$ for $\theta\in[s,t]$. Then we have

$$u_0(\eta(0))=\underline{u}(\xi_1(s),s)-\int_0^s g(\xi_1)$$

$$\begin{aligned}&\leq \rho_-(\xi_2, t, s, h, g) - \int_0^s g(\xi_1)\\ &= \operatorname*{ess\,inf}_{\theta\in[s,t]} \left\{ h(\dot\eta(\theta)) - \int_0^\theta g(\eta) \right\}.\end{aligned}$$

Since $u_0(\eta(0)) \leq \rho_-(\eta, s, h, g)$, it follows that $u_0(\eta(0)) \leq \rho_-(\eta, t, h, g)$. Therefore we have

$$\begin{aligned} v_1(x,t) &= \underline{u}(\eta(s), s) + \int_s^t g(\eta)\\ &= u_0(\eta(0)) + \int_0^t g(\eta) \geq \underline{u}(x,t). \end{aligned} \tag{4.29}$$

Let $\xi \in C(x,t)$ be such that $\underline{u}(x,t) = u_0(\xi(0)) + \int_0^t g(\xi)$ and $u_0(\xi(0)) \leq \rho_-(\xi, t, h, g) = \inf\{\operatorname{ess\,inf}_{\theta\in[0,s]}\{h(\dot\xi(\theta)) - \int_0^\theta g(\xi)\}, \operatorname{ess\,inf}_{\theta\in[s,t]}\{h(\dot\xi(\theta)) - \int_0^\theta g(\xi)\}\}$. Hence

$$\begin{aligned}\underline{u}(\xi(s), s) &\leq u_0(\xi(0)) + \int_0^s g(\xi)\\ &\leq \operatorname*{ess\,inf}_{\theta\in[s,t]} \left\{ h(\dot\xi(\theta) - \int_0^\theta g(\xi) + \int_0^s g(\xi) \right\}\\ &= \rho_-(\xi, t, s, h, g).\end{aligned}$$

This implies that $\underline{u}(x,t) = u_0(\xi(0)) + \int_0^s g(\xi) + \int_s^t g(\xi) \geq \underline{u}(\xi(s), s) + \int_s^t g(\xi) \geq v_1(x,t)$. Therefore from (4.29) $\underline{u}(x,t) = v_1(x,t)$ and since $|\underline{u}|_T < \infty$ and hence from (4.21), (4.26) follows.

Let

$$v_2(x,t) = \inf_{\xi\in C(x,t,s)} \left\{ \underline{u}^*(\xi(s), s) + \int_s^t g(\xi); \underline{u}^*(\xi(s), s) < \rho_-(\xi, t, s, h, g) \right\}. \tag{4.30}$$

Choose a sequence $(x_\kappa, t_\kappa) \to (x,t)$ such that $\underline{u}^*(x,t) = \lim_{\kappa\to\infty} \underline{u}(x_\kappa, t_\kappa)$. For $\xi \in C(x,t,s)$, let $s_\kappa = s + t_\kappa - t$ and define $\xi_\kappa \in C(x_\kappa, t_\kappa, s_\kappa)$ by $\xi_\kappa(\theta) = \xi(\theta - t_\kappa + t) + x_\kappa - x$. Then by change of variables $\alpha = \theta - t_\kappa + t$, we obtain

$$\operatorname*{ess\,inf}_{\theta\in[s_\kappa,t_\kappa]} \left\{ h(\dot\xi_\kappa(\theta)) - \int_{s_\kappa}^\theta g(\xi_\kappa) \right\} = \operatorname*{ess\,inf}_{\alpha\in[s,t]} \left\{ h(\dot\xi(\alpha)) - \int_s^\alpha g(\xi) \right\} + \beta_\kappa, \tag{4.31}$$

where $\beta_\kappa = O(\sup_{\alpha\in[s,t]}(\int_s^\alpha g(\xi) - \int_{s_\kappa}^{\alpha-t+t_\kappa} g(\xi_\kappa))) \to 0$ as $\kappa \to \infty$.

