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Non-existence of nodal solution for *m*-Laplace equation involving critical Sobolev exponents

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Abstract. In this paper we study the non-existence of nodal solutions for critical Sobolev exponent problem

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{m-2}\nabla u) &= |u|^{p-1}u + |u|^{q-1}u \text{ in } B(R) \\ u &= 0 \quad \text{on } \partial B(R) \end{aligned}$$

where $B(R)$ is a ball of radius R in $\mathbb{R}^n$.

Keywords. Critical exponent; eigenvalue; *m*-Laplacian.

1. Introduction

Consider the problem

$$\left.\begin{aligned} -\Delta_m u &= |u|^{p-1}u + |u|^{q-1}u \text{ in } B(R) \\ u &= 0 \qquad \text{on } \partial B(R), \end{aligned}\right\} \tag{1.1}$$

where $B(R)$ is a ball in $\mathbb{R}^n$ of radius R, $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$ and $1 < m < n$, $p + 1 = mn/(n-m)$ is the critical Sobolev exponent for the non-compact imbedding $H_0^{1,m} \to L^{p+1}$ and $1 \leqslant q \leqslant p-1$. In this paper we are interested in the radial solutions of (1.1) which change sign.

For $m = 2$, this problem has been discussed by many authors. It has been shown by Cerami *et al* [7] and Solomini [13] that (1.1) admits infinitely many radial solutions which change sign for $q = 1$ and $n \geqslant 7$. Atkinson *et al* [4, 5] and Adimurthi and Yadava [1] have proved that this result of infinitely many nodal solutions is optimal in the sense, when $3 \leqslant n \leqslant 6$, $q = 1$, then (1.1) does not admit any radial solution which changes sign for all R sufficiently small. For $p - 1 < q < p$, Jones [11] and Atkinsion-Peletier [3] have proved that (1.1) admits infinitely many radial solutions which change sign. It has been shown by Jones [11] for $1 < q < p-1$ and by Knaap [12] for $q = p-1$ that (1.1) does not admit any radial solution which changes sign provided R is sufficiently small. Atkinson *et al* [4, 5] have used asymptotic analysis to prove their non-existence result and Jones [11] has adapted dynamical system approach. In [1], the non-existence result has been obtained by Pohozaev's identity.

In this paper, following the method used in [1], we extend the non-existence result for the general m, $1 < m < n$. We prove

Theorem. *Let $m-1 \leqslant q \leqslant p-1$. Then there exists a $R_0 > 0$ such that for all $0 < R < R_0$,*

$$\left.\begin{aligned} -\Delta_m u &= |u|^{p-1}u + |u|^{q-1}u \quad \text{in } B(R) \\ u &= 0 \quad \text{on } \partial B(R) \end{aligned}\right\} \tag{1.2}$$

does not admit any radial solution which changes sign.

Remark 1. If $m = 2$, the above theorem gives all the above mentioned known results of the non-existence of solution which changes sign.

Remark 2. For $0 < q < m-1$, it has been shown in [9] that for sufficiently small R, (1.2) admits infinitely many solutions.

Remark 3. If $q = m-1$, then the above theorem is true for the range $m < n \leqslant m^2 + m$. For $m < n < m^2$, Atkinson *et al* [6] have proved a more stronger result, namely (1.2) does not admit any positive radial solution for R sufficiently small.

2. Proof of the theorem

Since we are looking for radial solution, we can set $u = u(r)$, $r = |x|$ and write (1.2) as

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}|u'|^{m-2}u') &= r^{n-1}(|u|^{p-1} + |u|^{q-1})u \quad \text{in } (0, R) \\ u'(0) &= u(R) = 0. \end{aligned}\right\} \tag{2.1}$$

To study the problem (2.1), we can consider the associated initial value problem

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}|v'|^{m-2}v') &= r^{n-1}(|v|^{p-1} + |v|^{q-1})v \quad \text{in } (0, \infty) \\ v'(0) &= 0, \quad v(0) = \gamma. \end{aligned}\right\} \tag{2.2}$$

Let $v(r, \gamma)$ be the unique solution of (2.2). Let $0 < R_1(\gamma) < R_2(\gamma) < \ldots$ be the zeros of $v(r, \gamma)$. In order to prove the Theorem, it is enough to show that there exists a $C_0 > 0$ such that

$$R_2(\gamma) \geqslant C_0 \tag{2.3}$$

for all $\gamma \in (0, \infty)$. To prove (2.3) we need the following.

Lemma. We have

$$\sup_{\gamma \in (0,\infty)} \{|v(r, \gamma)|; R_1(\gamma) \leqslant r \leqslant R_2(\gamma)\} \leqslant k_0 \tag{2.4}$$

where

$$k_0 = \left\{\begin{aligned} &\frac{1}{p} && \text{if } q = p-1 \\ &\frac{(p-q-1)^{(p-q-1)/(p-q)}}{(q+1)} && \text{if } q < p-1. \end{aligned}\right. \tag{2.5}$$

Proof. Suppose (2.4) is not true. Then there exists as $\gamma > 0$ and a $k > k_0$ such that $|v(r,\gamma)| = k$ has a solution in $[R_1(\gamma), R_2(\gamma)]$. Let $R > R_1(\gamma)$ be the first point at which $v(R,\gamma) = -k$. Let $w(r) = v(r,\gamma) + k$. Then w satisfies

