positive solutions of the semilinear Dirichlet problem with critical growth in the unit disc in $\mathbb{R}^2$

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Abstract. We prove the existence of a positive solution of the following problem

$$-\Delta u = f(r, u) \quad \text{in } D$$
$$u > 0$$
$$u = 0, \quad \text{on } \partial D$$

where $D$ is the unit disc in $\mathbb{R}^2$ and $f$ is a superlinear function with critical growth.

Keywords. Sub-critical growth, critical growth, super critical growth; Laplacian; Palais-Smale condition; Semilinear Dirichlet problem; unit disc.

Introduction

$D$ be the unit disc in $\mathbb{R}^2$. We are looking for positive radial solutions of the owing problem: Find $u$ in $C^2(D) \cap C^0(\overline{D})$ such that

$$-\Delta u = f(r, u) \quad \text{in } D$$
$$u > 0$$
$$u = 0, \quad \text{on } \partial D$$

(1.1)

cere $f$ is superlinear, $f(r, 0) = 0$, $(\partial f/\partial t)(r, 0) < \lambda_1$ with $\lambda_1$ being the first eigenvalue of the Dirichlet problem. For $n \geq 3$ and $f$ of critical growth, Brezis--Nirenberg [4] died the existence and non-existence of solutions of problem (1.1). For $n = 2$, the ical growth is of exponential type whereas in the case of $n \geq 3$, it is of polynomial e and the method adopted for $n \geq 3$ fails in the case of $n = 2$.

Carleson--Chang [5] obtained a positive solution for $f(u) = \lambda u \exp(\lambda u^2)$ with $\lambda < \lambda_1$ via a variational method. For growths of type $f(u) = u^n \exp(bu^2)$, Atkinson--etier [3] used the shooting argument to obtain a solution of (1.1). They assumed $\log f$ is strictly convex for large $u$.

In this paper we relax the conditions on $f$ and use a variational method to obtain solution of (1.1). Since we are interested in radial solutions, (1.1) is equivalent to
finding an \( u \) in \( C^2(D) \cap C^0(\overline{D}) \) with \( u \) radial and satisfying
\[
L_1 u \equiv -(ru')' = f(r,u)r \quad \text{in } [0,1)
\]
\[
u > 0 \quad \text{in } [0,1)
\]
\[
u(0) = u(1) = 0.
\] (1.2)

where \( u' = du/dr \).

The idea of the method is to approximate the energy functional by functionals satisfying Palais–Smale conditions. Then obtain the critical points of these approximate functionals by a constrained minimization problem similar to that of Zeev–Nehari [8] and then pass to the limit. The method of proof is in the spirit of Brezis–Nirenberg [4]. Here, we also get a constant \( "a" \) which is strictly less than the best possible constant and thereby the existence of solutions of (1.2) is guaranteed.

In [1] we also prove the existence of infinitely many solutions of (1.1) when \( f \) is odd and of critical growth. Also in [2] we prove the existence of solutions of (1.1) if \( D \) is replaced by an arbitrary smooth domain.

2. Statements

Let \( E = \{ u \in C^1[0,1]; u(1) = 0 \} \). For \( 0 \leq x \leq 1 \) and \( u \) in \( E \) define
\[
\| u \|_x^2 = \int_0^1 u^2(r)x^r \, dr
\]
\[
\| u \|_x^2 = \int_0^1 u'(r)^2r^s \, dr.
\]

Let \( H_x \) be the completion of \( E \) with respect to \( \| \cdot \|_x \). Define the operator \( L_x \) by
\[
L_x = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right).
\] (2.1)

Let \( (\lambda_x, \phi_x) \) be the first eigenvalue and the corresponding first eigenvector with \( \phi_x(0) = 1 \) of the following eigenvalue problem.
\[
L_x \phi = \lambda \phi \quad \text{in } [0,1]
\]
\[
\phi'(0) = \phi(1) = 0.
\] (2.2)

DEFINITION 2.1
Let \( f: [0, 1] \times [0, \infty) \rightarrow [0, \infty) \) be a \( C^1 \)-function. We say \( f \) is of class \( A \) if
(i) \( f(r, 0) = 0 \).
(ii) There exists a \( \delta_0 > 0 \) and for \( (r, t) \in Q_{\delta_0} = [0, \delta_0] \times [0, \infty) \) \( \partial f / \partial r(r, t) > 0 \).
(iii) There exists a \( t_0 > 0 \) such that \( f(r, t) < \lambda_1 t \) for all \( (r, t) \in [0, 1] \times [0, t_0] \).
(iv) There exist constants \( t_1 > 0, \beta > 2 \) such that \( \beta F(r, t) \leq f(r, t) t \) for all \( (r, t) \in [0, 1] \times [t_1, \infty) \) where \( F(r, t) = \int_0^t f(r, s) \, ds \).
Positive solutions of the semilinear Dirichlet

Let

$$A' = \left\{ f \in A : \frac{\partial f}{\partial t} > f \right\} \text{ in } [0,1] \times (0, \infty).$$

We consider the following three types of functions in our discussions.

Sub-critical: $f$ in $A$ is said to be sub-critical if there exists a $\delta > 0$ and for every $\epsilon > 0$

$$\sup_{(r,t) \in [0,1] \times (0, \infty)} f(r,t) \exp(-\epsilon t^2) < \infty$$

(2.3)

Critical: $f$ in $A'$ is said to be critical if there exists $\delta_1 > 0$ such that

(i) $f(r,t) = h(r,t) \exp(h(r)t^2) \quad \forall (r,t) \in \Omega_{\delta_1} = [0, \delta_1] \times [0, \infty)$

(ii) $\forall \epsilon > 0$,

$$\sup_{(r,t) \in \Omega_{\delta_1}} h(r,t) \exp(-\epsilon t^2) < \infty.$$  \hspace{1cm} (2.4)

(iii) For every $\epsilon > 0$, $h(0,t) \exp(\epsilon t^2) \to \infty$ as $t \to \infty$.

Super critical: $f \in A'$ is said to be super critical if for every $\epsilon > 0$

$$\sup_{x \in [0,1]} \int_0^1 f(r,x) r dr = \infty.$$  \hspace{1cm} (2.5)

For $f \in A$, $0 \leq x \leq 1$, let $\Sigma_x$ be the set of $C^2$-solutions of the following problem

$$L_x u = f(r,u) \quad \text{in } [0,1]$$

$$u > 0$$

$$u(0) = u(1) = 0.$$  \hspace{1cm} (2.6)

DEFINITION 2.2

$u$ in $H^1_0(D)$ is said to be a weak solution of (1.2) if

(i) $u > 0$ in $[0,1)$

(ii) $\int_0^1 f(r,u) u r dr < \infty$  \hspace{1cm} (2.7)

(iii) $\forall \phi \in C^2([0,1])$ with $\phi(1) = 0$

$$\int_0^1 u(L_1 \phi) r dr = \int_0^1 f(r,u) \phi r dr.$$

Since we are interested in only positive solutions of (1.2) and hence extending $f$ for $t \leq 0$ is irrelevant. Therefore we make the following conventions.

1) Whenever we say $f$ is in $A$, then we extend $f$ by $f(r,t) = 0$ for $t \leq 0$ and $r \in [0,1]$.

2) Whenever we say $f$ is in $A'$, then we extend $f$ by $f(r,t) = -f(r,-t)$ for $t \leq 0$.  \hspace{1cm} (2.8)
For \( u \) in \( H^0_\alpha \), define

\[
\bar{T}_\alpha(u) = \frac{1}{2} \| u \|_2^2 - \int_0^1 F(r, u)r^\alpha \, dr.
\]

\[
l_\alpha = \inf_{\Sigma_\alpha} \bar{T}_\alpha.
\]

Then we have

\[ \text{Theorem 2.1.} \quad \text{Let } f \text{ be in } A. \text{ Then there exists an } \alpha_0 < 1 \text{ such that for every } \alpha_0 \leq \alpha < 1, \Sigma_\alpha \text{ is non-empty and } \{l_\alpha\} \text{ is bounded. Let } l = \lim_{\alpha \to 1} l_\alpha. \text{ Suppose there exists } b > 0, M > 0 \text{ such that}
\]

(i) \( f(r, t) \leq M \exp(br^2) \) for all \( (r, t) \in [0, \delta] \times [0, \infty) \)

(ii) \( bl < 1. \)

Then there exists a solution \( u \) of (1.2).

\[ \text{COROLLARY 2.1} \]

If \( f \) is sub-critical, then there exists a solution.