Let $\xi \in C(x,t,s)$ such that $\underline{u}^*(\xi(s), s) < \rho_-(\xi, t, s, h, g)$. Hence from (4.29) and upper semicontinuity of $\underline{u}^*$ we can find a $\kappa_0 > 0$ such that for $\kappa \geq \kappa_0$, $\underline{u}^*(\xi_\kappa(s_\kappa), s_\kappa) < \rho_-(\xi_\kappa, t_\kappa, s_\kappa, h, g)$ and hence $\underline{u}(\xi_\kappa(s_\kappa), s_\kappa) \leq \underline{u}^*(\xi_\kappa(s_\kappa), s_\kappa) < \rho_-(\xi_\kappa, t_\kappa, s_\kappa, h, g)$. Therefore from (4.28) we have

$$\begin{aligned}\underline{u}^*(x,t) &= \lim_{\kappa\to\infty} \underline{u}(x_\kappa, t_\kappa) \leq \lim_{\kappa\to\infty} v_1(x_\kappa, t_\kappa)\\ &\leq \lim_{\kappa\to\infty} \left\{ \underline{u}(\xi_\kappa(s_\kappa), s_\kappa) + \int_{s_\kappa}^{t_\kappa} g(\xi_\kappa) \right\}\end{aligned}$$

$$\leq \lim_{\kappa\to\infty}\left\{\underline{u}^*(\xi_\kappa(s_\kappa),s_\kappa)+\int_{s_\kappa}^{t_\kappa} g(\xi_\kappa)\right\}$$
$$\leq \underline{u}^*(\xi(s),s)+\int_s^t g(\xi).$$

Since it is true for all ξ and hence $\underline{u}^*(x,t)\leq v_2(x,t)$. Combining this with $|\underline{u}^*|_T<\infty$ and (4.22) we obtain (4.27). This proves the Lemma.

Lemma 4.8 (*Dynamic programming principle*). *For every* $T>0, |\overline{u}|_T<\infty, \overline{u}$ *is upper semicontinuous and there exist* $M(T)>0$ *such that for* $0\leq s<t\leq T, x\in\mathbb{R}^n$,

$$\overline{u}(x,t)=\inf_{C_{M(T)}(x,t,s)}\left\{\overline{u}(\xi(s),s)+\int_s^t g(\xi);\ \overline{u}(\xi(s),s)<\rho_-(\xi,t,s,h,g)\right\}, \quad (4.32)$$

$$\overline{u}_*(x,t)\geq \inf_{C_{M(T)}(x,t,s)}\left\{\overline{u}_*(\xi(s),s)+\int_s^t g(\xi);\ \overline{u}_*(\xi(s),s)\leq\rho_-(\xi,t,s,h,g)\right\}. \quad (4.33)$$

Proof. Since $u_0\in W^{1,\infty}(\mathbb{R}^n)$, by taking $s=0$ in (4.18), $|\overline{u}|_T<\infty$ follows from (4.20). Let $(x_m,t_m)\to(x,t)$ as $m\to\infty$. Since u_0 is continuous, by (4.24) and (4.25) there exist $\eta,\eta_\kappa\in C_{M(T)}(x,t)$ such that $\eta_\kappa\to\eta$ uniformly and $\overline{u}(x,t)=u_0(\eta(0))+\int_0^t g(\eta)$ and $u_0(\eta_\kappa(0))<\rho_-(\eta_\kappa,t,h,g)$. Now for each κ, define $\eta_{m_\kappa}\in C(x_m,t_m)$ as follows:

$$\eta_{m_\kappa}(\theta)=\begin{cases}\eta_\kappa(\theta-t_m+t)+x_m-x & \text{if}\quad \theta\in[0\vee(t_m-t),t_m],\\ \eta_\kappa(0)+x_m-x & \text{if}\quad \theta\in[0,0\vee(t_m-t)].\end{cases} \quad (4.34)$$

