On the best constant of Hardy–Sobolev Inequalities

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Abstract

We obtain the sharp constant for the Hardy-Sobolev inequality involving the distance to the origin. This inequality is equivalent to a limiting Caffarelli–Kohn –Nirenberg inequality. In three dimensions, in certain cases the sharp constant coincides with the best Sobolev constant.

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1 Introduction

The standard Hardy inequality involving the distance to the origin, asserts that when $n \geq 3$ and $u \in C_0^\infty(I\!R^n)$ one has

$$\int_{I\!R^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{I\!R^n} \frac{u^2}{|x|^2} dx. \tag{1.1}$$

The constant $\left(\frac{n-2}{2}\right)^2$ is the best possible and remains the same if we replace $u \in C_0^\infty(I\!R^n)$ by $u \in C_0^\infty(B_1)$, where $B_1 \subset I\!R^n$ is the unit ball centered at zero. Brezis and Vázquez [BV] have improved it by establishing that for $u \in C_0^\infty(B_1)$,

$$\int_{B_1} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx + \mu_1 \int_{B_1} u^2 dx, \tag{1.2}$$

where $\mu_1 = 5.783...$ is the first eigenvalue of the Dirichlet Laplacian of the unit disc in $I\!R^2$. We note that μ_1 is the best constant in (1.2) independently of the dimension $n \geq 3$.

When taking distance to the boundary, the following Hardy inequality with best constant is also well known for $n \geq 2$ and $u \in C_0^\infty(B_1)$,

$$\int_{B_1} |\nabla u|^2 dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1-|x|)^2} dx. \tag{1.3}$$

Similarly to (1.2) this has also been improved by Brezis and Marcus in [BM] by proving that

$$\int_{B_1} |\nabla u|^2 dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1-|x|)^2} dx + b_n \int_{B_1} u^2 dx, \tag{1.4}$$

for some positive constant b_n. This time the best constant b_n depends on the space dimension with $b_n > \mu_1$ when $n \geq 4$, but in the $n = 3$ case, one has that $b_3 = \mu_1$, see [BFT].

On the other hand the classical Sobolev inequality

$$\int_{I\!R^n} |\nabla u|^2 dx \geq S_n \left(\int_{I\!R^n} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \tag{1.5}$$

is valid for any $u \in C_0^\infty(I\!R^n)$ where $S_n = \pi n(n-2) \left(\Gamma(\frac{n}{2})/\Gamma(n)\right)^{\frac{2}{n}}$ is the best constant, see [A], [T]. Maz'ya [M] combined both the Hardy and Sobolev term in one inequality valid in the upper half space. After a conformal transformation it leads to the following Hardy–Sobolev–Maz'ya inequality

$$\int_{B_1} |\nabla u|^2 dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1-|x|)^2} dx + B_n \left(\int_{B_1} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \tag{1.6}$$

valid for any $u \in C_0^\infty(B_1)$. Clearly $B_n \leq S_n$ and it was shown in [TT] that $B_n < S_n$ when $n \geq 4$. Again, the case $n = 3$ turns out to be special. Benguria Frank and Loss [BFL] have recently established that $B_3 = S_3 = 3(\pi/2)^{4/3}$ (see also Mancini and Sandeep [MS]).

When distance is taken from the origin the analogue of (1.6) has been established in [FT] by methods quite different to the ones we use in the present work. To state the result we first define

$$X_1(a, s) := (a - \ln s)^{-1}, \quad a > 0, \quad 0 < s \leq 1. \tag{1.7}$$

We then have:

$$\int_{B_1} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx + C_n(a) \left(\int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \tag{1.8}$$

We note that one cannot remove the logarithm X_1 in (1.8) and actually the exponent $\frac{2(n-1)}{n-2}$ is optimal. Our main concern in this note is to calculate the best constant $C_n(a)$ in (1.8). To this end we have:

Theorem A *Let $n \geq 3$. The best constant $C_n(a)$ in (1.8) satisfies*

$$C_n(a) = \begin{cases} (n-2)^{-\frac{2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\ a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. \end{cases}$$

When restricted to radial functions, the best constant in (1.8) is given by

$$C_{n,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n, \qquad \text{for all} \quad a \geq 0.$$

In all cases there is no $H_0^1(B_1)$ minimizer.

