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Global Compactness Properties of Semilinear Elliptic Equations with Critical Exponential Growth

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Sequences of positive solutions to semilinear elliptic equations of critical exponential growth in the plane either are precompact in the Sobolev H^1-topology or concentrate at isolated points of the domain. For energies allowing at most single-point blow-up, we establish a universal blow-up pattern near the concentration point and uniquely characterize the blow-up energy in terms of a geometric limiting problem. © 2000 Academic Press

1. INTRODUCTION

Let Ω be a smoothly bounded domain in $\mathbb{R}^2$. Consider the semilinear elliptic boundary value problem

$$-\Delta u = f(u) \text{ in } \Omega, \qquad u > 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is smooth and has critical exponential growth. For instance, let f be given by

$$f(s) = se^{4\pi s^2} \tag{2}$$

with primitive

$$F(s) = \int_0^s f(t)\, dt = \frac{1}{8\pi}\,(e^{4\pi s^2} - 1).$$

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Solutions u to (1) may be characterized as critical points of the functional

$$E(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2\,dx-\int_{\Omega}F(u)\,dx \tag{3}$$

in the Sobolev space $H_0^1(\Omega)$. Indeed, by the Moser–Trudinger inequality [14, 19] the functional E is well-defined and smooth on $H_0^1(\Omega)$, and critical points $u\in H_0^1(\Omega)$ are classical (smooth) solutions of (1). However, the functional E fails to satisfy the Palais–Smale condition (globally).

The situation is analogous to the case of semilinear elliptic equations of critical Sobolev growth on domains in $\mathbb{R}^n$, when $n\geqslant 3$. A well-studied model problem is the boundary value problem

$$-\Delta u=u\,|u|^{2^*-2}\text{ in }\Omega,\qquad u>0\text{ in }\Omega,\qquad u=0\text{ on }\partial\Omega, \tag{4}$$

on a domain $\Omega\subset\mathbb{R}^n$, where $2^*=\frac{2n}{n-2}$ is the Sobolev exponent.

In [4, 6, 16, 17] and elsewhere the compactness properties of the solution set of (4) and possible concentration phenomena have been analyzed in minute detail, and failure of the Palais–Smale condition has been traced to a universal mechanism, the "bubbling off" of spheres. Each sphere carries with it a certain quanta of energy related only to the Sobolev constant for the embedding $H_0^1(\Omega)\hookrightarrow L^{2^*}(\Omega)$, which is independent of the particular domain.

Our aim here is to establish similar results for critical semilinear equations on planar domains. In particular, analogous to [17] we would like to obtain a universal "geometric" characterization of possible blow-up and a quantization of the energy levels where blow-up may occur for the functional E in (3) above. First results in this direction were obtained in [18] in the radial case, and, in somewhat greater generality, but using the techniques from [18] and still in the radial case, in [2, 15].

However, in view of examples from [2] we cannot expect such results for arbitrary Palais–Smale sequences; see also Section 2. Therefore in the present paper we restrict our attention to *solutions* u_k of problems

$$-\Delta u_k=f_k(u_k)\text{ in }\Omega,\qquad u_k>0\text{ in }\Omega,\qquad u_k=0\text{ on }\partial\Omega, \tag{5}$$

where f_k are smooth of critical exponential growth with primitive F_k and associated energy functional E_k, $k\in\mathbb{N}$.

More precisely, we study nonlinearities of the form

$$f_k(s)=se^{\varphi_k(s)},\qquad k\in\mathbb{N},$$

where $\varphi_k \in C^\infty(\mathbb{R})$ is convex for $s \geqslant s_0$, with $s_0 \geqslant 0$ a fixed number independent of k, and such that

$$\varphi_k''(s) \leqslant 8\pi \qquad \text{for} \quad s \geqslant s_0. \tag{6}$$

Moreover, we assume that (φ_k) converges smoothly locally on $\mathbb{R}$ to a smooth limit φ, and, finally that

$$\lim_{s \to \infty} \varphi_k'(s)/s = 8\pi, \qquad \text{uniformly in } k. \tag{7}$$

Examples include suitable approximations (φ_k) of the function

$$\varphi(s) = 4\pi s^2 + \alpha \log(1+s^2),$$

giving rise to the nonlinearity

$$f(s) = s(1+s^2)^\alpha \, e^{4\pi s^2},$$

for arbitrary $\alpha \geqslant 0$.