Clearly $\eta_{m_\kappa}\to\eta_\kappa$ uniformly and by change of variables it follows that

$$\rho_-(\eta_{m_\kappa},t_m,h,g)=\rho_-(\eta_\kappa,t,h,g)+o(1), \quad (4.35)$$

where $o(1)\to 0$ as $m\to\infty$. Since u_0 is continuous, from (4.35) we can find a $m(\kappa)>0$ such that for $m>m(\kappa), u_0(\eta_{m_\kappa}(0))<\rho_-(\eta_{m_\kappa},t_m,h,g)$. This implies that $\overline{u}(x_m,t_m)\leq u_0(\eta_{m_\kappa}(0))+\int_0^{t_m} g(\eta_{m_\kappa})$. Now letting $m\to\infty,\kappa\to\infty$, we conclude that

$$\overline{\lim_{m\to\infty}}\ \overline{u}(x_m,t_m)\leq\lim_{\kappa\to\infty}\left\{u_0(\eta_\kappa(0))+\int_0^t g(\eta_\kappa)\right\}=\overline{u}(x,t).$$

This proves $\overline{u}$ is upper semicontinuous. Define

$$v_1(x,t)=\inf_{\xi\in C(x,t,s)}\left\{\overline{u}(\xi(s),s)+\int_s^t g(\xi)\ :\ \overline{u}(\xi(s),s)<\rho_-(\xi,t,s,h,g)\right\},$$

then

$$\begin{aligned}v_1(x,t)=\inf_{\xi\in C(x,t,s)}\Bigg[&\inf_{\eta\in C(\xi(s),s)}\left\{u_0(\eta(0))+\int_0^s g(\eta)+\int_s^t g(\xi);\ u_0(\eta(0))\right.\\ &\left.<\rho_-(\eta,s,h,g)\right\};\quad \inf_{\eta\in C(\xi(s),s)}\left\{u_0(\eta(0))+\int_0^s g(\eta):u_0(\eta(0))\right.\\ &\left.<\rho_-(\eta,s,h,g)\right\}<\rho_-(\xi,t,s,h,g)\Bigg]\end{aligned}$$

$$\begin{aligned}
&\le \inf_{\lambda\in C(x,t)}\left\{u_0(\lambda(0))+\int_0^t g(\lambda); u_0(\lambda(0))<\rho_-(\lambda,s,h,g), u_0(\lambda(0))\right.\\
&\qquad\left.+\int_0^s g(\lambda)<\rho_-(\lambda,t,s,h,g)\right\}\\
&=\inf_{\lambda\in C(x,t)}\left\{u_0(\lambda(0))+\int_0^t g(\lambda): u_0(\lambda(0))<\rho_-(\lambda,t,h,g)\right\}\\
&=\overline{u}(x,t). \qquad (4.36)
\end{aligned}$$

Since $|\overline{u}|_T<\infty$, hence from (4.22), there exist a $M_1(T)>0$ such that

$$v_1(x,t)=\inf_{\xi\in C_{M_1(T)}(x,t,s)}\left\{\overline{u}(\xi(s),s)+\int_s^t g(\xi);\ \overline{u}(\xi(s),s)<\rho_-(\xi,t,s,h,g)\right\}. \qquad (4.37)$$

Let $\varepsilon>0, r>0$ and $\xi\in C(x,t,s)$ such that $v_1(x,t)\ge\overline{u}(\xi(s),s)+\int_s^t g(\xi)-\varepsilon$ and $\overline{u}(\xi(s),s)+r<\rho_-(\xi,t,s,h,g)$. From (4.24), (4.25) and (4.4) there exist an $\eta\in C(\xi(s),s)$ such that $\overline{u}(\xi(s),s)>u_0(\eta(0))+\int_0^s g(\eta)-r,\quad u_0(\eta(0))<\rho_-(\eta,s,h,g)$. Let $\lambda\in C(x,t)$ be defined by $\lambda|_{[0,s]}=\eta, \lambda|_{[s,t]}=\xi$. Then $u_0(\lambda(0))=u_0(\eta(0))$ and

