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A probabilistic approach to second order variational inequalities with bilateral constraints

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Abstract. We study a class of second order variational inequalities with bilateral constraints. Under certain conditions we show the existence of a unique viscosity solution of these variational inequalities and give a stochastic representation to this solution. As an application, we study a stochastic game with stopping times and show the existence of a saddle point equilibrium.

Keywords. Variational inequalities; viscosity solution; stochastic game; stopping time; value; saddle point equilibrium.

1. Introduction and preliminaries

We study a class of second order nonlinear variational inequalities with bilateral constraints. This type of inequalities arises in zero sum stochastic differential games of mixed type where each player uses both continuous control and stopping times. Under a non-degeneracy assumption Bensoussan and Friedman [1, 4] have studied this type of problems. They proved the existence of a unique solution of these variational inequalities in certain weighted Sobolev spaces. This result together with certain techniques from stochastic calculus is then applied to show that the unique solution of these inequalities is the value function of certain stochastic differential games of mixed type. In this paper we study the same class of variational inequalities without the non-degeneracy assumption. The non-degeneracy assumption is crucially used in the analysis of the problem in [1, 4]. Thus the method used in [1, 4] does not apply to the degenerate case. We study the problem via the theory of viscosity solutions. We transform the variational inequalities with bilateral constraints to Hamilton–Jacobi–Isaacs (HJI for short) equations associated with a stochastic differential game problem with continuous control only. Then using standard results from the theory of viscosity solutions, we show that the value function of this stochastic differential game with continuous control is the unique viscosity solution of the corresponding variational inequalities. Then for a special case we identify this unique viscosity solution as the value function of the stochastic game with stopping times. We now describe our problem.

Let $U_i, i = 1, 2$, be the compact metric spaces. Let

$$b : \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}^d$$

and

$$a : \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}^{d\times d}.$$

We assume that:

(A1)

The functions b and a are bounded and continuous, $a(x, u_1, u_2)$ is C^2 in x uniformly with respect to u_1, u_2. The matrix a is symmetric and non-negative definite. Further there exists constant $C_1 > 0$ such that for all $u_i \in U_i$, $i = 1, 2$,

$$|b(x, u_1, u_2) - b(y, u_1, u_2)| \le C_1|x - y|.$$

Let

$$r : \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}$$

and

$$\psi_i : \mathbb{R}^d \to \mathbb{R}, \ i = 1, 2.$$

We assume that

(A2)

(i) r, ψ_1, ψ_2 are bounded and continuous.
(ii) There exists a constant $C_2 > 0$ such that for all $x, y \in \mathbb{R}^d$, $(u_1, u_2) \in U_1 \times U_2$,

$$\begin{aligned}&|r(x, u_1, u_2) - r(y, u_1, u_2)| + |\psi_1(x) - \psi_1(y)| + |\psi_2(x) - \psi_2(y)| \\ &\quad \le C_2|x - y|.\end{aligned}$$

(iii) $\psi_2 \le \psi_1$.

Let $H^+, H^- : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ be defined by

$$\begin{aligned}H^+(x, p, X) = \inf_{u_1\in U_1} \sup_{u_2\in U_2} \Big[&\frac{1}{2}\mathrm{tr}(a(x, u_1, u_2)X) \\ &+ b(x, u_1, u_2) \cdot p + r(x, u_1, u_2)\Big], \\ H^-(x, p, X) = \sup_{u_2\in U_2} \inf_{u_1\in U_1} \Big[&\frac{1}{2}\mathrm{tr}(a(x, u_1, u_2)X) \\ &+ b(x, u_1, u_2) \cdot p + r(x, u_1, u_2)\Big].\end{aligned}$$

Consider the following Hamilton–Jacobi–Isaacs variational inequalities with bilateral constraints

$$\left.\begin{array}{ll}\psi_2(x) \le v(x) \le \psi_1(x), & \forall\, x \\ \lambda v(x) - H^+(x, Dv(x), D^2v(x)) = 0, & \text{if } \psi_2(x) < v(x) < \psi_1(x) \\ \lambda v(x) - H^+(x, Dv(x), D^2v(x)) \ge 0, & \text{if } v(x) = \psi_2(x) \\ \lambda v(x) - H^+(x, Dv(x), D^2v(x)) \le 0, & \text{if } v(x) = \psi_1(x)\end{array}\right\} \quad (1.1)$$

and

$$\left.\begin{array}{ll} \psi_2(x) \le v(x) \le \psi_1(x), & \forall\, x \\ \lambda v(x) - H^-(x, Dv(x), D^2v(x)) = 0, & \text{if } \psi_2(x) < v(x) < \psi_1(x) \\ \lambda v(x) - H^-(x, Dv(x), D^2v(x)) \ge 0, & \text{if } v(x) = \psi_2(x) \\ \lambda v(x) - H^-(x, Dv(x), D^2v(x)) \le 0, & \text{if } v(x) = \psi_1(x) \end{array}\right\}. \quad (1.2)$$