Then we obtain the following result.

THEOREM 1.1. *Let $(u_k)_{k \in \mathbb{N}}$ solve (5) with $E_k(u_k) \to \beta < 1$. Also assume that $\varphi_k \to \varphi$ smoothly locally on $\mathbb{R}$ and (for simplicity) that Eq. (1) with $f(s) = se^{\varphi(s)}$ does not admit a solution $u > 0$ with energy less than $\frac{1}{2}$. Then either the family (u_k) accumulates strongly in $H_0^1(\Omega)$ at a solution u of (1) having energy $E(u) = \beta$, or $u_k \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, and for suitable sequences $k \to \infty$, $r_k \to 0$, $x_k \in \Omega$ there holds*

$$\varphi_k(u_k(x_k + r_k x)) + 2\log(r_k u_k(x_k)) + \log(8\pi) \to \log \frac{1}{(1+|x|^2/8)^2},$$

locally uniformly on $\mathbb{R}^2$, as $k \to \infty$. Moreover, in the latter case necessarily $\beta = \frac{1}{2}$.

Theorem 1.1 is a first step toward the universal description of concentration behavior for Eqs. (5), alluded to above. We expect that similar results hold for any $\beta \in \mathbb{R}$ and for general nonlinearities of critical exponential growth as defined in [1, Definition 2.1].

2. PALAIS–SMALE CONDITION

By definition, a C^1-functional E on a Banach space V with dual V^* satisfies the Palais–Smale condition at level β if the following holds.

(P.-S.)$_\beta$ Any sequence $(u_k)_{k \in \mathbb{N}}$ in V such that $E(u_k) \to \beta$, $\|dE(u_k)\|_{V^*} \to 0$ as $k \to \infty$ contains a convergent subsequence.

Critical variational problems are often characterized by the fact that $(\text{P.-S.})_\beta$ does not hold for large levels of β and that loss of compactness is associated with the "bubbling off" (after rescaling) of solutions to a certain limit equation. This limit equation often has a geometric interpretation that leads to a precise characterization of the energy levels β where $(\text{P.S.})_\beta$ fails. With regard to our problem (1) with associated energy E as a first result in this direction we have the following local compactness result from [1].

THEOREM 2.1 [1, Theorem, part (1), p. 394]. *Let $f(s)=se^{\varphi(s)}$, where φ satisfies* (6), (7) *and let E be the corresponding energy functional. Then any sequence $u_k \geqslant 0$ in $H_0^1(\Omega)$ with $E(u_k)\to\beta<\frac{1}{2}$ and $dE(u_k)\to 0$ in $H^{-1}(\Omega)$ as $k\to\infty$ is relatively compact in $H_0^1(\Omega)$.*

For energies $\beta\geqslant\frac{1}{2}$, we recall a non-compactness result from [2].

THEOREM 2.2 [2, Theorem A]. *For E given by* (3) *and f of critical exponential growth as in Theorem* 2.1 *above,* $(\text{P.-S.})_\beta$ *fails for any β of the form $\beta=k/2$, $k\in\mathbb{N}$.*

For $\beta=\frac{1}{2}$ a Palais–Smale sequence is constructed from the following family of scaled and truncated Green's functions also considered by Moser [14].

For $0<\rho<R$ let

$$m_{\rho,R}(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log\left(\dfrac{R}{\rho}\right)} & 0\leqslant|x|\leqslant\rho,\\ \log\left(\dfrac{R}{r}\right)\Big/\sqrt{\log\left(\dfrac{R}{\rho}\right)}, & \rho\leqslant|x|=r<R,\\ 0, & R\leqslant|x|,\end{cases}$$

and for $x_0\in\mathbb{R}^2$ let $m_{\rho,R,x_0}(x)=m_{\rho,R}(x-x_0)$.