$$\begin{aligned}
u_0(\lambda(0))&=\overline{u}(\xi(s),s)+r-\int_0^s g(\eta)\\
&<\rho_-(\xi,t,s,h,g)-\int_0^s g(\eta)\\
&=\operatorname*{ess\,inf}_{\theta\in[s,t]}\left\{h(\dot\lambda(\theta))-\int_0^\theta g(\lambda)\right\}.
\end{aligned}$$

Since $u_0(\lambda(0))=u_0(\eta(0))<\rho_-(\eta,s,h,g)$ and hence combining this with the above inequality implies that $u_0(\lambda(0))<\inf\{\operatorname{ess\,inf}_{\theta\in[s,t]}\{h(\dot\lambda(\theta))-\int_0^\theta g(\lambda)\},\ \operatorname{ess\,inf}_{\theta\in[0,s]}\ \{h(\dot\lambda(\theta))-\int_0^\theta g(\lambda)\}\}=\rho_-(\lambda,\ t,\ h,\ g)$. Therefore $v_1(x,\ t)\ge u_0(\lambda(0))+\int_0^t g(\lambda)-\epsilon\ge\ \overline{u}(x,\ t)-\epsilon$. Since ϵ is arbitrary, we obtain $v_1(x,t)\ge\overline{u}(x,t)$. This with (4.36), (4.37) implies (4.32)

$$v_2(x,t)=\inf_{\xi\in C(x,t,s)}\left\{\overline{u}_*(\xi(s),s)+\int_s^t g(\xi);\overline{u}_*(\xi(s),s)\le\rho_-(\xi,t,s,h,g)\right\}. \qquad (4.38)$$

Let $\lim_{\kappa\to\infty}(x_\kappa,t_\kappa)=(x,t), \lim_{\kappa\to\infty}\overline{u}(x_\kappa,t_\kappa)=\overline{u}_*(x,t)$. Let $\epsilon>0$. Then from (4.36), (4.37) and (4.38) we can choose a $\kappa(\epsilon)>0$ such that for every $\kappa>\kappa(\epsilon)$, there exist a $\xi_\kappa\in C_{M_1(T)}(x_\kappa,t_\kappa,s_\kappa)$ such that

$$\overline{u}_*(x,t)\ge\overline{u}(x_\kappa,t_\kappa)-\frac{\epsilon}{2}, \qquad (4.39)$$

$$\overline{u}(x_\kappa,t_\kappa)>\overline{u}(\xi_\kappa(s),s)+\int_s^{t_\kappa} g(\xi_\kappa)-\frac{\epsilon}{2}, \qquad (4.40)$$

$$\overline{u}(\xi_\kappa(s),s)<\rho_-(\xi_\kappa,t_\kappa,s,h,g). \qquad (4.41)$$

Extract a subsequence still denoted by ξ_κ converging to ξ uniformly. Then from (4.41), (4.4), (4.39) and (4.40)

$$\overline{u}_*(\xi(s),s)\le\varliminf_{\kappa\to\infty}\overline{u}_*(\xi_\kappa(s),s)\le\ \lim_{\kappa\to\infty}\overline{u}_*(\xi_\kappa(s),s)\le\rho_-(\xi,t,s,h,g), \qquad (4.42)$$

$$
\begin{aligned}
\overline{u}_*(x,t) &\geq \lim_{\kappa\to\infty} \overline{u}(x_\kappa, t_\kappa) - \frac{\epsilon}{2} \\
&\geq \underline{\lim}_{\kappa\to\infty} \left\{ \overline{u}(\xi_\kappa(s), s) + \int_s^{t_\kappa} g(\xi_\kappa) \right\} - \epsilon \\
&\geq \underline{\lim}_{\kappa\to\infty} \left\{ \overline{u}_*(\xi_\kappa(s), s) + \int_s^{t_\kappa} g(\xi_\kappa) \right\} - \epsilon \\
&= \overline{u}_*(\xi(s), s) + \int_s^t g(\xi) - \epsilon \\
&\geq v_2(x,t) - \epsilon,
\end{aligned}
$$