By a classical solution of (1.1), we mean a C^2-function v satisfying (1.1). Similarly a classical solution of (1.2) is defined.

The rest of our paper is structured as follows. In §2, we introduce the notion of viscosity solution and establish the existence of unique viscosity solutions of these variational inequalities by a probabilistic method. In §3, we apply these variational inequalities to treat a stochastic game with stopping times. We establish the existence of a saddle point equilibrium for this problem. Section 5 contains some concluding remarks.

2. Viscosity solutions

To motivate the definition of viscosity solutions of the variational inequalities we first prove the following result.

Theorem 2.1. *Assume* (A2)(iii). *A function $v \in C^2(\mathbb{R}^d)$ is a classical solution of* (1.1) *if and only if it is a classical solution of the equation*

$$\begin{aligned} &\max\{\min\{\lambda v(x) - H^+(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_2(x))\}; \\ &\qquad \lambda(v(x) - \psi_1(x))\} = 0. \end{aligned} \quad (2.1)$$

Similarly a function $v \in C^2(\mathbb{R}^d)$ is a classical solution of (1.2) *if and only if it is a classical solution of the equation*

$$\begin{aligned} &\min\{\max\{\lambda v(x) - H^-(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_1(x))\}; \\ &\qquad \lambda(v(x) - \psi_2(x))\} = 0. \end{aligned} \quad (2.2)$$

Proof. Let v be a classical solution of (1.1). Suppose x is such that $\psi_2(x) < v(x) < \psi_1(x)$. Then

$$\begin{aligned} &\lambda v(x) - H^+(x, Dv(x), D^2v(x)) = 0, \quad v(x) - \psi_2(x) > 0, \\ &\qquad v(x) - \psi_1(x) < 0. \end{aligned}$$

Thus (2.1) clearly holds in this case. Now if $v(x) = \psi_2(x)$, then

$$\min\{\lambda v(x) - H^+(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_2(x))\} = 0,$$

and hence (2.1) is satisfied. Finally assume $v(x) = \psi_1(x)$, then

$$\min\{\lambda v(x) - H^+(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_2(x))\} \le 0,$$

and hence

$$\begin{aligned} &\max\{\min\{\lambda v(x) - H^+(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_2(x))\}; \\ &\qquad \lambda(v(x) - \psi_1(x))\} = 0. \end{aligned}$$

Thus v satisfies (2.1). We now show the converse. It is clear from (2.1) that $v(x) \leq \psi_1(x)$. If $v(x) = \psi_1(x)$ for some x, then it clearly satisfies $v(x) \geq \psi_2(x)$ by (A2)(iii). Now let $v(x) < \psi_1(x)$. Then from (2.1), we have

$$\min\{\lambda v(x) - H^+(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_2(x))\} = 0$$

and hence $v(x) - \psi_2(x) \geq 0$. Thus for all x, we have $\psi_2(x) \leq v(x) \leq \psi_1(x)$. Now let $v(x) < \psi_1(x)$. Then from the above equation, we have

$$\lambda v(x) - H^+(x, Dv(x), D^2v(x)) \geq 0.$$

Similarly if $v(x) > \psi_2(x)$, we can show that

$$\lambda v(x) - H^+(x, Dv(x), D^2v(x)) \leq 0.$$

Thus v is a classical solution of (1.1). This concludes the proof of the first part. The second part of the theorem can be proved in a similar way. ■

Remark 2.2. Under (A2)(iii), we can also show that a function $v \in C^2(\mathbb{R}^d)$ is a classical solution of (1.1) if and only if it is a classical solution of the equation

$$\begin{aligned}\min\{&\max\{\lambda v(x) - H^+(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_1(x))\};\\ &\lambda(v(x) - \psi_2(x))\} = 0.\end{aligned}$$

Similarly a function $v \in C^2(\mathbb{R}^d)$ is a classical solution of (1.2) if and only if it is a classical solution of the equation

$$\begin{aligned}\max\{&\min\{\lambda v(x) - H^-(x, Dv(x), D^2v(x)); \lambda(v(x) - \psi_2(x))\};\\ &\lambda(v(x) - \psi_1(x))\} = 0.\end{aligned}$$

Theorem 2.1 motivates us to define viscosity solutions for (1.1) and (1.2) using (2.1) and (2.2) repsectively.