Proof. If \( f \) is sub-critical, we can take \( b \) as small as we want and satisfying (i) and (ii) of Theorem (2.1). Hence the solution exists.

\[ \text{Criterion to satisfy (2.10).} \quad \text{Let } f \text{ be in } A \text{ satisfying (i) of Theorem (2.1). Suppose there exists an } m > 0 \text{ such that}
\]

\[
\int_0^{1/2} F\left(r, \frac{m}{2}\right)r \, dr \geq 2m^2.
\]

\[ 2m^2b < 1. \]

Then \( f \) satisfies (ii) of Theorem (2.1).

For \( f \) in \( A^1 \) and for \( 0 \leq \alpha < 1 \), define

\[
B_\alpha = \left\{ u \in H^0_\alpha \setminus \{0\}; \| u \|_2^2 \leq \int_0^1 f(r, u)r^\alpha \, dr \right\}
\]

\[
\partial B_\alpha = \left\{ u \in B_\alpha; u \geq 0; \| u \|_2^2 = \int_0^1 f(r, u)r^\alpha \, dr \right\}
\]

\[
B_1 = \left\{ u \in H^1_0 \cap L^\infty \setminus \{0\}; \| u \|_2^2 = \int_0^1 f(r, u)r \, dr \right\}
\]

\[
B^*_1 = \left\{ u \in H^1_0 \setminus \{0\}; u \text{ is non-increasing, } \| u \|_2^2 = \int_0^1 f(r, u)r \, dr \right\}
\]

\[
\partial(B_1 \cup B^*_1) = \left\{ u \in B_1 \cup B^*_1; u \geq 0; \| u \|_2^2 = \int_0^1 f(r, u)r \, dr \right\}
\]

\[
B_{01} = \left\{ u \in B_1; u \text{ is constant in a nhd of zero} \right\}
\]

\[
\partial B_{01} = \left\{ u \in B_{01}; u \geq 0; \| u \|_2^2 = \int_0^1 f(r, u)r \, dr \right\}
\]
Positive solutions of the semilinear Dirichlet

For $0 \leq \alpha \leq 1$, $f \in A'$, $u$ in $H_0^\alpha$, define

$$I_u = \frac{1}{2} \int_0^1 f(r, u) u r^\alpha \, dr - \int_0^1 F(r, u) r^\alpha \, dr$$

(2.13)

since $f \in A'$; $f(r, t) - 2F(r, t) \geq 0$ for all $(r, t) \in [0, 1] \times \mathbb{R}$, hence $I_u(u) \geq 0$. Define $a_u$ by

$$\frac{a_u^2}{2} = \inf_{\Sigma_u} I_u.$$

(2.14)

**Theorem 2.2.** Let $f$ be in $A'$. Then there exists an $a_0 < 1$ such that for $\alpha_0 < \alpha < 1$, $\Sigma_u$ is non-empty and $\{a_u\}$ is bounded and satisfying

$$\frac{a_u^2}{2} = \inf_{\Sigma_u} I_u(u) = \inf_{\delta_a} I_u(u).$$

(2.15)

**Case 1.** If $f$ is super critical then $\lim_{a \to 1} a_u = 0$.

**Case 2.** If $f$ is critical and suppose there exists a $t_2 > 0$ such that

$$t_2 h \left( \left( \frac{2}{b(0)} \right)^{1/2}, t_2 \right) > 2 \left( \frac{2}{b(0)} \right)^{1/2} \exp(-t_2) < \delta_1 \quad \text{[see (2.4)]}$$

(2.16)

then $\lim_{a \to 1} a_u = a$ exists and is non-zero. Moreover there exists $u$ satisfying (1.2) such that

$$I_1(u) = \frac{a^2}{2} = \inf_{\Sigma_1} I_1 = \inf_{\delta_{t_0}} I_1 = \inf_{\delta_{t_0}} I_1.$$

(2.17)

**Remark 2.1.** Suppose there exists a sequence $t_\alpha \to \infty$ such that $h(0, t_\alpha) t_\alpha \to \infty$, then (2.16) is satisfied.

**Examples**

1. **Carlson–Chang.** Let $f_\alpha(t) = \lambda t \exp(\lambda t^2)$ for $0 < \lambda < \lambda_1$. Then $f_\alpha$ is in $A'$ and satisfies (2.16). Hence (1.1) has a solution.

2. **Atkinson–Peletier.** $f(t) = t^m \exp(br^2)$, $m > 1$, $b > 0$. Then $f$ is in $A'$ satisfying (2.16). Hence (1.1) has a solution.

3. $f(t) = \lambda t^m \exp(br^2 + \sin t^2), \quad b \geq 1$

   $m = 1, \quad 0 < \lambda < \lambda_1$,

   $m > 1, \quad \lambda > 0$.

Then $f$ is in $A'$ and satisfying (2.16). Hence (1.1) has a solution. Here log $f$ is not convex for large $t$.

4. Let $b(t)$ be a $C^1$-function on $[0, 1]$ such that $0 \leq b(r) \leq 1, b(r) \equiv 1$ in a neighbourhood of zero. Let $f(t, r) = r^m \exp(b(r) r^2 + (1 - b(r)) \exp(t))$. Then $f$ is in $A'$ satisfying (2.16). Hence (1.1) has a solution.
3. Proofs of theorems (2.1) and (2.2)

Lemma 3.1. For $0 \leq x < 1$, we have

(i) $H^2_0$ is compactly embedded in $C[0,1]$.

(ii) $\lambda_\delta < \lambda_1$ and $\lambda_\delta \to \lambda_1$ as $\delta \to 1$

(iii) $u$ in $H^2_0$, $r_1 \leq r_2$,

$$|u(r_1) - u(r_2)|^2 \leq \|u\|_1^2 \log \frac{r_2}{r_1}. \tag{3.1}$$

Proof. Let $r_1 \leq r_2$ and $u$ is in $H^2_0$. Then by integration by parts

$$|u(r_2) - u(r_1)|^2 = \left( \int_{r_1}^{r_2} u'(r) dr \right)^2 \leq \|u\|_1^2 \int_{r_1}^{r_2} r^{-\alpha} dr \leq \|u\|_1^2 \int_{r_1}^{r_2} \frac{r_2^{1-\alpha} - r_1^{1-\alpha}}{1-\alpha} \frac{dr}{r}. \tag{3.1}$$

Hence (i) follows from (3.1) and Arzela–Ascoli's theorem. Let $u$ is in $H^1_0$, then

$$|u(r_2) - u(r_1)|^2 = \left( \int_{r_1}^{r_2} u'(r) dr \right)^2 \leq \|u\|_1^2 \left( \int_{r_1}^{r_2} r^{-1} dr \right) \leq \|u\|_1^2 \log \frac{r_2}{r_1}. \tag{3.2}$$

This proves (iii).

We have

$$-(r\phi'_x)' = \lambda_\delta \phi_x r - (1 - \alpha)\phi'_x$$

$$-(r\phi'_x)' = \lambda_1 \phi_1 r.$$

Hence

$$\lambda_1 \int_0^1 \phi_1 \phi_x r dr = - \int_0^1 (r\phi'_x)' \phi_x dr$$

$$= - \int_0^1 (r\phi'_x)' \phi_x dr$$

$$= \lambda_\delta \int_0^1 \phi_x \phi_1 r dr - (1 - \alpha) \int_0^1 \phi'_x \phi_1 dr.$$

i.e.

$$(\lambda_1 - \lambda_\delta) \int_0^1 \phi_1 \phi_x r dr = - (1 - \alpha) \int_0^1 \phi'_x \phi_1 dr.$$
Since \( \phi_1 \leq 0 \) and hence \( \lambda_1 \leq \lambda_2 \), and \( \lambda_2 \to \lambda_1 \) as \( \alpha \to 1 \). This proves (ii).

**Lemma 3.2.** Let \( f \) be in \( A \), then there exists an \( \alpha_0 < 1 \) such that for \( \alpha_0 \leq \alpha < 1 \),

(i) \( \overline{T}_\alpha \) satisfies the Palais–Smale condition.

(ii) Let \( m > 0 \) be such that

\[
\int_0^{1/2} F \left( r, \frac{m}{2} \right) r \, dr \geq 2m^2
\]  \tag{3.3}

[Such a \( m \) exists because of the condition (iv) of definition (2.1)].