since (4.42) holds. Now letting $\epsilon \to 0$ to conclude that $\overline{u}_*(x,t) \geq v_2(x,t)$. Since $|\overline{u}_*|_T < \infty$ and hence from (4.21) there exist an $M(T) > 0$ such that (4.33) holds. This proves the lemma.

Proof of Theorem 2.2. Let $T > 0$ and $0 < t \leq T$ and $x \in \mathbb{R}^n$. From (4.23) and (4.24) there exist $M(T) > 0$ such that $\xi_t, \eta_t \in C_{M(T)}(x,t)$ and

$$
\begin{aligned}
\underline{u}(x,t) &= u_0(\xi_t(0)) + \int_0^t g(\xi_t), \\
\overline{u}(x,t) &= u_0(\eta_t(0)) + \int_0^t g(\eta_t).
\end{aligned}
$$

Since $|(x,x) - (\xi_t(\theta), \eta_t(\theta))| = |\int_\theta^t (\dot{\xi}_t(\lambda), \dot{\eta}_t(\lambda))\mathrm{d}\lambda| \leq M(T)|t-\theta|$ and hence $(\xi_t, \eta_t) \to (x,x)$ as $t \to 0$. This implies that $\lim_{t\to 0}(\underline{u}(x,t), \overline{u}(x,t)) = (u_0(x), u_0(x))$.

Suppose $\underline{u}$ is not a sub solution. Then there exist an $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+, \epsilon > 0, B$ a ball with centre (x_0, t_0) and a $\varphi \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ such that $\underline{u}^*(x_0, t_0) = \varphi(x_0, t_0), \underline{u}^* - \varphi \leq 0$ in $B, \varphi_t + H(\varphi, D\varphi) - g \geq 4\epsilon$ at (x_0, t_0). By continuity we can choose a $\delta > 0$ such that $\varphi_t + H(\varphi + \delta, D\varphi) - g \geq 3\epsilon$ at (x_0, t_0). Therefore from (A_6) of §3, there exist q with $\varphi(x_0, t_0) + \delta \leq h(q)$ and $\varphi_t + < q, D\varphi > -g \geq 2\epsilon$ at (x_0, t_0). By continuity, we can find a ball $B_1 \subset B$ around (x_0, t_0) such that for $(x,t) \in B_1$,

$$
\underline{u}^*(x,t) \leq \varphi(x,t) < h(q) - \frac{\delta}{2}, \tag{4.43}
$$

$$
\varphi_t + \langle q, D\varphi \rangle - g \geq \epsilon. \tag{4.44}
$$

Let $\xi(\theta) = x_0 + q(\theta - t_0)$ and choose a $s_0 < t_0$ such that for $\theta \in [s_0, t_0]$, $\sup_{\theta\in[s_0,t_0]} |\int_{s_0}^\theta g(\xi)| < \frac{\delta}{2}$ and $(\xi(\theta), \theta) \in B_1$. Then from (4.43), $\underline{u}^*(\xi(s_0), s_0) < h(q) - \frac{\delta}{2} \leq \inf_{\theta\in[s_0,t_0]} \{h(\dot{\xi}(\theta)) - \int_{s_0}^\theta g(\xi)\} = \rho_-(\xi, t_0, s_0, h, g)$. Hence from (4.26)

$$
\begin{aligned}
\varphi(x_0, t_0) = \underline{u}^*(x_0, t_0) &\leq \underline{u}^*(\xi(s_0), s_0) + \int_{s_0}^{t_0} g(\xi) \\
&\leq \varphi(\xi(s_0), s_0) + \int_{s_0}^{t_0} g(\xi).
\end{aligned} \tag{4.45}
$$