DEFINITION 2.3

An upper semicontinuous function $v : \mathbb{R}^d \to \mathbb{R}$ is said to be a viscosity subsolution of (1.1) if it is a viscosity subsolution of (2.1). Similarly a lower semicontinuous function $v : \mathbb{R}^d \to \mathbb{R}$ is said to be a viscosity supersolution of (1.1) if it is a viscosity supersolution of (2.1). A function which is both sub- and super-solution of (1.1) is called a viscosity solution of (1.1). Similarly, viscosity sub- and super-solutions of (1.2) are defined.

We now address the question of showing the existence of unique viscosity solutions of (1.1) and (1.2). This is done by showing that (1.1) and (1.2) are equivalent to Hamilton–Jacobi–Isaacs equations corresponding to a stochastic differential game.

Let ω_1, ω_2 be two symbols. We formulate a zero sum stochastic differential. In this game, $\bar{U}_i$ is the set of controls for player i, where $\bar{U}_i = U_i \cup \{\omega_i\}$, $i = 1, 2$. Let $\sigma(\cdot, \cdot, \cdot)$ be the non-negative square root of $a(\cdot, \cdot, \cdot)$. Extend b, σ, r to

$$\begin{aligned}&\bar{b} : \mathbb{R}^d \times \bar{U}_1 \times \bar{U}_2 \to \mathbb{R}^d, \quad \bar{\sigma} : \mathbb{R}^d \times \bar{U}_1 \times \bar{U}_2 \to \mathbb{R}^{d\times d},\\ &\bar{r} : \mathbb{R}^d \times \bar{U}_1 \times \bar{U}_2 \to \mathbb{R},\end{aligned}$$

respectively such that

$$\bar{b}(x, \omega_1, \cdot) \equiv 0, \quad \bar{b}(x, \cdot, \omega_2) \equiv 0, \quad \bar{\sigma}(x, \omega_1, \cdot) \equiv 0, \quad \bar{\sigma}(x, \cdot, \omega_2) \equiv 0,$$
$$\bar{r}(x, \omega_1, u_2) = \lambda\psi_1(x) \text{ for all } u_2 \in U_2 \text{ and } \bar{r}(x, \cdot, \omega_2) \equiv \lambda\psi_2(x).$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $W(t)$ a standard d-dimensional Brownian motion on it. Let $\bar{\mathcal{A}}_i$ denote the set of all $\bar{U}_i$-valued functions progressively measurable with respect to the process $W(t)$. For $(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \in \bar{\mathcal{A}}_1 \times \bar{\mathcal{A}}_2$, consider the controlled stochastic differential equation

$$\left.\begin{aligned} \mathrm{d}\bar{X}(t) &= \bar{b}(\bar{X}(t), \bar{u}_1(t), \bar{u}_2(t))\ \mathrm{d}t + \bar{\sigma}(\bar{X}(t), u_1(t), u_2(t))\ \mathrm{d}W(t) \\ \bar{X}(0) &= x \end{aligned}\right\}. \tag{2.3}$$

Let the payoff function be defined by

$$\bar{R}(x, \bar{u}_1(\cdot), \bar{u}_2(\cdot)) = E\left[\int_0^\infty \mathrm{e}^{-\lambda t}\bar{r}(\bar{X}(t), \bar{u}_1(t), \bar{u}_2(t))\ \mathrm{d}t\right].$$