One easily checks that $C_n(a) < S_n$ when $n \geq 4$. Surprisingly, in the $n = 3$ case one has that $C_3(a) = S_3 = 3(\pi/2)^{4/3} = B_3$, for $a \geq 1$, that is, inequalities (1.5), (1.6) and (1.8) share the same best constant.

Using the change of variables $u(x) = |x|^{-\frac{n-2}{2}} v(x)$ inequality (1.8) is easily seen to be equivalent to

$$\int_{B_1} |x|^{-(n-2)} |\nabla v|^2 dx \geq C_n(a) \left(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a, |x|)\, |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \qquad v \in C_0^\infty(B_1). \tag{1.9}$$

For later use we denote by $W_0^{1,2}(B_1;\ |x|^{-(n-2)})$ the completion of $C_0^\infty(B_1)$ under the norm $\left(\int_{B_1} |x|^{-(n-2)} |\nabla v|^2 dx\right)^{1/2}$.

Estimate (1.9) is a limiting case of a Caffarelli–Kohn–Nirenberg inequality. Indeed, for any $-\frac{n-2}{2} < b < \infty$, the following inequality holds:

$$\int_{I\!R^n} |x|^{2b} |\nabla v|^2 dx \geq S(b,n) \left(\int_{I\!R^n} |x|^{\frac{2bn}{n-2}} |v|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}, \qquad v \in C_0^\infty(I\!R^n); \tag{1.10}$$

see [CKN], Catrina and Wang [CW]. Moreover, for $b = -\frac{n-2}{2}$ estimate (1.10) fails. Clearly, estimate (1.9) is the limiting case of (1.10) for $b = -\frac{n-2}{2}$. Thus we have:

Theorem A' *Let $n \geq 3$. The best constant $C_n(a)$ in the limiting Caffarelli–Kohn–Nirenberg inequality (1.9) is given*

$$C_n(a) = \begin{cases} (n-2)^{-\frac{2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\ a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. \end{cases}$$

When restricted to radial functions, the best constant in (1.9) is given by

$$C_{n,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n, \qquad \text{for all} \quad a \geq 0.$$

In all cases there is no $W_0^{1,2}(B_1;\ |x|^{-(n-2)})$ minimizer.

We note that the nonexistence of a $W_0^{1,2}(B_1;\ |x|^{-(n-2)})$ minimizer of Theorem A' is stronger than the nonexistence of an $H_0^1(B_1)$ minimizer of Theorem A. This is due to the fact that the existence of an $H_0^1(B_1)$ minimizer for (1.8) would imply existence of a $W_0^{1,2}(B_1;\ |x|^{-(n-2)})$ minimizer for (1.9), see Lemma 2.1 of [FT].

The above results can be easily transformed to the exterior of the unit ball B_1^c. For instance we have:

Corollary *Let $n \geq 3$. For any $u \in C_0^\infty(B_1^c)$, there holds*

$$\int_{B_1^c} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{B_1^c} \frac{u^2}{|x|^2} dx + C_n(a) \left(\int_{B_1^c} X_1^{\frac{2(n-1)}{n-2}} \left(a, \frac{1}{|x|}\right) |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \tag{1.11}$$

where the best constant $C_n(a)$ is the same as in Theorem A.

Our method can also cover the case of a general bounded domain Ω containing the origin. In particular we have

Theorem B *Let $n \geq 3$ and $\Omega \subset I\!R^n$ be a bounded domain containing the origin. Set $D := \sup_{x\in\Omega} |x|$. For any $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + C_n(a) \left(\int_\Omega X_1^{\frac{2(n-1)}{n-2}} \left(a, \frac{|x|}{D}\right) |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad (1.12)$$

where the best constant $C_n(a)$ is independent of Ω and is given by

$$C_n(a) = \begin{cases} (n-2)^{-\frac{2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\ a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. \end{cases}$$

It follows easily from Theorem A' that there no minimizers for (1.11) and (1.12) in the appropriate energetic function space.