From (4.44) we have

$$
\varphi(x_0, t_0) - \varphi(\xi(s_0), s_0) = \int_{s_0}^{t_0} \frac{\mathrm{d}}{\mathrm{d}\theta}(\varphi(\xi(\theta), \theta))\mathrm{d}\theta
$$

$$= \int_{s_0}^{t_0} (\varphi_t + \langle q, D\varphi \rangle)(\xi(\theta), \theta) \mathrm{d}\theta$$
$$\geq \int_{s_0}^{t_0} g(\xi) + \epsilon(t_0 - s_0),$$

which contradicts (4.45). This proves $\underline{u}$ is a sub solution.

Next we will show that $\underline{u}$ is a super solution. Suppose not, since $\underline{u}$ is a lower semi-continuous function, hence there exist a $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+, \epsilon > 0$ a ball B centered at (x_0, t_0) and a $\varphi \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ such that $\underline{u}(x_0, t_0) = \varphi(x_0, t_0), \underline{u} - \varphi \geq 0$ in B, $\varphi_t + H(\varphi, D\varphi) - g \leq -4\epsilon$ at (x_0, t_0). Hence by continuity of H and (A_6) of § 3, we can find a ball $B_1 \subset B$ centered at (x_0, t_0) such that $\underline{u} \geq \varphi$ in B_1 and whenever $q \in \mathbb{R}^n, (x, t) \in B_1$ with $\varphi(x, t) \leq h(q)$, then at (x, t)

$$\varphi_t + \langle q, D\varphi \rangle - g \leq -2\epsilon. \tag{4.46}$$

For every $s \leq t_0$, from (4.23) and (4.26) choose a $\xi_s \in C_{M(T_0)}(x_0, t_0, s)$ such that $\underline{u}(\xi_s(s), s) \leq \rho_-(\xi_s, t_0, s, h, g)$ and $\underline{u}(x_0, t_0) = \underline{u}(\xi_s(s), s) + \int_s^{t_0} g(\xi_s)$. Now $|\xi_s(\theta) - x_0| \leq M(T_0)|t_0 - \theta|$ and hence we can find a $s_0 < t_0$ such that for any $s \in [s_0, t_0], (\xi_s(\theta), \theta) \in B_1$ for all $\theta \in (s_0, t_0]$. Therefore for $s \in [s_0, t_0]$

$$\begin{aligned} \varphi(x_0, t_0) = \underline{u}(x_0, t_0) &= \underline{u}(\xi_s(s), s) + \int_s^{t_0} g(\xi_s) \\ &\geq \varphi(\xi_s(s), s) + \int_s^{t_0} g(\xi_s) \end{aligned} \tag{4.47}$$

$$\varphi(\xi_s(s), s) \leq \underline{u}(\xi_s(s), s) \leq \rho_-(\xi_s, t_0, s, h, g). \tag{4.48}$$

Claim. There exist $s_1 \in [s_0, t_0)$ such that for almost every $\theta \in [s_1, t_0]$

$$\varphi_t(\xi_{s_1}(\theta), \theta) + \langle \dot{\xi}_{s_1}(\theta), D\varphi(\xi_{s_1}(\theta), \theta) \rangle - g(\xi_{s_1}(\theta), \theta) \leq -\epsilon. \tag{4.49}$$

Suppose not, then from (4.48) we can find a sequence $s_m \to t_0, \theta_m \in [s_m, t_0]$, $\xi_m = \xi_{s_m}$ such that

$$\varphi_t(\xi_m(\theta_m), \theta_m) + \langle \dot{\xi}_m(\theta_m), D\varphi(\xi_m(\theta_m), \theta_m) \rangle - g(\xi_m(\theta_m), \theta_m) \geq -\epsilon \tag{4.50}$$

$$\varphi(\xi_m(\theta_m), \theta_m) \leq h(\dot{\xi}_m(\theta_m))) - \int_{s_m}^{\theta_m} g(\xi_m). \tag{4.51}$$