A strategy for the player 1 is a 'non-antiticipating' map $\alpha : \bar{\mathcal{A}}_2 \to \bar{\mathcal{A}}_1$, i.e., for any $u_2, \bar{u}_2 \in \mathcal{A}_2$ such that $u_2(s) = \bar{u}_2(s)$ for all $0 \le s \le t$ then we have $\alpha[u_2](s) = \alpha[\bar{u}_2](s)$, $0 \le s \le t$. Let $\bar{\Gamma}$ denote the set of all non-anticipating strategies for player 1. Similarly strategies for player 2 are defined. Let the set of all non-anticipating strategies for player 2 be denoted by $\bar{\Delta}$. Then the upper and lower value functions are defined by

$$\bar{V}^+(x) = \sup_{\beta\in\bar{\Delta}} \inf_{\bar{u}_1\in\bar{\mathcal{A}}_1} \bar{R}(x, \bar{u}_1(\cdot), \beta[\bar{u}_1](\cdot)),$$
$$\bar{V}^-(x) = \inf_{\alpha\in\bar{\Gamma}} \sup_{\bar{u}_2\in\bar{\mathcal{A}}_2} \bar{R}(x, \alpha[\bar{u}_2](\cdot), \bar{u}_2(\cdot)).$$

Then we can closely follow the arguments in [3] to show that under (A1) and (A2), $\bar{V}^+$ and $\bar{V}^-$, respectively, are unique viscosity solutions of

$$\lambda v(x) - \bar{H}^+(x, Dv(x), D^2v(x)) = 0 \tag{2.4}$$

and

$$\lambda v(x) - \bar{H}^-(x, Dv(x), D^2v(x)) = 0 \tag{2.5}$$

in the class of bounded continuous functions, where $\bar{H}^+, \bar{H}^- : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ are defined as follows:

$$\bar{H}^+(x, p, X) = \inf_{\bar{u}_1\in\bar{U}_1} \sup_{\bar{u}_2\in\bar{U}_2} \left[\frac{1}{2}\mathrm{tr}(\hat{a}(x, u_1, u_2)X) + \bar{b}(x, \bar{u}_1, \bar{u}_2)\cdot p + r(x, \bar{u}_1, \bar{u}_2)\right],$$

$$\bar{H}^-(x, p, X) = \sup_{\bar{u}_2\in\bar{U}_2} \sup_{\bar{u}_1\in\bar{U}_1} \left[\frac{1}{2}\mathrm{tr}(\hat{a}(x, u_1, u_2)X) + \bar{b}(x, \bar{u}_1, \bar{u}_2)\cdot p + r(x, \bar{u}_1, \bar{u}_2)\right],$$

where $\hat{a} = \bar{\sigma}\bar{\sigma}^*$. We now establish the equivalence of (1.1) and (1.2) with (2.4) and (2.5) respectively.

Theorem 2.4. *Assume* (A2)(iii). *A continuous function* $v : \mathbb{R}^d \to \mathbb{R}$ *is a viscosity solution of* (2.4) *if and only if it is a viscosity solution of* (1.1). *Similarly a continuous function* $v : \mathbb{R}^d \to \mathbb{R}$ *is a viscosity solution of* (2.5) *if and only if it is a viscosity solution of* (1.2).

Proof. The proof of this theorem is a simple consequence of the observation that

$$\bar{H}^+(x, p, X) = (H^+(x, p, X) \vee \lambda\psi_2(x)) \wedge \lambda\psi_1(x) \tag{2.6}$$

and

$$\bar{H}^-(x, p, X) = (H^-(x, p, X) \wedge \lambda\psi_1(x)) \vee \lambda\psi_2(x). \tag{2.7}$$

■

As a consequence of this result we have the following existence and uniqueness result for the solutions of (1.1) and (1.2).

COROLLARY 2.5

Assume (A1) *and* (A2). *Then* $\bar{V}^+$ *and* $\bar{V}^-$ *are unique vicosity solutions of* (1.1) *and* (1.2) *respectively in the class of bounded continuous functions.*

Proof. Since (2.4) has a unique viscosity solution in the class of bounded continuous functions given by V^+, we get by Theorem 2.4, that $\bar{V}^+$ is the unique viscosity solution of (1.1) in the class of bounded continuous functions. Similarly V^- is the unique viscosity solution of (1.2) in the class of bounded continuous functions. ■

Remark 2.6. In the classical case, it is quite clear from the proof of Theorem 2.1 that under (A2)(iii), $\psi_2 \leq v \leq \psi_1$ if v is a classical solution of (2.1). In fact this remains true even in the case of viscosity solutions. Indeed assume (A2)(iii). Then any viscosity solution of (2.1) satisfies

$$\psi_2(x) \leq v(x) \leq \psi_1(x), \text{ for all } x \in \mathbb{R}^d.$$

Similar result holds for viscosity solutions of (2.2). Observe that ψ_2 is a viscosity sub-solution of (2.1) and ψ_1 is a viscosity super-solution of (2.1). Thus the desired result follows from a general comparison principle on viscosity solutions [2].