We next consider the k–improved Hardy–Sobolev inequality derived in [FT]. Let k be a fixed positive integer. For X_1 as in (1.7) we define for $s \in (0,1)$,

$$X_{i+1}(a,s) = X_1(a, X_i(a,s)), \qquad i = 1, 2, \dots, k. \quad (1.13)$$

Noting that $X_i(a,s)$ is a decreasing function of a we easily check that there exist unique positive constants $0 < a_k < \beta_{n,k} \leq 1$ such that :
(i) The $X_i(a_k, s)$ are well defined for all $i = 1, 2 \dots, k+1$, and all $s \in (0,1)$ and $X_{k+1}(a_k, 1) = \infty$. In other words, a_k is the minimum value of the constant a so that the X_i's, $i = 1, 2 \dots, k+1$, are all well defined in $(0,1)$.
(ii) $X_1(\beta_{n,k}, 1) X_2(\beta_{n,k}, 1) \dots X_{k+1}(\beta_{n,k}, 1) = n-2$.

For $n \geq 3$, k a fixed positive integer and $u \in C_0^\infty(B_1)$ there holds:

$$\begin{aligned} \int_{B_1} |\nabla u|^2 dx \ \geq \ & \left(\frac{n-2}{2}\right)^2 \int_{B_1} \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^k \int_{B_1} \frac{X_1^2(a,|x|) \dots X_i^2(a,|x|)}{|x|^2} u^2 dx \\ & + C_{n,k}(a) \left(\int_{B_1} (X_1(a,|x|) \dots X_{k+1}(a,|x|))^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \quad (1.14) \end{aligned}$$

In our next result we calculate the best constant $C_{n,k}(a)$ in (1.14).

Theorem C *Let $n \geq 3$ and $k = 1, 2, \dots$ be a fixed positive integer. The best constant $C_{n,k}(a)$ in (1.14) satisfies:*

$$C_{n,k}(a) = \begin{cases} (n-2)^{-\frac{2(n-1)}{n}} S_n, & a \geq \beta_{n,k} \\ \left(\prod_{i=1}^{k+1} X_i(a,1)\right)^{-\frac{2(n-1)}{n}} S_n, & a_k < a < \beta_{n,k}. \end{cases}$$

When restricted to radial functions, the best constant of (1.14) is given by

$$C_{n,k,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n, \qquad \text{for all} \quad a > a_k.$$

Again we notice that $C_{n,k}(a) < S_n$ for $n \geq 4$ but $C_{3,k} = S_3$ for $a \geq \beta_{3,k}$.

As in Theorem A, one can establish by similar arguments the nonexistence of an $H_0^1(B_1)$ minimizer to (1.14), as well as the analogues of Theorem A', Corollary and Theorem B in the case of the k–improved Hardy–Sobolev inequality.

2 The proofs

Theorem A follows from Theorem A', we therefore prove Theorem A':
Proof of Theorem A': At first we will show that

$$C_n(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n, \qquad \text{when} \quad a \geq \frac{1}{n-2}. \tag{2.1}$$

We have that

$$C_n(a) = \inf_{v \in C_0^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)} |\nabla v|^2 dx}{\left(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)\, |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}}. \tag{2.2}$$

We change variables by $(r = |x|)$

$$v(x) = y(\tau, \theta), \qquad \tau = \frac{1}{X_1(a,r)} = a - \ln r, \quad \theta = \frac{x}{|x|}. \tag{2.3}$$

This change of variables maps the unit ball $B_1 = \{x : |x| < 1\}$ to the complement of the ball of radius a, that is, $B_a^c = \{(\tau, \theta) : a < \tau < +\infty, \quad \theta \in S^{n-1}\}$. Noticing that $X_1'(a,r) = \frac{X_1^2(a,r)}{r}$, $d\tau = -\frac{X_1'(a,r)}{X_1^2(a,r)} = -\frac{dr}{r}$, we also have

$$|\nabla v|^2 = \left(\frac{\partial v}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_\theta v|^2 = e^{2(\tau - a)} (y_\tau^2 + |\nabla_\theta y|^2).$$