Then there exists a \( u_\alpha \) in \( C^2[0,1] \) satisfying

\[
L u_\alpha = f(r, u_\alpha) \quad \text{in} \quad [0,1] \]
\[u_\alpha > 0 \]
\[u'_\alpha(0) = u'_\alpha(1) = 0. \tag{3.4}\]

and

\[\overline{T}_\alpha(u_\alpha) \leq 2m^2.\]

**Proof.** Proof of this lemma is standard (see [7]). For the sake of completeness we will prove it.

**Step 1.** Let \( u_n \) in \( H_0^1 \) be a sequence such that

\[
|\overline{T}_\alpha(u_n)| \leq M
\]
\[\overline{T}_\alpha(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}\]

\[
\beta \overline{T}_\alpha(u_n) - \langle \overline{T}_\alpha(u_n), u_n \rangle
\]
\[\geq \left( \frac{\beta}{2} - 1 \right) \int_0^1 u'_n(r)^2 r^3 \, dr - \int_0^1 \left[ \beta F(r, u_n) - f(r, u_n)u_n \right] r^2 \, dr
\]
\[\geq \left( \frac{\beta}{2} - 1 \right) \int_0^1 u'_n(r)^2 r^3 \, dr - \int_{|u_n| \leq t_1} \left[ \beta F(r, u_n) - f(r, u_n)u_n \right] r^2 \, dr
\]
\[\geq \left( \frac{\beta}{2} - 1 \right) \int_0^1 u'_n(r)^2 r^3 \, dr + C, \tag{3.6}\]

where \( C \) is a constant depending only on \( F \). Since \( \beta > 2 \), (3.5) and (3.6) imply \( \{ u_n \| \_ \_ \} \) is bounded. Let \( u_n \) converge to \( u \) weakly in \( H_0^1 \) and strongly in \( C[0,1] \).

\[
\langle \overline{T}(u_n), u_n - u \rangle = \int_0^1 u'_n(r)^2 r^3 \, dr - \int_0^1 u'_n(r)u'(r)r^3 \, dr
\]
\[- \int_0^1 f(r, u_n)(u_n - u)r^2 \, dr \tag{3.7}\]

(3.5) and (3.7) imply

\[
\int_0^1 u'_n(r)^2 r^3 \, dr \to \int_0^1 u'(r)^2 r^3 \, dr.
\]
Hence \( u_\alpha \) converges strongly to \( u \) and this proves (i).

**Step 2.** From (ii) of Lemma (3.1) and (iii) of Definition (2.1) there exists an \( \alpha_0 < 1 \) and a \( \lambda > 0 \) such that

\[
F(r,t) \leq \frac{\lambda t^2}{2} < \frac{\lambda_\alpha t^2}{2} \quad \text{for all } r \in [0,1], \quad 0 < |t| < t_0.
\] (3.8)

Let \( u \) in \( H_0^\alpha \) be such that

\[
\|u\|_\alpha^2 \leq \frac{(1 - \alpha)}{2} t_0^2.
\] (3.9)

From (3.1) and (3.9) we have

\[
|u(r)|^2 \leq t_0^2.
\] (3.10)

Hence (3.8) and (3.10) give

\[
F(r,u(r)) \leq \frac{\lambda u(r)^2}{2}
\] (3.11)

\[
I_\alpha(u) = \frac{1}{2} \|u\|_\alpha^2 - \int_0^1 F(r,u)r^\alpha \, dr \\
\geq \frac{1}{2} \|u\|_\alpha^2 - \frac{\lambda}{2} \int_0^1 u(r)^2 r^\alpha \, dr \\
\geq \frac{1}{2} \left[ \|u\|_\alpha^2 - \frac{\lambda}{\lambda_\alpha} \|u\|_\alpha^2 \right] \\
= \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_\alpha} \right) \|u\|_\alpha^2.
\] (3.12)

Hence zero is a local minima.

**Step 3.** Define \( u_0 \) in \( H_0^\alpha \) by

\[
u_0(r) = \begin{cases} \frac{m}{2} & 0 \leq r < \frac{1}{2} \\ m(1 - r) & \frac{1}{2} \leq r \leq 1 \end{cases}
\] (3.13)

Then

\[
I_\alpha(u_0) = \frac{1}{2} \int_{1/2}^1 m^2 r^\alpha \, dr - \int_0^{1/2} F(r,u_0)r^\alpha \, dr \\
\leq \frac{m^2}{2(1 + \alpha)} \left( 1 - \frac{1}{2^{1+\alpha}} \right) - \int_0^{1/2} F(r,u_0)r^\alpha \, dr \\
\leq \frac{m^2}{2(1 + \alpha)} \left( 1 - \frac{1}{2^{1+\alpha}} \right) - \int_0^{1/2} F \left( r, \frac{m}{2} \right) r^\alpha \, dr \\
\leq \frac{m^2}{2(1 + \alpha)} \left( 1 - \frac{1}{2^{1+\alpha}} \right) - 2m^2 < 0
\] (3.14)
and for $0 \leq t \leq 1$,
\[
\tilde{I}_a(tu_0) \leq \frac{t^2}{2} \|u_0\|_2^2 \\
\leq \frac{m^2}{2(1 + \alpha)} \left(1 - \frac{1}{2^{1+\alpha}}\right) \leq 2m^2
\]
(3.15)

Hence $\tilde{I}_a$ satisfies all the hypotheses of Mountain pass theorem and hence there exists a critical point $u_*$ of $\tilde{I}_a$ such that
\[
\tilde{I}_a(u_*) \leq \sup_{t \in [0,1]} \tilde{I}_a(tu_0).
\]

Now from (3.15) it follows that
\[
\tilde{I}_a(u_*) \leq 2m^2
\]
and $u_*$ satisfies (3.4).

**Lemma 3.3.** Let $f$ be in $A'$, then there exists $\alpha_0 < 1$ such that for all $\alpha < 1$, $\Sigma_\alpha$ is non-empty and an $u_* \in \Sigma_\alpha$ satisfying
\[
\frac{\alpha^2}{2} = I_\alpha(u_*) = \inf_{u \in \Sigma_\alpha} I_\alpha(u) = \inf_{w \in B} I_\alpha(w)
\]
(3.16)

and for all $w$ in $H^1_0$, $\|w\|_2 = 1$,
\[
\int_0^1 f(r, a_\alpha w)wr^\alpha dr \leq a_\alpha.
\]
(3.17)

**Proof.** Let $u$ be in $B_\alpha$. Define $\gamma \leq 1$ such that
\[
\|u\|_2^2 = \frac{1}{\gamma} \int_0^1 f(r, \gamma u)u^\alpha dr.
\]
(3.18)

Such a $\gamma$ exists because $f(r, t)/t$ is an increasing function and $u$ is in $B_\alpha$ and $|f(r, t)| < \lambda_\alpha |t|$ for $|t| < t_\alpha$, $\alpha_0 \leq \alpha < 1$.

Define $v = \gamma u$, then
\[
\|v\|_2^2 = \|u\|_2^2 = \int_0^1 f(r, \gamma u)(\gamma u)^\alpha dr \\
= \int_0^1 f(r, v)v^\alpha dr.
\]
(3.19)

Hence $v$ is in $\partial B_\alpha$ and since $\gamma \leq 1$, and $f \in A'$, we have
\[
I_\alpha(v) = I_\alpha(\gamma u) \leq I_\alpha(u).
\]

This together with $\partial B_\alpha \subset B_\alpha$ imply that
\[
d_\alpha = \inf_{\partial B_\alpha} I_\alpha = \inf_{B_\alpha} I_\alpha.
\]
(3.20)
Adimurthi

Let \( u_n \) in \( \partial B_{a_n} \) be a sequence such that \( u_n \to 0 \) and \( I_{a_n}(u_n) \to d_a \). Such a sequence exists because for \( u \) in \( \partial B_{a_n} \) implies \( |u| \) is in \( \partial B_{a_n} \) and \( I_{a_n}(u) = I_{a_n}(|u|) \).

We claim that \( \{ \| u_n \|_{a_n} \} \) is bounded. Let \( N \) be such that for all \( n \geq N \),

\[
\begin{align*}
    d_a &\leq I_{a_n}(u_n) \leq d_a + 1 \\
    d_a + 1 &\geq I_{a_n}(u_n) = \frac{1}{2} \int_0^1 \left[ f(r, u_n)u_n - 2F(r, u_n) \right] r^s \, dr \\
        &\geq \frac{1}{2} \int_0^1 \left[ f(r, u_n)u_n - \beta F(r, u_n) \right] r^s \, dr \\
        &\geq \left( \frac{\beta}{2} - 1 \right) \int_0^1 F(r, u_n) r^s \, dr.
\end{align*}
\]

From (iv) of Definition (2.1), there exists a constant \( C \) depending only on \( f \) such that for all \( v \) in \( H_0^1 \),

\[
\int_0^1 \left[ f(r, v) v - \beta F(r, v) \right] r^s \, dr \geq C.
\]

From (3.22) and (3.23) there exists a constant \( C_1 \) independent of \( n \) such that

\[
\int_0^1 F(r, u_n) r^s \, dr \leq C_1.
\]

From (3.21) and (3.24) we have

\[
\| u \|_{a_n}^2 = 2I_{a_n}(u_n) + 2 \int_0^1 F(r, u_n) r^s \, dr \\
\leq 2(d_a + 1) + 2C_1
\]

and this proves the claim.