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}|w'|^{m-2}w') = r^{n-1}f(w) \quad \text{in } (0,R), \\ w > 0 \\ w'(0) = w(R) = 0. \end{aligned}\right\} \tag{2.6}$$

where $f(w) = (|w-k|^{p-1} + |w-k|^{q-1})(w-k)$. Let F denote the primitive of f. Then by Pohozaev's identity [8] and [10], we have

$$\begin{aligned} 0 &\leqslant \int_0^R \left\{\left(\frac{mn}{n-m}\right)F(w) - f(w)w\right\} r^{n-1}\,\mathrm{d}r \\ &= \int_0^R g(w-k)r^{n-1}\,\mathrm{d}r - \left\{k^{p+1} + \left(\frac{p+1}{q+1}\right)k^{q+1}\right\}\frac{R^n}{n}, \end{aligned} \tag{2.7}$$

where

$$g(s) = \left(\frac{p-q}{q+1}\right)|s|^{q+1} - k|s|^{p-1}s - k|s|^{q-1}s.$$

Now observe that for $-k \leqslant s \leqslant 0$, $g(s)$ is decreasing and non-negative. Therefore

$$g(s) \leqslant g(-k) = k^{p+1} + \left(\frac{p+1}{q+1}\right)k^{q+1} \tag{2.8}$$

for all $s \in [-k, 0]$.

For $s > 0$ we have

Claim. $g(s) < 0$ for all $s > 0$.

For $s > 0$, let $h(s) = g(s)/s^q$. Then

$$h(s) = \left(\frac{p-q}{q+1}\right)s - ks^{p-q} - k.$$

Case 1. Let $q = p - 1$. Then

$$h(s) = -\left(k - \frac{1}{p}\right)s - k.$$

Since $k > 1/p$, we get $h(s) < 0$.

Case 2. Let $q < p - 1$. Then h has a maximum at

$$s_0 = \left(\frac{1}{k(q+1)}\right)^{1/(p-q-1)}$$

Since $k > k_0$, we get

$$h(s_0) = \frac{p-q-1}{(q+1)^{p-q/(p-q-1)}} \frac{1}{k^{1/(p-q-1)}} - k < 0$$

and this proves the claim.

Now from (2.7), (2.8) and Claim, we have

$$\begin{aligned} 0 &\leqslant \int_{0\leqslant w\leqslant k} g(w-k)r^{n-1}\,\mathrm{d}r + \int_{w>k} g(w-k)r^{n-1}\,\mathrm{d}r \\ &\quad - \left\{k^{p+1} + \left(\frac{p+1}{q+1}\right)k^{q+1}\right\}\frac{R^n}{n} \\ &< g(-k)\frac{R^n}{n} - \left\{k^{p+1} + \left(\frac{p+1}{q+1}\right)k^{q+1}\right)\frac{R^n}{n} \\ &= 0 \end{aligned}$$

which is a contradiction. This proves the lemma.

Before going into the proof of (2.3), we recollect some known results about the first eigenvalue for Δ_m (see [2]).

Let Ω be a bounded domain with $C^{2,\beta}$ boundary and let $\alpha \in L^\infty(\Omega)$ be such that meas $\{x \in \Omega; \alpha(x) > 0\} \neq 0$. Then there exists a unique $\lambda(\alpha, \Omega) > 0$ such that

$$\begin{aligned} -\Delta_m \phi &= \lambda(\alpha, \Omega)\alpha|\phi|^{m-2}\phi \quad \text{in } \Omega \\ \phi &> 0 \\ \phi &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.9}$$

admits a unique (up to multiplication by a constant) solution.

Obviously, if $0 \leqslant \alpha_1 \leqslant \alpha_2$ and $\alpha_i \in L^\infty(\Omega)$, then

$$\lambda_1(\alpha_1, \Omega) \geqslant \lambda_1(\alpha_2, \Omega). \tag{2.10}$$

Moreover,

$$\lambda_1(1, \Omega) \to \infty \text{ as meas } (\Omega) \to 0. \tag{2.11}$$

Proof of (2.3). We claim that there exists a $\delta > 0$ such that

$$R_2(\gamma) - R_1(\gamma) \geqslant \delta \tag{2.12}$$

for all $\gamma \in (0, \infty)$.

Since $m - 1 \leqslant q$, by the lemma there exists a $C > 1$ such that

$$\sup_{\gamma \in (0,\infty)} \{|v|^{p-m+1} + |v|^{q-m+1}; R_1(\gamma) \leqslant r \leqslant R_2(\gamma)\} < C. \tag{2.13}$$

Now suppose (2.12) is not true. Choose a $\gamma_0 > 0$ such that

$$\lambda_1(C, B(R_1(\gamma_0), R_2(\gamma_0))) \geqslant 2, \tag{2.14}$$

where $B(R_1(\gamma_0), R_2(\gamma_0)) = \{x \in \mathbb{R}^n; R_1(\gamma_0) \leqslant |x| \leqslant R_2(\gamma_0)\}$. On the other hand, from (2.2),

$$\lambda_1(|v|^{p-m+1} + |v|^{q-m+1}, B(R_1(\gamma_0), R_2(\gamma_0)) = 1. \tag{2.15}$$

This, together with (2.13) and (2.10), contradicts (2.14). This completes the proof of the Theorem.

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