A straightforward calculation shows that for $y \in C^\infty([a,\infty) \times S^{n-1})$ under Dirichlet boundary condition on $\tau = a$ we have

$$C_n(a) = \inf_{y(a,\theta)=0} \frac{\int_a^\infty \int_{S^{n-1}} (y_\tau^2 + |\nabla_\theta y|^2) dS d\tau}{\left(\int_a^\infty \int_{S^{n-1}} \tau^{-\frac{2(n-1)}{n-2}} |y|^{\frac{2n}{n-2}} dS d\tau \right)^{\frac{n-2}{n}}}. \tag{2.4}$$

In the sequel we will relate $C_n(a)$ with the best Sobolev constant S_n. It is well known that for any R with $0 < R \leq \infty$,

$$S_n = \inf_{u \in C_0^\infty(B_R)} \frac{\int_{B_R} |\nabla u|^2 dx}{\left(\int_{B_R} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}}. \tag{2.5}$$

We also know that $S_n = S_{n,radial}$ the latter being the infimum when taken over radial functions. Changing variables in (2.5) by

$$u(x) = z(t, \theta), \qquad t = |x|^{-(n-2)}, \qquad \theta = \frac{x}{|x|}, \tag{2.6}$$

it follows that for any $R \in (0, \infty]$,

$$(n-2)^{-\frac{2(n-1)}{n}} S_n = \inf_{z(R^{-(n-2)},\theta)=0} \frac{\int_{R^{-(n-2)}}^{\infty} \int_{S^{n-1}} (z_t^2 + \left(\frac{1}{n-2}\right)^2 \frac{1}{t^2} |\nabla_\theta z|^2) dS dt}{\left(\int_{R^{-(n-2)}}^{\infty} \int_{S^{n-1}} t^{-\frac{2(n-1)}{n-2}} |z|^{\frac{2n}{n-2}} dS dt\right)^{\frac{n-2}{n}}}. \quad (2.7)$$

We note that a function u is radial in x if and only if the function z is a function of t only. Looking at (2.4) and (2.7) we have that

$$C_n(a) \le C_{n,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_{n,radial} = (n-2)^{-\frac{2(n-1)}{n}} S_n. \quad (2.8)$$

On the other hand let us take $R = a^{-\frac{1}{n-2}}$ (so that $a = R^{-(n-2)}$) and assume that $a \ge \frac{1}{n-2}$. Then $\left(\frac{1}{n-2}\right)^2 \frac{1}{t^2} \le 1$ since $t \ge a \ge \frac{1}{n-2}$, and therefore

$$C_n(a) \ge \left(\frac{1}{n-2}\right)^{\frac{2(n-1)}{n}} S_n.$$

Combining this with (2.8) we conclude our claim (2.1).

Our next step is to prove the following: For any $a > 0$ we have that

$$C_n(a) \le a^{\frac{2(n-1)}{n}} S_n. \quad (2.9)$$

To this end let $0 \ne x_0 \in B_1$ and consider the sequence of functions

$$U_\varepsilon(x) = (\varepsilon + |x - x_0|^2)^{-\frac{n-2}{2}} \phi_\delta(|x - x_0|), \quad (2.10)$$

where $\phi_\delta(t)$ is a C_0^∞ cutoff function which is zero for $t > \delta$ and equal to one for $t < \delta/2$; δ is small enough so that $|x_0| + \delta < 1$ and therefore $U_\varepsilon \in C_0^\infty(B_\delta(x_0)) \subset C_0^\infty(B_1)$.