3. Application to stochastic games

In this section we consider a stochastic game with stopping times. We show the existence of a value and a saddle point equilibrium for this problem. We now describe the stochastic game with stopping times.

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. Assume that b, σ are bounded and Lipschitz continuous. Consider the stochastic differential equation

$$\left.\begin{aligned} \mathrm{d}X(t) &= b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t),\ t > 0 \\ X(0) &= x \end{aligned}\right\}. \tag{3.1}$$

Here W is a d-dimensional Brownian motion on an underlying complete probability space $(\Omega, \mathcal{F}, P)$. Let $r, \psi_1, \psi_2 : \mathbb{R}^d \to \mathbb{R}$ be bounded and Lipschitz continuous functions and $\psi_1 \geq \psi_2$. Let $\lambda > 0$. Define

$$R(x, \theta, \tau) = E\left[\int_0^{\theta\wedge\tau} e^{-\lambda s} r(X(s))\, ds + e^{-\lambda(\theta\wedge\tau)}[\psi_1(X(\theta))\chi_{\theta<\tau} + \psi_2(X(\tau))\chi_{\tau\leq\theta}]\right], \tag{3.2}$$

where θ, τ are the stopping times with respect to the σ-field generated by $W(t)$.

Let $\tilde{W}$ denote another d-dimensional Brownian motion independent of W, constructed on an augmented probability space which is also denoted by $(\Omega, \mathcal{F}, P)$ by an abuse of notation. Let $\sigma^\gamma : \mathbb{R}^d \to \mathbb{R}^{d\times 2d}$ be defined by $\sigma^\gamma = [\sigma \quad \gamma I_d]$, where I_d is the $d \times d$ identity matrix. Now consider the following stochastic differential equation

$$\left.\begin{aligned} dX^\gamma(t) &= b(X^\gamma(t))\, dt + \sigma^\gamma(X^\gamma(t))\, d\overline{W}(t),\ t > 0 \\ X^\gamma(0) &= x \end{aligned}\right\}, \tag{3.3}$$

where $\overline{W} = [W, \tilde{W}]^*$. Player 1 tries to minimize $R(x, \cdot, \cdot)$, as in (3.2), over stopping times θ (with respect to the σ-field generated by $\overline{W}(t)$), whereas player 2 tries to maximize the same over stopping times τ (with respect to the σ-field generated by $\overline{W}(t)$). Note that these stopping times need not be finite a.s.. In other words, each player has the option of not stopping the game at any time. We refer to [7] for a similar treatment to stochastic games with stopping times. We now define the lower and upper value functions. Let

$$V^-(x) = \sup_{\tau\geq 0}\inf_{\theta\geq 0} R(x, \theta, \tau),$$
$$V^+(x) = \inf_{\theta\geq 0}\sup_{\tau\geq 0} R(x, \theta, \tau).$$

The stochastic game with stopping times is said to have a value if $V^+ \equiv V^-$.

We now establish the existence of a value for this problem. Let $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ be defined by

$$H(x, p, X) = \frac{1}{2}\,\mathrm{tr}\,(a(x)X) + b(x)\cdot p + r(x),$$

where $a = \sigma\sigma^*$.

Theorem 3.1. *The stochastic game with stopping times has a value and a saddle point equilibrium. The value of this game is the unique viscosity solution in the class of bounded and continuous functions of the variational inequalities with bilateral constraints given by*

$$\left.\begin{aligned} &\psi_2(x) \leq w(x) \leq \psi_1(x), && \forall x \\ &\lambda w(x) - H(x, Dw(x), D^2w(x)) = 0, && \text{if } \psi_2(x) < w(x) < \psi_1(x) \\ &\lambda w(x) - H(x, Dw(x), D^2w(x)) \geq 0, && \text{if } w(x) = \psi_2(x) \\ &\lambda w(x) - H(x, Dw(x), D^2w(x)) \leq 0, && \text{if } w(x) = \psi_1(x) \end{aligned}\right\}. \tag{3.4}$$

Proof. Using the results of §2, it follows that there is a unique viscosity solution w of (3.4) in the class of bounded continuous functions. We now identify this solution as the value function of the stochastic game with stopping times.