Let \( u_\alpha \) be the weak limit of \( u_n \) and \( \alpha_0 \) be as in Lemma (3.2). We claim that for \( \alpha \leq \alpha_0 < 1 \), \( u_\alpha \in \Sigma_\alpha \) satisfying (3.16).

First we will show that \( u_\alpha \) is non-zero. Suppose \( u_\alpha \equiv 0 \), then from Lemma (3.1). \( u_\alpha \) converges to 0 in \( C[0, 1] \). Let \( N \) be an integer such that

\[
u_n(r) < \alpha_0 \quad \text{for all} \quad n \geq N, \quad r \in [0, 1].
\]

Then from (iii) of Definition (2.1) and the choice of \( \alpha_0 \),

\[
f(r, u_\alpha(r), u_\alpha(r)) < \lambda \alpha_0 u_\alpha(r).
\]

Since \( u_\alpha \in \partial B_{\alpha_0} \) we have from (3.26)

\[
\| u_\alpha \|_{a_0}^2 = \int_0^1 f(r, u_\alpha) u_\alpha r^s \, dr \\
< \lambda \alpha_0 \int_0^1 u_\alpha(r) r^s \, dr \leq \| u_\alpha \|_{a_0}^2
\]
which is a contradiction and hence $u_\alpha \neq 0$ and

$$I_\alpha(u_\alpha) = \lim_{\alpha \to \infty} I_\alpha(u_\alpha) = d_\alpha$$

$$\|u_\alpha\|^2 = \lim_{\alpha \to \infty} \|u_\alpha\|^2 = \int_0^1 f(r, u_\alpha) u_\alpha r^\alpha dr,$$

(3.27)

$u_\alpha$ is in $\partial B_\delta$. If not, then by (3.27) we can choose a $\gamma < 1$ such that

$$\|u_\alpha\|^2 = \frac{1}{\gamma} \int_0^1 f(r, u_\alpha) u_\alpha r^\alpha dr.$$

Then $\gamma u_\alpha$ is in $\partial B_\delta$ and

$$d_\alpha < I(\gamma u_\alpha) < I(u_\alpha) = d_\alpha.$$

This proves that $u_\alpha$ is in $\partial B_\delta$. Since $u_\alpha$ is a minimizer and hence there exists a real number $\rho$ such that for all $\phi$ in $H_0^\delta$,

$$\int_0^1 u_\alpha'(r)\phi'(r)r^\alpha dr - \int_0^1 f(r, u_\alpha)\phi r^\alpha dr = \rho \left\{ 2 \int_0^1 u_\alpha'(r)\phi'(r)r^\alpha dr - \int_0^1 f(r, u_\alpha)\phi r^\alpha dr - \int_0^1 \frac{\partial f}{\partial r}(r, u_\alpha) u_\alpha r^\alpha \phi r^\alpha dr \right\}.$$

(3.28)

Putting $\phi = u_\alpha$ in (3.28) and using the fact that $u_\alpha \in \partial B_\delta$, we have

$$\rho \left\{ 2 \int_0^1 u_\alpha'(r)r^\alpha dr - \int_0^1 f(r, u_\alpha) u_\alpha r^\alpha dr - \int_0^1 \frac{\partial f}{\partial r}(r, u_\alpha) u_\alpha r^\alpha \phi r^\alpha dr \right\} = 0.$$

Since $u_\alpha$ is in $\partial B_\delta$, we have

$$\rho \int_0^1 \left[ f(r, u_\alpha) - \frac{\partial f}{\partial r}(r, u_\alpha) u_\alpha \right] u_\alpha(r)^2 r^\alpha dr = 0.$$

Since $f$ is in $A'$, and $u$ is not zero, it implies that $\rho = 0$. Hence from (3.28) and by regularity of elliptic operator, it follows that $u_\alpha$ is in $\Sigma_\alpha$ and $I_\alpha(u_\alpha) = d_\alpha$. Since $\Sigma_\alpha \subset \partial B_\delta$, we have $\frac{d^2}{2} = \inf_{\Sigma_\alpha} I_\alpha = I_\alpha(u_\alpha) = d_\alpha$ and this proves (3.16). Let $\|w\|_\alpha = 1$. Choose $\gamma > 0$ such that

$$1 = \frac{1}{\gamma} \int_0^1 f(r, \gamma w) wr^\alpha dr.$$

(3.29)

Then $\gamma w$ is in $\partial B_\delta$. Hence

$$\frac{d^2}{2} \leq I_\alpha(\gamma w) \leq \frac{\gamma^2}{2} \|w\|^2 = \frac{\gamma^2}{2}$$


implies \( a_\gamma \leq \gamma \). Since \( f \) is in \( A' \), we have

\[
\frac{1}{a_\gamma} \int_0^1 f(r, a_\gamma w) r^\alpha \, dr \leq \frac{1}{\gamma} \int_0^1 f(r, \gamma w) r^\alpha \, dr = 1
\]

i.e.

\[
\int_0^1 f(r, a_\gamma w) r^\alpha \, dr \leq a_\gamma
\]

proving (3.17).

**Lemma 3.4.** Let \( f \) be in \( A' \) and \( a_\gamma \) is as in Lemma (3.3). Then \( \{a_\gamma\} \) is bounded on \([a_\gamma, 1)\).

Let \( a = \lim_{\gamma \to 1} a_\gamma \). Then for all \( w \in H_0^1 \) with \( \|w\|_1 = 1 \), we have

\[
\int_0^1 f(r, aw) r^\alpha \, dr \leq a.
\]  

(3.30)

**Proof.** From Lemma (3.2) and (3.3) we have \( l_\alpha = a_\gamma^2/2 \) and \( l_\alpha \leq 2m^2 \). Hence \( \{a_\gamma\} \) is bounded on \([a_\gamma, 1)\). Let \( \alpha_\gamma \) be a sequence such that \( a_\alpha_\gamma \to a \) as \( \alpha_\gamma \to 1 \) and \( w \) be in \( E \) with \( \|w\|_1 = 1 \). Let \( v_\gamma = \|w\|_{H_0^1}^\alpha \). Then from (3.17) we have

\[
\int_0^1 f(r, a_\gamma w) v_\gamma r^\alpha \, dr \leq a_\gamma.
\]

Letting \( \alpha_\gamma \to 1 \), \( v_\gamma \to w \), \( a_\alpha_\gamma \to a \), we get

\[
\int_0^1 f(r, aw) r^\alpha \, dr \leq a.
\]  

(3.31)

Since \( f \) is odd, and hence by Fatou’s (3.31) holds for all \( w \) in \( H_0^1 \).

**Lemma 3.5.** Let \( f \) be in \( A, 0 \leq \alpha < 1, 0 \leq \epsilon \leq 1 \), and \( u \) in \( \Sigma_\alpha \). Then we have

\[
u(r) = \frac{1 - r^{1 - \alpha}}{1 - \alpha} \int_0^r f(t, u(t)) t^\alpha \, dt + \int_0^1 t^\alpha \left( \frac{1 - t^{1 - \alpha}}{1 - \alpha} \right) f(t, u(t)) \, dt \]  

(3.32)

\[1/2 \epsilon^{1 + \alpha} u'(\epsilon)^2 = (1 + \alpha) \int_0^\epsilon F(r, u) r^\alpha \, dr + \int_0^\epsilon \frac{\partial F}{\partial r} (r, u) r^{1 + \alpha} \, dr + \frac{1 - \alpha}{2} \int_0^\epsilon u'(r)^2 r^\alpha \, dr - e^{1 + \alpha} F(\epsilon, u(\epsilon)).
\]  

(3.33)

**Proof.** If \( v(r) \) is the right hand side of (3.32), then by differentiating twice, \( v \) satisfies

\[
L_\alpha v = f(r, u)
\]

\[
v'(0) = v(1) = 0.
\]  

(3.34)