Choose x_0, $R>0$ such that $B_R(x_0)\subset\Omega$. Shift $x_0=0$. Observe that

$$\int_{B_R(0)}|\nabla m_{\rho,R}|^2\,dx=1$$

for any $0<\rho<R$. Moreover, for fixed R we have, as $\rho\to 0$,

$$\int_{B_R(0)} m_{\rho,R}^2 e^{4\pi m_{\rho,R}^2}\,dx\geqslant\int_0^\rho\left(\frac{R}{\rho}\right)^2\log\left(\frac{R}{\rho}\right)r\,dr\to\infty,$$

while for any $a<1$ there holds

$$\int_{B_R(0)} m_{\rho,R}^2 e^{4\pi a^2 m_{\rho,R}^2}\,dx\to 0.$$

Thus, for small $\rho > 0$ there exists $a_\rho > 0$ such that the function

$$u_\rho = a_\rho m_{\rho, R}$$

satisfies

$$\int_\Omega |\nabla u_\rho|^2 \, dx = \int_\Omega f(u_\rho) \, u_\rho \, dx \tag{8}$$

and $a_\rho \to 1$ as $\rho \to 0$.

As $\rho \to 0$ then $u_\rho \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, $F(u_\rho) \to 0$ in L^1, and $E(u_\rho) \to \frac{1}{2}$. Moreover, $dE(u_\rho) \to 0$ in $H^{-1}(\Omega)$, and $(u_\rho)_{\rho > 0}$ is a Palais–Smale sequence for E at level $\beta = \frac{1}{2}$. (For the reader's convenience we give a short proof of convergence $dE(u_\rho) \to 0 (\rho \to 0)$ in Appendix A.) Choosing disjoint balls $B_R(x_k) \subset \Omega$, $1 \leqslant k \leqslant K$, similarly the function

$$u_\rho = \sum_{k=1}^{K} a_\rho m_{\rho, R, x_k}$$

is a (P.-S.)-sequence at level $\beta = \frac{K}{2}$; see [2]. We can also stack bubbles on bubbles, for instance, by letting

$$u_{\rho, \sigma} = a_\rho m_{\rho, R} + a_{\rho, \sigma} m_{\sigma, \rho},$$

for radii $0 < \sigma < \rho < R$, where $\rho \to 0$, $\sigma/\rho \to 0$, with suitable numbers $a_\rho \to 1$ and $a_{\rho, \sigma} \to 1$ as above, to obtain a Palais–Smale sequence $(u_{\rho, \sigma})$ blowing up at energy level $\beta = 1$, and similarly at any level $\beta = \frac{K}{2}$, $K \in \mathbb{N}$.

The asymptotic scaling behavior of $u_\rho = a_\rho m_{\rho, R}$, captured in the formula

$$\lim_{\rho \to 0} (4\pi u_\rho^2(\rho x) + 2a_\rho^2 \log \rho) = \begin{cases} 2 \log R, & \text{for} \quad |x| \leqslant 1, \\ 2 \log R - 4 \log |x|, & \text{else}, \end{cases}$$

is in contrast with Theorem 1.1. This shows that, in contrast to the higher-dimensional case $n \geqslant 3$, we cannot expect a universal characterization of blow-up for Palais–Smale sequences, in general.

Moreover, from Theorem 2.2 we see that our characterization of the concentration behavior of sequences of solutions applies to energy levels which are large compared to the energy threshold for blow-up.

3. PROOF OF THEOREM 1.1: COMPACTNESS

First recall the Moser–Trudinger inequality [14, 19]. Let $\fint \cdots$ denote mean value.

THEOREM 3.1. *There exists a constant C such that for any smoothly bounded domain $\Omega \subset\subset \mathbb{R}^2$ there holds*

$$\sup_{u \in H_0^1(\Omega),\ \|\nabla u\|_{L^2(\Omega)} \leqslant 1} \fint_\Omega e^{4\pi u^2}\, dx \leqslant C.$$

Also the following variant of the Moser–Trudinger inequality, due to Chang and Yang [7], will play a fundamental role in our argument.