Then, it is well known, cf [BN], that

$$S_n = \lim_{\varepsilon \to 0} \frac{\int_{B_1} |\nabla U_\varepsilon|^2 dx}{\left(\int_{B_1} |U_\varepsilon|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}. \quad (2.11)$$

From (2.2) we have that for any $\varepsilon > 0$ small enough,

$$\begin{aligned} C_n(a) &= \inf_{v \in C_0^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)} |\nabla v|^2 dx}{\left(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |v|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}} \\ &\le \frac{\int_{B_\delta(x_0)} |x|^{-(n-2)} |\nabla U_\varepsilon|^2 dx}{\left(\int_{B_\delta(x_0)} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |U_\varepsilon|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}} \\ &\le \left(\frac{|x_0| + \delta}{|x_0| - \delta}\right)^{n-2} \frac{1}{X_1^{\frac{2(n-1)}{n}}(a, |x_0| - \delta)} \frac{\int_{B_\delta(x_0)} |\nabla U_\varepsilon|^2 dx}{\left(\int_{B_\delta(x_0)} |U_\varepsilon|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}, \end{aligned}$$

where we used the fact that $X_1(a,s)$ is an increasing function of s. Taking the limit $\varepsilon \to 0$ we conclude:

$$C_n(a) \le \left(\frac{|x_0|+\delta}{|x_0|-\delta}\right)^{n-2} \frac{S_n}{X_1^{\frac{2(n-1)}{n}}(a,|x_0|-\delta)}.$$

This is true for any $\delta > 0$ small enough, therefore

$$C_n(a) \le X_1^{-\frac{2(n-1)}{n}}(a,|x_0|)\, S_n.$$

Since $|x_0| < 1$ is arbitrary and $X_1(a,s)$ is an increasing function of s, we end up with

$$C_n(a) \le X_1^{-\frac{2(n-1)}{n}}(a,1)\, S_n = a^{\frac{2(n-1)}{n}}\, S_n, \tag{2.12}$$

and this proves our claim (2.9).

To complete the calculation of $C_n(a)$ we will finally show that

$$C_n(a) \ge a^{\frac{2(n-1)}{n}}\, S_n, \qquad \text{when} \quad 0 < a < \frac{1}{n-2}. \tag{2.13}$$

To prove this we will relate the infimum $C_n(a)$ to a Caffarelli–Kohn–Nirenberg inequality. We will need the following result:

Proposition 2.1 *Let $b > 0$ and*

$$S_n(b) := \inf_{v \in C_0^\infty(I\!\!R^n)} \frac{\int_{I\!\!R^n} |x|^{2b} |\nabla u|^2 dx}{\left(\int_{I\!\!R^n} |x|^{\frac{2bn}{n-2}} |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}. \tag{2.14}$$

Then $S_n(b) = S_n$ and this constant is not achieved in the appropriate function space.

This is proved in Theorem 1.1 of [CW].

We change variables in (2.14) by

$$u(x) = z(t,\theta), \qquad t = |x|^{-(n-2)-2b}, \qquad \theta = \frac{x}{|x|}. \tag{2.15}$$

A straightforward calculation shows that for any R',

$$(n-2+2b)^{-\frac{2(n-1)}{n}}\, S_n \le \inf_{z(R',\theta)=0} \frac{\int_{R'}^\infty \int_{S^{n-1}} \left(z_t^2 + \frac{1}{(n-2+2b)^2 t^2} |\nabla_\theta z|^2\right) dSdt}{\left(\int_{R'}^\infty \int_{S^{n-1}} t^{-\frac{2(n-1)}{n-2}} |z|^{\frac{2n}{n-2}} dSdt\right)^{\frac{n-2}{n}}}. \tag{2.16}$$

Taking $R' = a$ and comparing (2.16) with (2.4) we have that if

$$1 \ge \frac{1}{(n-2+2b)^2\, t^2}, \qquad \text{for} \quad t \ge a, \tag{2.17}$$

then

$$C_n(a) \ge (n-2+2b)^{-\frac{2(n-1)}{n}}\, S_n. \tag{2.18}$$

Condition (2.17) is satisfied if we choose $b \in (0, +\infty)$ such that

$$\frac{1}{n-2} > a = (n - 2 + 2b)^{-1} > 0. \tag{2.19}$$

For such a b it follows from (2.18) that

$$C_n(a) \geq a^{\frac{2(n-1)}{n}} S_n,$$

and this proves our claim (2.13).