Hence by uniqueness, \( v = u \). This proves (3.32). \( u \) is in \( \Sigma_\alpha \), hence

\[
(r^\alpha u')' = - f(r, u(r)) r^\alpha.
\]  

(3.35)
multiply (3.35) by ru(r) and integrate from 0 to \( \varepsilon \) we get

\[
\int_0^\varepsilon \left( r^a u'(r) \right)' u'(r) r dr = - \int_0^\varepsilon f(r, u) u' r^{1+a} dr.
\]

(3.36)

Since \( (\frac{dF}{dr})(r, u(r)) = (\frac{\partial F}{\partial r})(r, u(r)) + f(r, u(r)) u'(r) \), we have

\[
\frac{1}{2} e^{1+a} u'(\varepsilon)^2 - \frac{(1-a)}{2} \int_0^\varepsilon u'(r)^2 r^a dr = - \int_0^\varepsilon \frac{dF}{dr} r^{1+a} dr + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+a} dr
\]

\[
= - F(\varepsilon, u(\varepsilon)) e^{1+a} + (1+a) \int_0^\varepsilon F(r, u) r^a dr
\]

\[
+ \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+a} dr.
\]

Hence

\[
\frac{1}{2} e^{1+a} u'(\varepsilon)^2 = (1+a) \int_0^\varepsilon F(r, u) r^a dr + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+a} dr + \frac{1-a}{2} \int_0^\varepsilon u'(r)^2 r^a dr
\]

\[- F(\varepsilon, u(\varepsilon)) e^{1+a}.
\]

This proves (3.33).

**Lemma 3.6.** Let \( f \) be in \( A, x_n \rightarrow 1, u_n \) is in \( \Sigma_{\alpha_a} \) and a constant \( M \) independent of \( n \) such that

(i) \( \| u_n \|_{\alpha_a} \leq M \)

(ii) \( \lim_{n \rightarrow \infty} u_n(1) = \eta \neq 0 \).

(3.37)

Then there exists a subsequence (still denoted by \( u_n \)) such that the weak limit \( u \) of \( u_n \) in \( H_0^1 \) is a weak solution of (1.2). Furthermore

\[
\lim_{n \rightarrow \infty} \int_0^1 F(r, u_n) r^a dr = \int_0^1 F(r, u) r dr.
\]

(3.38)

**Proof.** \( \| u_n \|_1 \leq \| u_n \|_{\alpha_a} \leq M \), hence by going to a subsequence the weak limit \( u \) of \( u_n \) in \( H_0^1 \) exists. From (iii) of Lemma (3.1), \( u_n \) converges to \( u \) uniformly on compact subsets of \( [0, 1] \). We claim \( u \) is not identically zero. For, if \( u \equiv 0 \), then, since \( u_n \) in \( \Sigma_{\alpha_a} \), we have for \( 0 < r \leq 1 \),

\[
r^a u_n'(r) = u_n'(1) + \int_r^1 f(r, u_n) r^a dr.
\]

(3.39)

From (ii) of (3.37) and (3.39) and using \( u_n \rightarrow 0 \) on \( [r, 1] \) uniformly

\[
r \lim_{n \rightarrow \infty} u_n'(r) = \eta.
\]

(3.40)

Hence by Fatou's lemma, and (3.40)

\[
\infty = \eta^2 \int_0^1 \frac{r dr}{r^2} < \int_0^1 \lim_{n \rightarrow \infty} u_n'(r)^2 r^a dr \leq \lim_{n \rightarrow \infty} \| u_n \|_{\alpha_a}^2 \leq M
\]
which is a contradiction. Hence \( u \neq 0 \) and \( u \) satisfies

\[
-(ru') = f(r, u)r \quad \text{in } (0, 1]
\]

\[
u(1) = 0.
\] (3.41)

Now by Fatous, we have

\[
\int_0^1 f(r, u)ur \, dr \leq \lim_{n \to \infty} \int_0^1 f(r, u_n)u_n r^2 \, dr \leq M.
\] (3.42)

Hence

\[
\int_0^1 f(r, u)u \, dr \leq \int_{u \leq 1} f(r, u)u \, dr + \int_{u > 1} f(r, u)ur \, dr < \infty.
\] (3.43)

For any \( 0 < r \leq 1 \), integrating (3.41) from \( r \) to 1, we get

\[
r u'(r) = u'(1) + \int_r^1 f(t, u)t \, dt.
\] (3.44)

(3.44) gives \( ru'(r) \) is monotone and hence limit \( r \to 0 \) exists. We claim that

\[
\lim_{r \to 0} ru'(r) = 0.
\] (3.45)

For, if \( \lim_{r \to 0} ru'(r) = C < 0 \), then there exists \( \varepsilon > 0 \) such that \( -u'(r) \geq C/r \) for \( 0 < r \leq \varepsilon \).

Hence

\[
\infty = C^2 \int_0^\varepsilon \frac{r \, dr}{r^2} \leq \int_0^\varepsilon ru'(r)^2 \, dr < \infty.
\]

Hence (3.45) is true. Using (3.44) and (3.45) we get

\[
u'(1) = -\int_0^1 f(t, u)t \, dt.
\] (3.46)

Let \( \phi \) be in \( C^2[0, 1] \) with \( \phi(1) = 0 \). Multiply \( \phi' \) to (3.44) and integrate from 0 to 1, and using (3.46) we have

\[
\int_0^1 u'(r)\phi'(r) \, dr = u'(1)(\phi(1) - \phi(0)) + \int_0^1 \phi'(r) \int_r^1 f(t, u)t \, dt \, dr
\]

\[
= u'(1)(\phi(1) - \phi(0)) + \int_0^1 f(t, u)\phi(t) \, dt
\]

\[
- \phi(0) \int_0^1 f(t, u)t \, dt
\]

\[
= \int_0^1 f(t, u)\phi(t) \, dt
\]

and hence \( u \) is a weak solution of (1.2).
From (3.33) and (3.37) we have
\[
\lim_{s \to -\infty} \left\{ (1 + \alpha_s) \int_0^1 F(r, u_s) r^{\alpha_s} \, dr + \int_0^1 \frac{\partial F}{\partial r} r^{1 + \alpha_s} \, dr \right\} = \frac{1}{2} \eta^2 \tag{3.47}
\]

Now multiply \(ru(r)\) to (3.41) and integrate from \(r = 1\), we have
\[
- \frac{1}{2} r^2 u'(r)^2 + \frac{1}{2} u'(1)^2 = - \int_1^r \frac{dF}{dt} t^2 \, dt + \int_{t_1}^1 \frac{\partial F}{\partial t} t^2 \, dt
\]
\[
= F(r, u(r)) r^2 + 2 \int_t^1 F(t, u) t \, dt + \int_t^1 \frac{\partial F}{\partial t} t^2 \, dt. \tag{3.48}
\]

Since \(ru(r) \to 0\), \(\int_0^1 F(t, u) t \, dt < \infty\), \(\frac{\partial F}{\partial r} > 0\) in \([0, \delta_0]\) and \(\int_{\delta_0} \frac{\partial F}{\partial t} t^2 \, dt < \infty\), we conclude that \(\lim_{r \to -0} F(r, u(r)) r^2\) exists and claim that
\[
\lim_{r \to -0} F(r, u(r)) r^2 = 0. \tag{3.49}
\]

If not, there exists a constant \(C > 0\) and \(\epsilon > 0\) such that
\[F(r, u(r)) r^2 \geq C \quad \text{for all } 0 < r < \epsilon.
\]

Hence
\[
\infty = \int_0^r \frac{C}{r} \, dr \leq \int_0^r F(r, u(r)) r \, dr < \infty
\]

which is a contradiction.