THEOREM 3.2. *There exists a constant $C > 0$ such that for any $R > 0$, any $w \in H^1(B_R(0))$ satisfying $\int_{B_R(0)} w\, dx = 0$, $\int_{B_R(0)} |\nabla w|^2\, dx \leqslant 1$ there holds*

$$\fint_{B_R(0)} e^{2\pi w^2}\, dx \leqslant C.$$

Let $u_k \in H_0^1(\Omega)$ solve (5) with $E_k(u_k) \to \beta < 1$. Then, as in [1, p. 404 f.] with error $o(1) \to 0$ as $k \to 0$ we have

$$\begin{aligned} 2\beta + o(1) = 2E_k(u_k) - \langle dE_k(u_k), u_k \rangle &= \int_\Omega (f_k(u_k)\, u_k - 2F_k(u_k))\, dx \\ &\geqslant \tfrac{1}{2} \int_\Omega f_k(u_k)\, u_k\, dx - C = \tfrac{1}{2} \|u_k\|^2_{H_0^1(\Omega)} - C \end{aligned}$$

with some constant $C = C(\Omega)$ independent of δ.

Here we used that

$$\langle dE_k(u_k), u_k \rangle = \int_\Omega |\nabla u_k|^2\, dx - \int_\Omega f_k(u_k)\, u_k\, dx = 0 \tag{9}$$

in view of (5), and we used the pointwise estimate

$$\varepsilon f_k(s)\, s \geqslant F_k(s) - C(\varepsilon) \tag{10}$$

for any $\varepsilon > 0$ and all $s > 0$, implied by exponential growth, with $\varepsilon = \frac{1}{4}$.

Thus $(u_k)_{k \in \mathbb{N}} \subset H_0^1(\Omega)$ is bounded, and, as $k \to \infty$ suitably, we may assume that $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$, and pointwise almost everywhere. Moreover, by (9), (10) above, also $F_k(u_k) \to F(u)$, $f_k(u_k) \to f(u)$ in $L^1(\Omega)$, and $u \in H_0^1(\Omega)$ solves (1) with

$$E(u) + \tfrac{1}{2} \int_\Omega |\nabla(u_k - u)|^2\, dx = E_k(u_k) + o(1) \leqslant \beta + o(1) < 1,$$

where $o(1) \to 0$ as $k \to \infty$. Hence, if we assume that Eq. (1) does not admit a solution $u > 0$ with $E(u) < 1/2$ it follows that

$$\limsup_{k \to \infty} \int_\Omega |\nabla(u_k - u)|^2 \, dx < 2. \tag{11}$$

The following result, generalizing a result of Lions [12, Theorem I.6], characterizes the possible loss of compactness macroscopically.

Lemma 3.3. *Under the above assumptions, either $u_k \to u$ strongly in $H_0^1(\Omega)$, or there is $x_0 \in \bar{\Omega}$ and a sequence $k \to \infty$ such that*

$$|\nabla u_k|^2 \, dx \rightharpoonup \mu \geqslant \delta_{x_0} \qquad (k \to \infty) \tag{12}$$

weakly in the sense of measures, where δ_{x_0} is the Dirac mass distribution centered at x_0, and $u_k \to u$ strongly in $H^1_{\mathrm{loc}}(\Omega \backslash \{x_0\})$.