We finally establish the nonexistence of an energetic minimizer. We will argue by contradiction. Suppose that $\bar{v} \in W_0^{1,2}(B_1;\ |x|^{-(n-2)})$ is a minimizer of (2.2). Through the change of variables (2.3), the quotient in (2.4) admits also a minimizer $\bar{y}$.

Consider first the case when $a \geq \frac{1}{n-2}$. Comparing (2.4) and (2.7) with $R = a^{-\frac{1}{n-2}}$, we conclude that $\bar{y}$ is a radial minimizer of (2.7) as well. It then follows that (2.5) admits a radial $H_0^1(B_R)$ minimizer $\bar{u}(r) = \bar{y}(t)$, $t = r^{-(n-2)}$, which contradicts the fact that the Sobolev inequality (2.5) has no H_0^1 minimizers.

In the case when $0 < a < \frac{1}{n-2}$, we use a similar argument comparing (2.4) and (2.16) to conclude the existence of a radial minimizer to (2.16) with b as in (2.19). This contradicts the nonexistence of minimizer for (2.14). The proof of Theorem A' is now complete.

Proof of Corollary: One can argue in a similar way as in the previous proof, or apply Kelvin transform to the inequality of Theorem A.

Proof of Theorem B: The lower bound on the best constant follows from Theorem A, the fact that if $u \in C_0^\infty(\Omega)$ then $u \in C_0^\infty(B_D)$ (since $\Omega \subset B_D$) and a simple scaling argument.

To establish the upper bound in the case where $0 < a < \frac{1}{n-2}$ we argue exactly as in the proof of (2.9) using the test functions (2.10) that concentrate near a point of the boundary of Ω, that realizes the $\max_{x\in\Omega} |x|$. Let us now consider the case where $a \geq \frac{1}{n-2}$. For $a > 0$ and $0 < \rho < 1$, we set

$$\tilde{C}_n(a, \rho) := \inf_{u \in C_0^\infty(B_\rho)} \frac{\int_{B_\rho} |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_{B_\rho} \frac{u^2}{|x|^2} dx}{\left(\int_{B_\rho} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}.$$

A simple scaling argument and Theorem A shows that:

$$\tilde{C}_n(a, \rho) = C_n(a - \ln \rho).$$

Thus, for ρ small enough we have that

$$\tilde{C}_n(a, \rho) = (n-2)^{-\frac{2(n-1)}{n}} S_n.$$

Since for ρ small, $B_\rho \subset \Omega$ the upper bound follows easily in this case as well.

Proof of Theorem C: To simplify the presentation we will write $X_i(|x|)$ instead of $X_i(a, |x|)$. Let k be a fixed positive integer. We first consider the case $a \geq \beta_{k,n}$. We change variables in (1.14) by

$$u(x) = |x|^{-\frac{n-2}{2}} X_1^{-1/2}(|x|) X_2^{-1/2}(|x|) \dots X_k^{-1/2}(|x|) v(x),$$

to obtain

$$\int_{B_1} |x|^{-(n-2)} X_1^{-1}(|x|)\dots X_k^{-1}(|x|)|\nabla v|^2 dx \geq$$

$$C_{n,k}(a)\left(\int_{B_1} |x|^{-n} X_1(|x|)\dots X_k(|x|) X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}, \qquad v\in C_0^\infty(B_1). \quad (2.20)$$

We further change variables by

$$v(x) = y(\tau,\theta), \qquad \tau = \frac{1}{X_{k+1}(r)}, \qquad \theta = \frac{x}{|x|} \qquad (r=|x|).$$