Now using (3.49), (3.48) becomes
\[
\frac{1}{2} u'(1)^2 = 2 \int_0^1 F(r, u) r \, dr + \int_0^1 \frac{\partial F}{\partial r} (r, u) r^2 \, dr. \tag{3.50}
\]

Since \(u'(1) = \lim_{s \to -\infty} u'(1)\), and hence from (3.47) and (3.50) we have
\[
2 \int_0^1 F(r, u) r \, dr + \int_0^1 \frac{\partial F}{\partial r} (r, u) r^2 \, dr
\]
\[
= \lim_{s \to -\infty} \left\{ (1 + \alpha_s) \int_0^1 F(r, u_s) r^{\alpha_s} \, dr + \int_0^1 \frac{\partial F}{\partial r} (r, u_s) r^{1 + \alpha_s} \, dr \right\}. \tag{3.51}
\]

By Fatou's and using (ii) of Definition (2.1) we have
\[
2 \int_0^1 F(r, u) r \, dr \leq \lim_{s \to -\infty} (1 + \alpha_s) \int_0^1 F(r, u_s) r^{\alpha_s} \, dr
\]
\[
\int_0^1 \frac{\partial F}{\partial r} (r, u) r^2 \, dr \leq \lim_{s \to -\infty} \int_0^1 \frac{\partial F}{\partial r} (r, u_s) r^{\alpha_s + 1} \, dr. \tag{3.52}
\]

By going to a subsequence, we conclude from (3.51) and (3.52) that
\[
\lim_{s \to -\infty} (1 + \alpha_s) \int_0^1 F(r, u_s) r^{\alpha_s} \, dr = 2 \int_0^1 F(r, u) r \, dr
\]
Adimurthi

and

$$\lim_{s \to 0} \int_0^1 \frac{\partial F}{\partial r}(r, u_s) r^{n+1} dr = \int_0^1 \frac{\partial F}{\partial r}(r, u) r^2 dr.$$ 

**Lemma 3.7.** Let $f$ in $A'$ be critical. Then

$$\frac{2}{b(0)} = \sup \left\{ c^2 ; \sup_{\|w\|_1 \leq 1} \int_0^1 f(r, cw) wr dr < \infty \right\}$$

(3.53)

**Proof.** $f = h(r, t) \exp[b(r)t^2]$ for $(r, t)$ in $Q_n$.

Let

$$C_0^2 = \sup \left\{ c^2 ; \sup_{\|w\|_1 \leq 1} \int_0^1 f(r, cw) wr dr < \infty \right\}$$

**Step 1.** $C_0^2 > 2/b(0)$.

If not, then choose $\epsilon > 0$, $c > 0$ and a $\delta < (\delta_1, \delta_0)$ such that

$$\frac{2}{b(0)} < c^2 < (c + \epsilon)^2 < C_0^2.$$  

(3.54)

For $r_0 \in [0, \delta_1]$, define

$$W_{r_0}(r) = \frac{\log \frac{1}{r}}{\left( \log \frac{1}{r_0} \right)^{1/2}} \text{ for } r_0 \leq r \leq 1$$

$$W_{r_0}(r) = \left( \frac{1}{r_0} \right)^{1/2} \text{ for } 0 \leq r \leq r_0.$$  

(3.55)

Then $\|w_{r_0}\|_1 = 1$. Since $(\partial f/\partial r)(r, t) \geq 0$ in $Q_{2c}$, we have

$$h(0, t) \exp[b(0)t^2] \leq h(r, t) \exp[b(r)t^2] \text{ in } Q_{2c}.$$  

$(c + \epsilon)^2 < C_0^2$ implies that there exists an absolute constant $M$ depending only on $c$ and $f$ such that

$$M \geq \int_0^1 f(r, (c + \epsilon)w_{r_0}) w_{r_0} r dr \geq \int_0^1 f(r, (c + \epsilon)w_{r_0}) w_{r_0} r dr$$

$$\geq \int_0^{r_0} f \left( 0, (c + \epsilon) \left( \frac{1}{\log \frac{1}{r_0}} \right)^{1/2} \right) \left( \log \frac{1}{r_0} \right)^{1/2} r dr$$

$$= \frac{1}{2} \left( \log \frac{1}{r_0} \right)^{1/2} h \left( 0, (c + \epsilon) \left( \frac{1}{\log \frac{1}{r_0}} \right)^{1/2} \right) \exp \left[ b(0)(c + \epsilon)^2 \log \frac{1}{r_0} \right] r_0^2$$

$$\geq \frac{1}{2} \left( \log \frac{1}{r_0} \right)^{1/2} h \left( 0, (c + \epsilon) \left( \frac{1}{\log \frac{1}{r_0}} \right)^{1/2} \right) \exp \left[ \epsilon^2 \left( \frac{1}{\log \frac{1}{r_0}} \right) \right] r_0^{2(b(0) - 2)}$$

as $r_0 \to 0$.  


Hence \( C_0^2 < 2/b(0) \).

**Step 2.** \( C_0^2 = 2/b(0) \).

Suppose not, then choose \( \varepsilon > 0, \delta > 0 \) such that \( \delta \leq \min(\delta_1, \delta_0) \) and for all \( r \) in \([0, \delta] \),

\[
C_0^2 < (C_0 + \varepsilon)^2 < \frac{2 - \varepsilon}{b(r)}.
\]

Let \( \|w\|_1 \leq 1 \), then

\[
\int_0^1 f(r, (C_0 + \varepsilon)w) wr \, dr = \int_0^\delta + \int_\delta^1.
\]  

(3.56)

Since \( \|w\|_1 = 1 \) implies from Lemma (3.1)

\[
|w(r)| \leq \log \frac{1}{r},
\]

hence there exists a constant \( M_1 \) such that

\[
\sup_{|w|_1 \leq 1} \int_0^1 f(r, (C_0 + \varepsilon)w) wr \, dr \leq M_1 (3.57)
\]

and

\[
\int_0^\delta f(r, (C_0 + \varepsilon)w) wr \, dr \leq \int_0^\delta h(r, (C_0 + \varepsilon)w)[\exp(C_0 + \varepsilon)^2 b(r)w^2] wr \, dr
\]

\[
\leq \int_0^\delta h(r, (C_0 + \varepsilon)w)[\exp(2 - \varepsilon)w^2] wr \, dr
\]

\[
\leq M_2 \int_0^\delta [\exp(2 - \varepsilon/2)w^2] r \, dr
\]

\[
\leq M_2 \int_0^\delta r e^{1/2} \, dr \leq M_3 (3.58)
\]

where

\[
M_2 = \sup_{r \geq 0, t \geq 0} h(r, t) \exp \frac{t^2}{2}.
\]

This implies \( C_0 > (C_0 + \varepsilon) \) which is a contradiction. Hence \( C_0^2 = 2/b(0) \).

**Lemma 3.8.** Let \( f \) in \( A' \) be critical and suppose there exists a \( t_0 > 0 \) satisfying

\[
\exp - t_0^2 < \delta_1,
\]

\[
h\left(0, \left(\frac{2}{b(0)}\right)^{1/2}\right) t_0 > 2\left(\frac{2}{b(0)}\right)^{1/2} (3.59)
\]
Let $a \geq 0$ such that
\[
\sup_{1 \leq r \leq 1} \int_0^1 f(r, aw) wr \, dr \leq a \tag{3.60}
\]
then $a^2 < 2/b(0)$.

**Proof.** From Lemma (3.7), $a^2 \leq 2/b(0)$. Suppose $a^2 = 2/b(0)$, then take $r_0 = \exp -t_0^2$, $w_{t_0}$ as in (3.55) and from (3.60) we have
\[
\left( \frac{2}{b(0)} \right)^{1/2} = a \geq \int_0^{r_0} f(r, aw_{t_0}) wr \, dr \\
= \int_0^{r_0} f(0, aw_{t_0}) w_{t_0} \, dr \\
= f(0, at_0) t_0 \frac{r_0^2}{2} \\
= t_0 h(0, at_0) \exp \left( \log \frac{1}{r_0} \right) \frac{r_0^2}{2} \\
= \frac{1}{2} t_0 h \left( 0, \left( \frac{2}{b(0)} \right)^{1/2} t_0 \right) \geq \left( \frac{2}{b(0)} \right)^{1/2}
\]
which is a contradiction. Hence the result.