We first assume that w is C^2. Let $w(x) > \psi_2(x)$. Let θ be any stopping times. Define the stopping time $\hat{\tau}$ by

$$\hat{\tau} = \inf\{t \geq 0 : \psi_2(X(t)) = w(X(t))\},$$

where $X(t)$ is a solution of (3.1) with the initial condition $X(0) = x$. Since w is a smooth viscosity solution of (3.4), by Ito's formula, we have for any $T > 0$ and stopping time θ,

$$w(x) \leq E\left\{\int_0^{T\wedge\theta\wedge\hat{\tau}} \mathrm{e}^{-\lambda s} r(X(s))\, \mathrm{d}s + \mathrm{e}^{-\lambda(T\wedge\theta\wedge\hat{\tau})} w(X(T\wedge\theta\wedge\hat{\tau})\right\}.$$

Letting $T \to \infty$ in the above equation, we obtain

$$w(x) \leq E\left\{\int_0^{\theta\wedge\hat{\tau}} \mathrm{e}^{-\lambda s} r(X(s))\, \mathrm{d}s + \mathrm{e}^{-\lambda(\theta\wedge\hat{\tau})} w(X(\theta\wedge\hat{\tau}))\right\}.$$

Now using the first inequality in (3.4) and the definition of $\hat{\tau}$ in the above, it follows that

$$w(x) \leq R(x, \theta, \hat{\tau}).$$

Since θ is arbitrary, we get

$$w(x) \leq V^-(x).$$

Next let w is C^2 and $w(x) < \psi_1(x)$. Define the stopping time $\hat{\theta}$ by

$$\hat{\theta} = \inf\{t \geq 0 : \psi_1(X(t)) = w(X(t))\},$$

where $X(t)$ is a solution of (3.1) with the initial condition $X(0) = x$. Now using the foregoing arguments we can show that

$$w(x) \geq V^+(x).$$

Thus

$$w \equiv V^+ \equiv V^-.$$

We now prove this result for the general case. Let w_ϵ be the sup-convolution of w, i.e.,

$$w_\epsilon(x) = \sup_{\xi\in\mathbb{R}^d}\left\{w(\xi) - \frac{|\xi - x|^2}{2\epsilon}\right\}.$$

Then $w_\epsilon \to w$ uniformly in $\mathbb{R}^d$ as $\epsilon \to 0$, w_ϵ are bounded, Lipschitz continuous, semi-convex and satisfy a.e. on $\mathbb{R}^d$,

$$\lambda w_\epsilon(x) - H(x, Dw_\epsilon(x), D^2 w_\epsilon(x)) \leq \rho_0(x),$$

for a modulus ρ_0 (see [8]). Now let w_ϵ^δ be the standard mollification of w_ϵ. Then w_ϵ^δ are $C^{2,1}$, $w_\epsilon^\delta \to w_\epsilon$ uniformly in $\mathbb{R}^d$ and $Dw_\epsilon^\delta(x) \to Dw_\epsilon(x)$, $D^2 w_\epsilon^\delta(x) \to D^2 w_\epsilon(x)$ for a.e. $x \in \mathbb{R}^d$. Also w_ϵ^δ have the same Lipschitz constant as w_ϵ and for any $\gamma > 0$, we have

$$\lambda w_\epsilon^\delta(x) - \frac{\gamma^2}{2} \operatorname{tr}(D^2 w_\epsilon^\delta(x)) - H(x, Dw_\epsilon^\delta(x), D^2 w_\epsilon^\delta(x)) \le \rho_0(\epsilon) + g_\delta(x) + \frac{\gamma^2 d}{\epsilon}, \tag{3.5}$$

where g_δ are uniformly continuous. Let

$$r_\delta(x, u_1, u_2) = r(x, u_1, u_2) + \rho_0(\epsilon) + \frac{\gamma^2 d}{\epsilon}.$$

We now assume $w(x) > \psi_2(x)$. Define the stopping time $\hat{\tau}$ as before. Then $w_\epsilon^\delta(x) > \psi_2(x)$ for sufficiently small ϵ and δ. Applying Ito's formula for w_ϵ^δ, we obtain for any $T > 0$ and any stopping time θ,

$$w_\epsilon^\delta(x) \le E\left\{\int_0^{T\wedge\theta\wedge\hat{\tau}} \mathrm{e}^{-\lambda s} r_\delta(X^\gamma(s))\, \mathrm{d}s + \mathrm{e}^{-\lambda(T\wedge\theta\wedge\hat{\tau})} w_\epsilon^\delta(X^\gamma(T\wedge\theta\wedge\hat{\tau}))\right\}, \tag{3.6}$$

where $X^\gamma(t)$ is the solution of (3.3) with the initial condition $X^\gamma(0) = x$.