This change of variables maps the unit ball $B_1 = \{x : |x| < 1\}$ to the complement of the ball of radius $r_a := X_{k+1}^{-1}(1)$, that is, $B_{r_a}^c = \{(\tau,\theta) : X_{k+1}^{-1}(1) < \tau < +\infty, \quad \theta \in S^{n-1}\}$. Note that

$$d\tau = -\frac{X'_{k+1}(r)}{X_{k+1}^2(r)} dr = -\frac{X_1(r)\dots X_k(r)}{r} dr.$$

Let us denote by $f_1(t)$ the inverse function of $X_1(t)$. We also set $f_{i+1}(t) = f_1(f_i(t))$, $i = 1,2,\dots,k$. Consequently, $r = f_{k+1}(\tau^{-1})$. Also, $X_1(r) = f_k(\tau^{-1})$, $X_2(r) = f_{k-1}(\tau^{-1})$, $\dots$ $X_k(r) = f_1(\tau^{-1})$.

We then find

$$C_{n,k}(a) = \inf_{y(r_a,\theta)=0} \frac{\int_{r_a}^\infty \int_{S^{n-1}} (y_\tau^2 + \left(f_1(\tau^{-1})\dots f_k(\tau^{-1})\right)^{-2} |\nabla_\theta y|^2) dS d\tau}{\left(\int_{r_a}^\infty \int_{S^{n-1}} \tau^{-\frac{2(n-1)}{n-2}} |y|^{\frac{2n}{n-2}} dS d\tau\right)^{\frac{n-2}{n}}}. \quad (2.21)$$

Again, we will relate this with the best Sobolev constant S_n. From (2.7) we have that

$$(n-2)^{-\frac{2(n-1)}{n}} S_n = \inf_{z(r_a,\theta)=0} \frac{\int_{r_a}^\infty \int_{S^{n-1}} (z_t^2 + \frac{1}{(n-2)^2 t^2} |\nabla_\theta z|^2) dS dt}{\left(\int_{r_a}^\infty \int_{S^{n-1}} t^{-\frac{2(n-1)}{n-2}} |z|^{\frac{2n}{n-2}} dS dt\right)^{\frac{n-2}{n}}}. \quad (2.22)$$

Comparing this with (2.21) we have that

$$C_{n,k}(a) \leq C_{n,k,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_{n,radial} = (n-2)^{-\frac{2(n-1)}{n}} S_n. \quad (2.23)$$

On the other hand for $a \geq \beta_{k,n}$ and $\tau \geq r_a$ we have that

$$\begin{aligned}\left(\tau^{-1} f_1(\tau^{-1})\dots f_k(\tau^{-1})\right)^{-2} &\geq \left(r_a^{-1} f_1(r_a^{-1})\dots f_k(r_a^{-1})\right)^{-2} \\ &= (X_1(a,1)\dots X_k(a,1) X_{k+1}(a,1))^{-2} \\ &\geq \frac{1}{(n-2)^2},\end{aligned}$$

therefore

$$\left(f_1(\tau^{-1})\dots f_k(\tau^{-1})\right)^{-2} \geq \frac{1}{(n-2)^2 \tau^2}, \qquad \tau \geq r_a,$$

and consequently,

$$C_{n,k}(a) \geq (n-2)^{-\frac{2(n-1)}{n}} S_n.$$

From this and (2.23) it follows that

$$C_{n,k}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n, \qquad \text{when} \quad a \geq \beta_{k,n}.$$

The case where $a_k < a < \beta_{k,n}$ is quite similar to the case $0 < a < \frac{1}{n-2}$ in the proof of Theorem A'. That is, testing in (2.20) the sequence U_ε as defined in (2.10), we first prove that

$$C_{n,k}(a) \leq \left(\prod_{i=1}^{k+1} X_i(a,1) \right)^{\frac{-2(n-1)}{n}} S_n,$$

by an argument quite similar to the one leading to (2.12). Finally, in the case $a_k < a < \beta_{k,n}$, we obtain the opposite inequality by comparing the infimum in (2.21) with the infimum in (2.16). This time we take $R' = r_a$ and $b > 0$ is chosen so that

$$\prod_{i=1}^{k+1} X_i(a,1) = n - 2 + 2b.$$

We omit further details.

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