**Lemma 3.9.** For any $\varepsilon > 0$, $0 \leq \alpha < 1$,
\[
\sup_{0 < r \leq 1} r^\alpha \left( \frac{1 - r^{1 - \alpha}}{1 - \alpha} \right) \leq \frac{1}{\varepsilon}. \tag{3.61}
\]

**Proof.** Let $g(r) = r^\alpha (1 - r^{1 - \alpha})$. Then $g(0) = g(1) = 0$. Let $0 < r_0 < 1$ such that
\[
g(r_0) = \sup_{0 \leq r \leq 1} g(r)
\]
then
\[
0 = g'(r_0) = \varepsilon t_0^{\alpha - 1} \left( \frac{1 - r_0^{1 - \alpha}}{1 - \alpha} \right) - r_0^{1 - \alpha}.
\]
Hence
\[
\frac{1 - r_0^{1 - \alpha}}{1 - \alpha} = \frac{r_0^{1 - \alpha}}{\varepsilon}.
\]
Therefore
\[
g(r) \leq g(r_0) \leq \frac{r_0^{1 - \alpha + \varepsilon}}{\varepsilon} \leq \frac{1}{\varepsilon}.
\]
Positive solutions of the semilinear Dirichlet

Lemma 3.10. Let $f$ in $A$ be critical, then

$$\inf_{B_1 \cup B_1^*} I_1 = \inf_{B_1 \cup B_1^*} I_1 = \inf_{B_01} I_1$$  \hspace{1cm} (3.62)

Proof. $u$ is in $B_1 \cup B_1^*$ implies $|u|$ also in $B_1 \cup B_1^*$ and $I_1(u) = I_1(|u|)$. Let $u \in B_1 \cup B_1^*$; choose a $\gamma < 1$ such that

$$\|u\|_1^2 = \int_0^1 f(r, ru)r dr.$$ 

Then $yu$ is in $\partial(B_1 \cup B_1^*)$ and $I_1(yu) \leq I_1(u)$. Hence

$$\inf_{B_1 \cup B_1^*} I_1 = \inf_{B_1 \cup B_1^*} I_1.$$ 

Now let $u \geq 0$ is in $\partial(B_1 \cup B_1^*)$. Since $f$ is critical, we have for any $s > 1$

$$\int_0^1 f(r, su)r dr < \infty.$$ 

Let $v = su$, then

$$\|v\|_1^2 = s^2 \|u\|_1^2 = s^2 \int_0^1 f(r, u)r dr = s \int_0^1 f\left(r, \frac{r}{s}\right)r dr < s \int_0^1 f(r, v)r dr$$  \hspace{1cm} (3.63)

because $s > 1$ and $f(r, t)/t$ is increasing.

Choose an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$

$$\|v\|_1^2 < \int_\varepsilon^1 f(r, v)r dr \leq \int_0^1 f(r, v)r dr$$  \hspace{1cm} (3.64)

and define

$$v_\varepsilon = \begin{cases} v(r) & \text{if } 0 \leq r \leq \varepsilon \\ v(r) & \text{if } \varepsilon \leq r \leq 1. \end{cases}$$  \hspace{1cm} (3.65)

Then from (3.64) $v_\varepsilon$ is in $B_01$.

Now we claim that $I_1(v_\varepsilon) \rightarrow I_1(v)$ as $\varepsilon \rightarrow 0$.

Case 1. If $v$ is in $B_1$, then $\|v_\varepsilon\|_\infty \leq \|v\|_\infty$ and hence by dominated convergence theorem $I_1(v_\varepsilon) \rightarrow I_1(v)$.

Case 2. If $v$ is in $B_1^*$, then $v_\varepsilon \uparrow v$ and hence by Monotone convergence theorem, $I_1(v_\varepsilon) \rightarrow I_1(v)$. Hence

$$\inf_{B_01} I_1 \leq I_1(v_\varepsilon) \rightarrow I_1(v) \text{ as } \varepsilon \rightarrow 0.$$  \hspace{1cm} (3.66)
Adimurthi

$f$ is critical and is in $A'$, we have for $1 \leq s \leq 2$

$$f(r, su) su - 2F(r, su) \leq 2f(r, 2u)u - 2F(r, 2u)$$

and is in $L^1$. Hence by dominated convergence theorem,

$$I_1(v) \to I_1(u) \quad {\text{as}} \ s \to 1.$$ 

(3.67)

Combining (3.66) and (3.67) we have

$$\inf_{B_1} I_1 \leq \inf_{B_0} I_1 \leq \inf_{B_1} I_1$$

and hence the result.

Proof of theorem (2.1). From Lemma (3.2), there exists $\alpha_0 < 1$ such that $\Sigma_n$ is non-empty for $\alpha \leq x < 1$ and \{l_n\} is bounded by $2m^2$ where $m$ is given by (3.3). Let $l = \lim_{n \to \infty} l_n$.

Let $f$ satisfies (2.10). Let $\eta > 0$, $\gamma > 0$, $\alpha_n \to 1$, $u_n$ in $\Sigma_n$ such that

(i) $l_n \to l$ as $\alpha_n \to 1$

(ii) $l_n \leq \bar{u}_n, u_n < \left( l_n + \frac{\eta}{2} \right)$

(iii) $(l_n + \eta)b \leq \gamma < 1$.

(3.68)

We claim that

$$\lim_{n \to \infty} u_n(1) \neq 0.$$ 

(3.69)

If not, then $u_n(1) \to 0$. Since $u_n \in \Sigma_n$, we have

$$u_n(1) = -\int_0^1 f(r, u_n) r \, dr \to 0 \quad {\text{as}} \alpha_n \to 1.$$

Since for any $0 \leq r \leq 1$ we have

$$r^\alpha u_n(r) = u_n(1) + \int_0^r f(t, u_n) t^{\alpha} \, dt$$

we have

$$\sup_{r \in [0, 1]} |r^\alpha u_n(r)| \to 0 \quad {\text{as}} \alpha_n \to 1.$$ 

This shows for any $0 < r_0 \leq 1$,

$$\sup_{r \in [r_0, 1]} |u_n(r)| \to 0 \quad {\text{as}} \alpha_n \to 1.$$ 

(3.70)

This in turn implies

$$\sup_{r \in [r_0, 1]} |u_n(r)| \leq \int_{r_0}^1 |u_n'(t)| \, dt \to 0 \quad {\text{as}} \alpha_n \to 1.$$ 

(3.71)
Positive solutions of the semilinear Dirichlet

From (ii) of definition (2.1) and (3.33) we have

\[
\frac{1}{2}\delta^2_\alpha u_\alpha(\delta_0)^2 = (1 + \alpha) \int_0^{\delta_0} F(r, u_\alpha) r^{m-2} dr + \int_0^{\delta_0} \frac{\partial F}{\partial r}(r, u_\alpha) r^{1 + \alpha} dr \\
+ \left(1 - \frac{\alpha}{2}\right) \int_0^{\delta_0} u_\alpha(r)^2 r^{m-2} dr - \delta^{1 - \alpha}_0 \delta_0^{-\alpha} F(\delta_0, u_\alpha(\delta_0)) \\
\geq (1 + \alpha) \int_0^{\delta_0} F(r, u_\alpha) r^{m-2} dr - \delta^{1 + \alpha}_0 \delta_0^{1 + \alpha} F(\delta_0, u_\alpha(\delta_0))
\]  
(3.72)

Hence by (3.70) and (3.72) we have

\[
\int_0^{\delta_0} F(r, u_\alpha) r^{m-2} dr \to 0 \quad \text{as } \alpha \to 1.
\]  
(3.73)

From (3.71) and by dominated convergence theorem

\[
\int_\delta_0^1 F(r, u_\alpha) r^{m-2} dr \to 0 \quad \text{as } \alpha \to 1.
\]  
(3.74)

Combining (3.73) and (3.74) we have

\[
\int_0^1 F(r, u_\alpha) r^{m-2} dr \to 0 \quad \text{as } \alpha \to 1.
\]  
(3.75)

Let \(\delta_0\) be such that for all \(n \geq \delta_0\),

\[
\int_0^1 F(r, u_\alpha) r^{m-2} dr < \frac{\eta}{2}.
\]  
(3.76)

From (ii) and (iii) of (3.68) and (3.66)

\[
\frac{1}{2}\|u_\alpha\|_\infty^2 = \int_\delta_\alpha (u_\alpha) + \int_0^1 F(r, u_\alpha) r^{m-2} dr \\
< \left(l_\alpha + \frac{\eta}{2}\right) + \frac{\eta}{2} = (l_\alpha + \eta)
\]

\[
\leq \frac{\eta}{b}.
\]

Hence

\[
|u_\alpha(r)|^2 \leq \|u\|_\infty^2 \log \frac{1}{r} \\
< 2(l_\alpha + \eta) \log \frac{1}{r} \\
\leq \frac{2\eta}{b} \log \frac{1}{r}.
\]  
(3.77)
From (3.32), (3.70) and (3.77) we have

\[
\begin{align*}
\lambda_n u_n(0) &= \int_0^1 t^{\alpha_n} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) f(t, u_n) \, dt \\
&= \int_0^{s_1} t^{\alpha_n} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) f(t, u_n) \, dt + \int_{s_1}^1 t^{\alpha_n} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) f(t, u_n) \, dt \\
&\leq M \int_0^{s_1} t^{\alpha_n} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) \exp \left(\frac{\epsilon}{1 - \alpha_n}\right) \, dt + M_1 \\
&\leq M \int_0^{s_1} t^{\alpha_n} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) \exp \left(2\gamma \log \frac{1}{t}\right) \, dt + M_1 \\
&\leq M \int_0^{s_1} t^{\alpha_n - 2\gamma} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) \, dt + M_1 \\
&= \frac{2M}{\epsilon} \left(\frac{1}{\alpha_n - 2\gamma + 1} + \frac{1}{2}\right) + M_1. 
\end{align*}
\]  