Proof. Let $x_0 \in \Omega$. Suppose there is $r_{x_0} > 0$ such that

$$\limsup_{k \to \infty} \int_{B_{r_{x_0}}(x_0)} |\nabla u_k|^2 \, dx < 1.$$

For $r_0 \leqslant \min\{r_{x_0}, e^{-e}\}$ define the cut-off function

$$\psi(r) = \min\left\{1, \log\log\log\left(\frac{1}{r}\right) - \log\log\log\left(\frac{1}{r_0}\right)\right\},$$

if $r \leqslant r_0$, $\psi(r) = 0$ else. Note that $\psi(r) = 1$ for $r \leqslant r_1 = r_1(r_0)$. Computing

$$|\psi'(r)|^2 = \frac{1}{r^2(\log(1/r)\log\log(1/r))^2}$$

for $r \in [r_1, r_0]$, moreover, we easily see that

$$\int_0^\infty |\psi'(r)|^2 \,(1 + \log(1 + |\psi'(r)|^2))\, r\, dr$$

$$\leqslant C \int_{r_1}^{r_0} \frac{dr}{r\log(1/r)(\log\log(1/r))^2} \leqslant \frac{C}{\log\log(1/r_0)} \to 0$$

as $r_0 \to 0$.

Given $x_0 \in \Omega$, $r_0 > 0$ as above, then let

$$v_k(x) = \psi(|x - x_0|)\, u_k(x) \in H_0^1(\Omega).$$

Observe that by Young's inequality for any $\varepsilon>0$ with a constant $C_1(\varepsilon)$ we can bound

$$\begin{aligned}|\nabla v_k|^2 &\leqslant |\nabla u_k|^2 + 2\,|\nabla\psi|\,\psi\,|\nabla u_k|\,u_k + u_k^2\,|\nabla\psi|^2\\ &\leqslant (1+\varepsilon)\,|\nabla u_k|^2 + C_1(\varepsilon)\,u_k^2\,|\nabla\psi|^2.\end{aligned}$$

Define

$$g_k(t) = \sup_{s>0}\{s^2 t - s f_k(s)\}$$

and observe that for any $t>0$ we have

$$\begin{aligned}g_k(t) &= \tfrac{1}{2}\varphi_k'(s)\,s^2 f_k(s)_{|2st=2f_k(s)+\varphi_k'(s)\,sf_k(s)} \leqslant s^2 t_{|t=(1/2)\,\varphi_k'(s)\,f_k(s)}\\ &\leqslant Ct\log(1+t).\end{aligned}$$

Then, letting $s=u_k$, $t=\varepsilon^{-1}C_1(\varepsilon)\,|\nabla\psi|^2$, for any $\varepsilon>0$ we can estimate

$$C_1(\varepsilon)\,u_k^2\,|\nabla\psi|^2 \leqslant \varepsilon u_k f_k(u_k) + C(\varepsilon)\,|\nabla\psi|^2\,(1+\log(1+|\nabla\psi|^2))$$

and hence

$$\begin{aligned}\int_{B_{r_0}(x_0)} |\nabla v_k|^2\,dx &\leqslant (1+\varepsilon)\int_{B_{r_0}(x_0)} |\nabla u_k|^2\,dx + \varepsilon\int_{B_{r_0}(x_0)} u_k f_k(u_k)\,dx\\ &\quad + C(\varepsilon)\int_{B_{r_0}(x_0)} |\nabla\psi|^2\,(1+\log(1+|\nabla\psi|^2))\,dx\\ &\leqslant C<1,\end{aligned}$$

if we first choose $\varepsilon>0$ and then $r_0 \leqslant r_{x_0}$ sufficiently small.

By the Moser–Trudinger inequality, applied to $v_k \in H_0^1(\Omega)$, we then conclude that the family $(e^{4\pi v_k^2})_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ for some $p>1$. Hence also the functions $f(u_k)$ are bounded in $L^q(B_{r_1}(x_0))$ for some $q>1$, where $r_1 = \exp(-(\log(1/r_0))^e)$. In particular, if (12) does not holds for any $x_0 \in \bar{\Omega}$ and any sequence $k\to\infty$ upon covering $\bar{\Omega}$ with finitely many such balls $B_{r_i}(x_i)$, from pointwise convergence $u_k \to u$ we then conclude that $f(u_k) \to f(u)$ strongly in $H^{-1}(\Omega)$, and $u_k \to u$ in $H_0^1(\Omega)$.