Since $|\dot{\xi}_m(\theta_m)| \leq M(t_0)$, hence for a subsequence still denoted by ξ_m, let $q = \lim_{m\to\infty} \dot{\xi}_m(\theta_m)$. Now letting $m \to \infty$ in (4.50) and (4.51) and using upper semicontinuity of h to obtain

$$\varphi_t(x_0, t_0) + \langle q, D\varphi(x_0, t_0) \rangle - g(x_0, t_0) \geq -\epsilon$$
$$\varphi(x_0, t_0) \leq \overline{\lim_{m\to\infty}} \left\{ h(\dot{\xi}_m(\theta_m)) - \int_{s_m}^{\theta_m} g(\xi_m) \right\} \leq h(q),$$

which contradicts (4.46) and hence the claim. From (4.49)

$$\varphi(x_0, t_0) - \varphi(\xi_{s_1}(s_1), s_1) = \int_{s_1}^{t_0} \frac{\mathrm{d}}{\mathrm{d}\theta} \varphi(\dot{\xi}_{s_1}(\theta), \theta) \mathrm{d}\theta$$

$$= \int_{s_1}^{t_0} (\varphi_t + \langle \dot{\xi}_{s_1}, D\varphi \rangle)(\xi_{s_1}(\theta), \theta) \mathrm{d}\theta$$
$$\leq \int_{s_1}^{t_0} g(\xi_{s_1}) - \epsilon(t_0 - s_1),$$

which contradicts (4.47). This proves that $\underline{u}$ is a super solution and hence it is a viscosity solution. Similarly from Lemma 4.8, it follows that $\overline{u}$ is a viscosity solution. This together with Lemma 4.6 completes the proof of the Theorem.

Remark 4.9. In Lemma 4.6, assumptions on g are only sufficient but not necessary. For example consider the problem

$$u_t + \mathrm{e}^{-u}|u_x| = g(t)$$
$$u(x,0) = u_0(x).$$

Solutions of this problem are given by

$$\underline{u}(x,t) = \inf_y \left\{ u_0(y) + \int_0^t g(s)\mathrm{d}s;\ u_0(y) \leq -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s} \right) \right\} \tag{4.52}$$

and

$$\overline{u}(x,t) = \inf_y \left\{ u_0(y) + \int_0^t g(s)\mathrm{d}s;\ u_0(y) < -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s} \right) \right\}. \tag{4.53}$$

Furthermore $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$. Here $g(t) \leq 0$ is not required.

Proof. By formula

$$\underline{u}(x,t) = \inf_{\xi \in C(x,t)} \left\{ u_0(y) + \int_0^t g(s)\mathrm{d}s;\ u_0(y) \leq \operatorname*{ess\,inf}_{0 \leq s \leq t} \left\{ h(\dot{\xi}(s)) - \int_0^s g(\theta)\mathrm{d}\theta \right\} \right\},$$

where $h(q) = \log\left(\frac{1}{|q|}\right)$. Let

$$\overline{\xi}(s) = \left(\frac{x-y}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s} \right) \left(\int_0^s \exp\left(-\int_0^\theta g(\eta)\mathrm{d}\eta \right) \mathrm{d}\theta \right) + y.$$

Then $\overline{\xi}(t) = x, \overline{\xi}(0) = y$ and $\dot{\xi}(s) = ((x-y)/(\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s)) \exp(-\int_0^s g(\eta)\mathrm{d}\eta)$. Also

$$h(\dot{\xi}(s)) - \int_0^s g(\eta)\mathrm{d}\eta = -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s} \right).$$

Therefore

$$\underline{u}(x,t) \leq \inf_y \left\{ u_0(y) + \int_0^t g(s)\mathrm{d}s;\ u_0(y) \leq -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s} \right) \right\}.$$