(3.78)

Now choose \( \epsilon > 0 \) such that

\[\alpha_n > 2\gamma - 1 + \epsilon \quad \text{for all } n, \text{ large.}\]

Then from (3.61) and (3.78) we have

\[
\begin{align*}
\lambda_n u_n(0) &\leq M \int_0^{s_1} t^{\alpha_n - 2\gamma - \epsilon/2} t^{\alpha_n/2} \left(1 - \frac{t^{\alpha_n}}{1 - \alpha_n}\right) \, dt + M_1 \\
&\leq \frac{2M}{\epsilon} \left(\frac{1}{\alpha_n - 2\gamma + 1 + \frac{\epsilon}{2}}\right) + M_1 \\
&\leq \frac{4M}{\epsilon} + M_1. 
\end{align*}
\]  

Hence

\[\| u_n \|_{\infty} = u_n(0) \leq \frac{4M}{\epsilon} + M_1.\]

Since \( u_n \) is in \( \sum_{r} \alpha_n \) and \( \{ \| u_n \|_{\infty} \} \) is bounded and hence \( u_n \) converges strongly in \( C[0, 1] \) and in \( H^1_0 \) to a function \( u \). From (3.71) \( u(r) \to 0 \) as \( \alpha_n \to \infty \) for every \( r \neq 0 \), we have \( u \equiv 0 \) and hence \( u_n(0) \to 0 \). Now choose \( N \) large such that \( \| u_n \|_{\infty} \leq \epsilon_0 \) for all \( n \geq N \).

From (iii) of Definition (2.1) we have

\[
\begin{align*}
\lambda_n \int_0^1 u_n \phi_{\alpha_n} r^{\alpha_n} \, dr &= - \int_0^1 (r^{\alpha_n} \phi_{\alpha_n}) u_n \, dr \\
&= - \int_0^1 (r^{\alpha_n} u_n') \phi_{\alpha_n} \, dr \\
\lambda_n \int_0^1 u_n \phi_{\alpha_n} r^{\alpha_n} \, dr &= - \int_0^1 (r^{\alpha_n} u_n') \phi_{\alpha_n} \, dr \\
&= \int_0^1 f(r, u_n) \phi_{\alpha_n} r^{\alpha_n} \, dr \\
&< \lambda_n \int_0^1 u_n \phi_{\alpha_n} r^{\alpha_n} \, dr. 
\end{align*}
\]
and hence a contradiction. This proves the claim. Hence by going to a subsequence, we assume that

$$
\lim_{n \to 1} u_n(1) \neq 0 \\
l_n \leq I_n(u_n) < 2l_n \leq 4m^2.
$$

(3.80)

Now

$$
4m^2 \geq \frac{1}{2} \int_0^1 [f(r, u_n)u_n - 2F(r, u_n)]r^n \, dr \\
= \frac{1}{2} \int_0^1 [f(r, u_n)u_n - \beta F(r, u_n)]r^n \, dr + \frac{\beta - 2}{2} \int_0^1 F(r, u_n)r^n \, dr \\
\geq M_1 + (\frac{\beta - 2}{2}) \int_0^1 F(r, u_n)r^n \, dr
$$

where $M_1$ is constant independent of $n$. Hence $\exists M_2 > 0$ such that

$$
\int_0^1 F(r, u_n)r^n \, dr \leq M_2 \\
\frac{1}{2} \|u_n\|_{2,1}^2 = I_n(u_n) + \int_0^1 F(r, u_n)r^n \, dr \leq 4m^2 + M_2.
$$

(3.81)

Hence $\{\|u_n\|_{2,1}\}$ is uniformly bounded. Hence from (3.80) and Lemma (3.6), $u_n$ converges weakly to a non-zero solution $u$ of (1.2).

From condition (i) of Theorem (2.1), we have for every $1 \leq p \leq \infty$, $f(u) \in L^p(D)$ (see Moser [6]). Hence by regularity of elliptic operators, $u \in W^{2,p}(D)$ and hence by Sobolev imbedding $u$ is in $C^1(D)$ and hence in $C^2(D)$. This proves the result.

Remark 3.1. From the proof of Theorem (2.1) it follows that if $m > 0$ is satisfying (2.11), then from Lemma (3.2) $l_n \leq 2m^2$ and hence $l \leq 2m^2$. Therefore if $2m^2b < 1$ implies $lb < 1$. This proves the criterion (2.10).

Proof of Theorem 2.2. From Lemma (3.2) there exists $a_0 < 1$ such that $\Sigma_0$ is non-empty and $\{a_0\}$ is bounded for $a_0 < a < 1$. Lemma (3.3) gives (2.15).

Case 1. Let $f$ be super critical and $\lim_{a \to 1} a_n = a \neq 0$. Then from Lemma (3.4) we have

$$
\sup_{|w| \leq 1} \int_0^1 f(r, aw)r \, dr \leq a.
$$

contradicting the fact that $f$ is super critical. Hence $a = 0$.

Case 2. If $f$ is critical, let $a = \lim_{a \to 1} a_n$. Then from (3.30) it follows that

$$
\sup_{|w| \leq 1} \int f(r, aw)r \, dr \leq a
$$
and from Lemma (3.8),

\[
\frac{a^2}{2} b(0) < 1. \tag{3.82}
\]

Now choose an \( \varepsilon \) and \( \delta \) positive such that

(i) \( f(r, t) \leq M \exp [(b(0) + \varepsilon)t^2] \) for all \( (r, t) \in Q_s \).

(ii) \( \frac{a^2}{2} (b(0) + \varepsilon) < 1. \) \tag{3.83}

Such a choice is possible because of (3.82) and the condition that \( f \) is critical.

Since \( a^2 / 2 = I_{\alpha} \), and hence \( f \) satisfies (2.10) of Theorem (2.1) with \( b \) replaced by \( (b(0) + \varepsilon) \) and hence there exists a sequence \( u_n \) in \( \Sigma_{\alpha} \) and a weak solution \( u \) of (1.2) such that

(iii) \( I_{\alpha}(u_n) \to \frac{a^2}{2} \) as \( \alpha \to 1 \)

(iv) \( u_n \to u \) in \( H_0^1 \).

(v) \( \lim_{n \to \infty} \int_0^1 F(r, u_n)r^s dr = \int_0^1 F(r, u)r dr. \) \tag{3.84}

In fact (iii) follows from Lemma (3.6). From weak lower semicontinuity of the norm we have

\[
\| u \|_2^2 \leq \lim_{\alpha \to 1} \| u_n \|_{a\alpha}.
\]

and hence from (iii) we have

\[
I_1(u) \leq \lim_{\alpha \to 1} I_{\alpha}(u_n) = \frac{a^2}{2}. \tag{3.85}
\]

Let \( w \) be in \( B_{0\alpha} \). Choose \( \gamma_s \) such that

\[
\| w \|_2^2 = \frac{1}{\gamma_s} \int_0^1 f(r, \gamma_s w)wr^s dr.
\]

Such a \( \gamma_s \) exists and \( \lim_{\alpha \to 1} \gamma_\alpha = \gamma_1 \) exists and is \( \leq 1 \) because \( w \) is in \( B_{0\alpha} \) and \( \gamma_s w \) is in \( B_\alpha \). Hence

\[
\frac{a^2}{2} \leq I(\gamma_s w).
\]

Taking the \( \lim \) as \( \alpha \to 1 \), we get

\[
\frac{a^2}{2} \leq I_1(\gamma w) \leq I_1(w)
\]
Positive solutions of the semilinear Dirichlet

This implies

\[ \frac{a^2}{2} \leq \inf_{\mathcal{B}_{01}} I_1. \]  \hspace{1cm} (3.86)

From Lemma (3.10), (3.85) and (3.86) and using the fact that \( u \) is in \( \mathcal{B}_1^* \), we get

\[ I_1(u) = \frac{a^2}{2} = \inf_{\mathcal{B}_{01}} I_1 \]

and \( a \neq 0 \) because \( u \neq 0 \). This proves Theorem (2.2).

Remark 3.2. Suppose \( f(r,t) \leq 0 \) for \( r \in [0,1] \) and \( 0 \leq t \leq t_0 \) and satisfying all other hypothesis on \( f \), then also the Theorems (2.1) and (2.2) are valid.

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References

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