In general, by (11), for any subsequence there can be at most one concentration point x_0 in the sense of (12). Given $\rho>0$ we may then cover $\bar{\Omega}\backslash B_\rho(x_0)$ by finitely many balls $B_{r_i}(x_i)$ as above to see that the sequence $(f_k(u_k))_{k\in\mathbb{N}}$ is bounded in L^q on $\Omega\backslash B_\rho(x_0) =: \Omega_\rho$ for some $q>1$. Fixing a cut-off function $\psi \in C_0^\infty(B_{2\rho}(x_0))$ such that $\psi \equiv 1$ on $B_\rho(x_0)$ and truncating $v_k = (1-\psi)\, u_k \in H_0^1(\Omega_\rho)$, then we see that

$$-\Delta v_k = (1-\psi)\, f_k(u_k) + 2\nabla\psi\, \nabla u_k + \Delta\psi u_k$$

is bounded in $L^q(\Omega_\rho)$ and hence precompact in $H^{-1}(\Omega_\rho)$. It follows that a subsequence $v_k \to v = (1-\psi)\, u$ strongly in $H_0^1(\Omega_\rho)$ and thus $u_k \to u$ in $H^1(\Omega \backslash B_{2\rho}(x_0))$. Letting $\rho = \rho_k \to 0$ $(k \to \infty)$ and choosing a diagonal subsequence (u_k) we then obtain that $u_k \to u$ in $H^1_{\mathrm{loc}}(\Omega \backslash \{x_0\})$, as desired. ∎

If $u_k \to u$ in $H_0^1(\Omega)$ the proof of Theorem 1.1 is complete. For the remainder of the proof we may pass to subsequences, whenever necessary. For ease of notation, these will always be relabelled (u_k). We thus may assume that (u_k) satisfies (12). In this case, with error $o(1) \to 0$ as $k \to \infty$, we can estimate

$$\begin{aligned} E(u) &= E_k(u_k) - \tfrac{1}{2} \int_\Omega |\nabla(u_k - u)|^2\, dx + o(1) \\ &\leqslant \beta - \tfrac{1}{2} + o(1) < 1 - \tfrac{1}{2} = \tfrac{1}{2} \end{aligned}$$

for sufficiently large $k \in \mathbb{N}$. By hypothesis, then, $u = 0$. Hence in the following we may assume that $u_k \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, $F_k(u_k) \to 0$, $f_k(u_k) \to 0$ in $L^1(\Omega)$, and, by Lemma 3.3, that

$$|\nabla u_k|^2\, dx \rightharpoonup 2\beta\delta_{x_0},\; u_k f_k(u_k)\, dx \rightharpoonup 2\beta\delta_{x_0} \tag{13}$$

weakly in the sense of measures as $k \to \infty$, where $1 \leqslant 2\beta < 2$.

4. BLOW-UP ANALYSIS

For a suitable number $0 < a < 1$ determined in Lemmas 4.3 and 4.6 below, we (tentatively) choose $r_k > 0$, $x_k \in \Omega$ such that

$$\frac{a}{2} \leqslant \int_{B_{r_k}(x_k)} f_k(u_k)\, u_k\, dx = \sup_{x_0 \in \Omega} \int_{B_{r_k}(x_0)} f_k(u_k)\, u_k\, dx \leqslant a.$$

Observe that $r_k \to 0$ as $k \to \infty$ on account of (13).
Scale

$$\Omega_k = \{x \in \mathbb{R}^2;\, x_k + r_k x \in \Omega\},$$

and let

$$v_k(x) = u_k(x_k + r_k x) \in H_0^1(\Omega_k),$$

satisfying the equation

$$-\Delta v_k = r_k^2 f_k(v_k) \qquad \text{in } \Omega_k \tag{14}$$

and the normalization condition

$$\frac{a}{2} \leqslant \int_{B_1(0)} r_k^2 f_k(v_k)\, v_k\, dx = \sup_{x_0 \in \Omega_k} \int_{B_1(x_0)} r_k^2 f_k(v_k)\, v_k\, dx \leqslant a. \tag{15}$$

We extend v_k as $v_k \equiv 0$ on $\mathbb{R}^2 \setminus \Omega_k$. Passing to a sub-sequence $k \to \infty$, we may assume that $\Omega_k \to \Omega_\infty$ where $\Omega_\infty = \mathbb{R}^2$ or Ω_∞ is a half-space.