By a standard martingale inequality, for any $\eta > 0$, we can find a constant R_η such that

$$P\left(\sup_{0\le s\le T} |X^\gamma(s)| \ge R_\eta\right) \le \eta.$$

Let $\Omega_\eta \subset \mathbb{R}^d$ be such that $|\Omega_\eta| \le \eta$ and $g_\eta \to 0$ uniformly on $B_{R_\eta} \setminus \Omega_\eta$. Using this we can find a local modulus ρ_1 such that

$$\left| E\int_0^{T\wedge\theta\wedge\hat{\tau}} g_\delta(X^\gamma(s))\, \mathrm{d}s \right| \le \rho_1(\delta, \gamma).$$

Using this in (3.6), we obtain

$$w_\epsilon^\delta(x) \le E\left\{\int_0^{T\wedge\theta\wedge\hat{\tau}} \mathrm{e}^{-\lambda s} r(X^\gamma(s))\, \mathrm{d}s + \mathrm{e}^{-\lambda(T\wedge\theta\wedge\hat{\tau})} w_\epsilon^\delta(X^\gamma(T\wedge\theta\wedge\hat{\tau}))\right\} + \rho_2(\delta, \gamma) + T\left(\rho_0(\epsilon) + \frac{\gamma^2 d}{\epsilon}\right), \tag{3.7}$$

where ρ_2 is a local modulus. Now using moment estimates [8], we have

$$E\left(\sup_{0\le s\le T} |X(s) - X^\gamma(s)|^2\right) \le C\gamma^2$$

for some constant $C > 0$ which may depend on T and x. Now using this in (3.7), and passing to the limits $\gamma \to 0$ and then $\delta, \epsilon \to 0$, we obtain

$$w(x) \le E\left\{\int_0^{T\wedge\theta\wedge\hat{\tau}} \mathrm{e}^{-\lambda s} r(X(s))\, \mathrm{d}s + \mathrm{e}^{-\lambda(T\wedge\theta\wedge\hat{\tau})} w(X(T\wedge\theta\wedge\hat{\tau}))\right\}.$$

Now letting $T \to \infty$, as before, we obtain

$$w(x) \leq V^-(x).$$

Similarly we can show that

$$w(x) \geq V^+(x)$$

for all x such that $\psi_1(x) > w(x)$. Thus the stochastic game with stopping times has a value.

We now show that $(\hat{\theta}, \hat{\tau})$ is a saddle point equilibrium. We need to prove this only when $\psi_2(x) < w(x) < \psi_1(x)$. Using the foregoing arguments, we can show that

$$w(x) = E\left[\int_0^{\hat{\theta}\wedge\hat{\tau}} \mathrm{e}^{-\lambda s} r(X(s))\, \mathrm{d}s + \mathrm{e}^{-\lambda\hat{\theta}\wedge\hat{\tau}} w(X(\hat{\theta}\wedge\hat{\tau}))\right] = R(x, \hat{\theta}, \hat{\tau}).$$

Clearly $(\hat{\theta}, \hat{\tau})$ constitutes a saddle point equilibrium. ■

Remark 3.2. The above result generalizes the optimal stopping time problem for degenerate diffusions studied by Menaldi [6]. Menaldi has characterized the value function in the optimal stopping time problem as a maximal solution of the corresponding variational inequalities with one sided constraint. He has used the penalization arguments to obtain his results. Here we have used the method of viscosity solutions to generalize the optimal stopping time problem to stochastic games with stopping times. We also wish to mention that Stettner [7] has studied stochastic games with stopping times for a class of Feller Markov processes. He has employed a penalization argument using semigroup theory. Thus our approach is quite different from that of Stettner.

4. Conclusions

We have studied a class of second order variational inequalities with bilateral constraints. Under certain conditions, we have showed the existence and uniqueness of viscosity solutions by transforming the variational inequalities to HJI equations corresponding to a stochastic differential game. Here we have confined our attention to a particular form of H which arises in stochastic differential games of mixed type. A general form of H can be reduced to this particular form by a suitable representation formula as in [5]. Thus our probabilistic method can be used to prove the existence and uniqueness of viscosity solutions for more general class of second order nonlinear variational inequalities with bilateral constraints.

As an application, we have showed the existence of a value and a saddle point equilibrium for a stochastic game with stopping times. We now give a brief description of a stochastic differential game of mixed type where each player uses both continuous control and stopping times.