On the other hand,

$$u_0(y) \le h(\dot{\xi}(\theta)) - \int_0^\theta g(\eta)\mathrm{d}\eta \quad \forall\, \theta \in [0,t]$$

implies

$$\exp\left(-u_0(y) - \int_0^\theta g(\eta)\mathrm{d}\eta\right) \ge |\dot{\xi}(\theta)|.$$

On integration over $[0,t]$, we have since $\int_0^t |\dot{\xi}(\theta)| \ge |x-y|$,

$$u_0(y) \le -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s}\right).$$

This implies

$$\underline{u}(x,t) \ge \inf_y \left\{ u_0(y) + \int_0^t g(s)\mathrm{d}s;\ u_0(y) \le -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s}\right)\right\}.$$

Hence (4.52). Similarly (4.53) follows.

Choose $M_0 > 0$ be such that $\int_{t_1}^t g(\eta)\mathrm{d}\eta \le M_0(t-t_1)$ for all $t_1 < t$. Since $\int_0^t \exp(-\int_0^s g(\theta)\,\mathrm{d}\theta)\mathrm{d}s$ is an increasing function of t, it follows that

$$\begin{aligned}
\underline{u}(x,t_1) &= \inf_y \left\{ u_0(y) + \int_0^{t_1} g(s)\mathrm{d}s;\ u_0(y) \le -\log\left(\frac{|x-y|}{\int_o^{t_1} \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s}\right)\right\} \\
&\ge \inf_y \left\{ u_0(y) + \int_0^t g(s)\mathrm{d}s - \int_{t_1}^t g(s)\mathrm{d}s; \right. \\
&\qquad \left. u_0(y) < -\log\left(\frac{|x-y|}{\int_0^t \exp(-\int_0^s g(\theta)\mathrm{d}\theta)\mathrm{d}s}\right)\right\} \\
&\ge \overline{u}(x,t) - M_0(t-t_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
\underline{u}^*(x,t) &\ge \lim_{t_1 \to t} \underline{u}(x,t_1) \\
&\ge \lim_{t_1 \to t} (\overline{u}(x,t) - M_0(t-t_1)) \\
&= \overline{u}(x,t).
\end{aligned}$$

Hence we have $\underline{u}^* = \overline{u}$ and similarly $\overline{u}_* = \underline{u}$.

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References

[1] Adimurthi and Veerappa Gowda G D, Hopf Lax type formula for sub and supersolutions, *Adv. Diff. Eq.* **5** (2000) 97–119
[2] Barron E N, Evans L C and Jensen R, Viscosity solutions of Isaacs equations and differential games with Lipschitz controls, *J. Diff. Eq.* **53** (1984) 213–233
[3] Barron E N and Ishii H, The Bellman equation for minimizing maximum cost, *Nonlinear Analysis TMA* **13(9)** (1989) 1067–1090
[4] Barron E N and Liu W, Calculus of variations in L^∞, *Appl. Math. Opt.* **35** (1997) 237–263
[5] Barron E N, Jensen R and Liu W, Hopf-Lax type formula for $u_t + H(u, Du) = 0$, *J. Diff. Eq.* **126** (1996) 48–64
[6] Barles G and Perthame B, Discontinuous solutions of deterministic optimal stopping time problems, *Math. Modelling Numer. Anal.* **21** (1987) 57–579
[7] Evans L C, *Partial differential equations*, Berkeley Mathematics Lecture Notes, (1994) Vols 3A, 3B
[8] Evans L C and Souganidis P, Differential games and representation formulas for Hamilton Jacobi equations, *Indiana Univ. Math. J.* **33** (1984) 773–795
[9] Hitoshi Ishii, Perron's method for Hamilton Jacobi equations, *Duke. Math. J.* **55** (1987) 369–364
[10] Lions P L, *Generalized solutions of Hamilton Jacobi equations*, Research notes in Mathematics (Pitmann) (1982)
[11] Rudin W, *Functional Analysis* (Tata McGraw Hill Pub.) (1974)