For $y \in \mathbb{R}^2$, $r > 0$ decompose

$$v_k = w_k + c_k \qquad \text{on} \quad B_r(y), \tag{16}$$

where c_k denotes the mean value

$$c_k = c_k(y, r) = \fint_{B_r(y)} v_k(x)\, dx.$$

Observe that

$$\int_{B_r(y)} |\nabla w_k|^2\, dx = \int_{B_r(y)} |\nabla v_k|^2\, dx \leqslant \int_{\Omega} |\nabla u_k|^2\, dx \leqslant 2 \tag{17}$$

for large $k \in \mathbb{N}$.

In fact, for $r < 1$ we have a sharper upper bound.

LEMMA 4.1. *For any $y \in \mathbb{R}^2$, any $r > 0$ we have $w_k \rightharpoonup 0$ in $H^1(B_r(y))$ as $k \to \infty$. Moreover, for any $r < 1$ there holds*

$$\limsup_{k \to \infty} \int_{B_r(y)} |\nabla w_k|^2\, dx \leqslant a.$$

Proof. Fix $y \in \mathbb{R}^2$, $r > 0$, $c_k = c_k(y, r)$. Suppose first that $\Omega_\infty = \mathbb{R}^2$. For any $R > 0$ consider the function $\tilde{v}_k = v_k - c_k \in H^1(B_R(y))$. Since the mean value of $\tilde{v}_k$ on $B_r(y)$ vanishes, by Poincaré's inequality and (17) the family $(\tilde{v}_k)_{k \in \mathbb{N}}$ is bounded in $H^1(B_R(y))$, and, as $k \to \infty$,

$$-\Delta \tilde{v}_k = -\Delta v_k = r_k^2 f_k(v_k) \to 0 \qquad \text{in} \quad L^1(B_R(y)).$$

Choosing $R = R^{(k)} \to \infty$ suitably, we may assume that $\tilde{v}_k \rightharpoonup \tilde{v}$ weakly locally in H^1, where $\tilde{v} \in H^1_{loc}(\mathbb{R}^2)$ is harmonic with

$$\int_{\mathbb{R}^2} |\nabla \tilde{v}|^2 \, dx \leqslant \liminf_k \int_{B_{R^{(k)}}(y)} |\nabla \tilde{v}_k|^2 \, dx \leqslant 2.$$

It follows that $\tilde{v} \equiv \text{const.} = 0$ and thus that $\tilde{v}_k \rightharpoonup 0$ weakly locally in H^1 as $k \to \infty$. In particular, $w_k = \tilde{v}_k|_{B_r}(y) \rightharpoonup 0$ weakly in $H^1(B_r(y))$.

For any cut-off function $\psi \in C_0^\infty(B_1(y))$ with $0 \leqslant \psi \leqslant 1$, upon testing Eq. (14) by $\psi \tilde{v}_k \in H_0^1(B_1(y))$, moreover, we obtain

$$\begin{aligned}\int_{B_1(y)} |\nabla \tilde{v}_k|^2 \psi \, dx &= \int_{B_1(y)} r_k^2 f_k(v_k) \, \tilde{v}_k \psi \, dx - \int_{B_1(y)} \nabla \tilde{v}_k \, \nabla \psi \tilde{v}_k \, dx \\ &\leqslant \int_{B_1(y)} r_k^2 f_k(v_k) \, v_k \, dx + o(1) \leqslant a + o(1),\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$. Given $r < 1$, we may then choose ψ such that $\psi \equiv 1$ on $B_r(y)$ to obtain

$$\int_{B_r(y)} |\nabla w_k|^2 \, dx \leqslant \int_{B_1(y)} |\nabla \tilde{v}_k|^2 \psi \, dx \leqslant a + o(1),$$

as desired.