Consider the following controlled stochastic differential equation

$$\left.\begin{aligned} \mathrm{d}X(t) &= b(X(t), u_1(t), u_2(t))\mathrm{d}t + \sigma(X(t), u_1(t), u_2(t))\mathrm{d}W(t), \quad t > 0 \\ X(0) &= x \end{aligned}\right\}. \tag{4.1}$$

Here $W(t)$ is a standard d-dimensional Brownian motion on an underlying complete probability space $(\Omega, \mathcal{F}, P)$ and σ is the non-negative square root of a. The processes $u_1(t)$ and $u_2(t)$ are U_1 and U_2-valued control processes of players 1 and 2 respectively which are progressively measurable with respect to the σ-field generated by $W(t)$. Let $\lambda > 0$ be the discount factor. The payoff function is given by

$$R(x, u_1(\cdot), \theta, u_2(\cdot), \tau) = E\left[\int_0^{\theta\wedge\tau} \mathrm{e}^{-\lambda s} r(X(s), u_1(s), u_2(s))\, \mathrm{d}t + \mathrm{e}^{-\lambda(\theta\wedge\tau)}[\psi_1(x(\theta))\chi_{\theta<\tau} + \psi_2(x(\tau))\chi_{\tau\le\theta}]\right], \tag{4.2}$$

where θ, τ are the stopping times with respect to the filtration generated by $W(\cdot)$.

An admissible control for player 1 is a map $u_1(\cdot) : [0, \infty) \to U_1$ which is progressively measurable with respect to the σ-field generated by $W(\cdot)$. The set of all admissible controls for player 1 is denoted by $\mathcal{A}_1$. Similarly an admissible control for player 2 is defined. Let $\mathcal{A}_2$ denote the set of all admissible controls for player 2.

We identify two controls $u_1(\cdot), \tilde{u}_1(\cdot)$ in $\mathcal{A}_1$ on $[0, t]$ if $P(u_1(s) = \tilde{u}_1(s)$ for a.e. $s \in [0, t]) = 1$. Similarly we identify the controls in $\mathcal{A}_2$.

An admissible strategy for player 1 is a map $\alpha : \mathcal{A}_2 \to \mathcal{A}_1$ such that if $u_2 = \tilde{u}_2$ on $[0, s]$ then $\alpha[u_2] = \alpha[\tilde{u}_2]$ for all $s \in [0, \infty)$. The set of all admissible strategies for player 1 is denoted by Γ. Similarly admissible strategies for player 2 are defined. Let Δ denote the set of all admissible strategies for player 2.

Let $\mathcal{S}$ denote the set of all stopping times. Let $\hat{\Gamma}$ denote the set of all non-anticipating maps $\hat{\alpha} : \mathcal{A}_2 \to \mathcal{A}_1 \times \mathcal{S}$ for player 1. Similarly let $\hat{\Delta}$ denote the set of all non-anticipating maps $\hat{\beta} : \mathcal{A}_1 \to \mathcal{A}_2 \times \mathcal{S}$ for player 2. Player 1 tries to minimize $R(x, u_1(\cdot), \theta, u_2(\cdot), \tau)$ over his admissible control $u_1(\cdot)$ and stopping times θ, whereas player 2 tries to maximize the same over his admissible control $u_2(\cdot)$ and stopping times τ. We now define the upper and lower value functions of stochastic differential game of mixed type. Let

$$V^+(x) = \sup_{\hat{\beta}\in\hat{\Delta}} \inf_{u_1\in\mathcal{A}_1,\theta\ge 0} R(x, u_1(\cdot), \theta, \hat{\beta}[u_1](\cdot)),$$

$$V^-(x) = \inf_{\hat{\alpha}\in\hat{\Gamma}} \sup_{u_2\in\mathcal{A}_2,\tau\ge 0} R(x, \hat{\alpha}[u_2](\cdot), u_2(\cdot), \tau).$$

The functions V^+ and V^- are respectively called upper and lower value functions of the stochastic differential game of mixed type. This differential game is said to have a value if both upper and lower value functions coincide. Now we make the following conjecture.

Conjecture. The value functions V^+ and V^- are unique viscosity solutions of (1.1) and (1.2) respectively in the class of bounded continuous functions.

Note that the above conjecture is true for the special case treated in §3. Analogous results also holds when the matrix a is independent of the control variables and is uniformly elliptic [1, 4].

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