If, on the other hand, Ω_∞ is a half-space, then for every $y \in \mathbb{R}^2$ and $R > 2 \operatorname{dist}(y, \partial\Omega_\infty)$ by Poincaré's inequality $(v_k)_{k \in \mathbb{N}}$ is bounded in $H^1(B_R(y))$. As $R = R^{(k)} \to \infty$ suitably, a diagonal sequence $v_k \rightharpoonup v$ weakly in $H^1_{loc}(\mathbb{R}^2)$, where v is harmonic in Ω_∞ with $\nabla v \in L^2(\mathbb{R}^2)$ and $v = 0$ on $\partial\Omega_\infty$. Hence $v \equiv 0$. Testing (14) by $\psi v_k \in H_0^1(\Omega_k)$, where $\psi \in C_0^\infty(B_1(y))$, as above we deduce that

$$\limsup_{k \to \infty} \int_{B_1(y)} |\nabla v_k|^2 \psi \, dx \leqslant a$$

and conclude as before. ∎

The following result is related to the embedding $H^1 \hookrightarrow BMO$, the space of functions on $\mathbb{R}^2$ having bounded mean oscillation.

Lemma 4.2. *For $y_1, y_2 \in \mathbb{R}^2$, and $r_1, r_2 > 0$, letting $|y_1 - y_2| + r_1 + r_2 = 2r$, there holds*

$$|c_k(y_1, r_1) - c_k(y_2, r_2)| \leqslant C + 2 \log\left(\frac{r^2}{r_1 r_2}\right) \tag{18}$$

with an absolute constant C.

Proof. Choose y on the segment joining y_1 and y_2 so that $B_{r_1}(y_1) \cup B_{r_2}(y_2) \subset B_r(y)$, and let $v_k = w_k + c_k$ be the decomposition of v_k on $B_r(y)$, $v_k = w_k^i + c_k^i$ on $B_{r_i}(y_i) \subset B_r(y)$, $i = 1, 2$.

Then, by Jensen's inequality, for each $i = 1, 2$ we obtain

$$\begin{aligned}
|c_k^i - c_k| &= \left| \fint_{B_{r_i}(y_i)} (w_k - w_k^i)\, dx \right| = \left| \fint_{B_{r_i}(y_i)} w_k\, dx \right| \\
&\leqslant \fint_{B_{r_i}(y_i)} |w_k|\, dx \leqslant \log\left(\fint_{B_{r_i}(y_i)} e^{|w_k|}\, dx \right) \\
&\leqslant 2 \log\left(\frac{r}{r_i}\right) + \log\left(\fint_{B_r(y)} e^{|w_k|}\, dx \right).
\end{aligned}$$

Estimating

$$|w_k| \leqslant 2\pi w_k^2 / \|\nabla w_k\|^2_{L^2(B_r(y))} + \frac{1}{8\pi} \|\nabla w_k\|^2_{L^2(B_r(y))},$$

in view of Theorem 3.2 and (17) we have

$$\log\left(\fint_{B_r(y)} e^{|w_k|}\, dx \right) \leqslant C$$

uniformly for all k, y, and r, and thus

$$|c_k^i - c_k| \leqslant C + 2 \log\left(\frac{r}{r_i}\right), \qquad i = 1, 2.$$

The claim follows. ∎

LEMMA 4.3. *Suppose* $\sup_k |c_k| < \infty$, *where* $c_k = c_k(y, r)$ *for some* $y \in \mathbb{R}^2$ *and some* $0 < r < 1$. *Then a subsequence* $v_k \to v_0$ *in* $H^1_{\text{loc}}(B_r(y))$.

Proof. By uniform boundedness of (w_k) in $H^1(B_r(y))$, boundedness of (c_k) implies that (v_k) is bounded in $H^1(B_r(y))$. Hence we may assume that $v_k \rightharpoonup v_0$ weakly in $H^1(B_r(y))$ as $k \to \infty$, and strongly in $L^2(B_r(y))$.

Moreover, since $\int_{B_r(y)} w_k\, dx = 0$, by Lemma 4.1 for $a \leqslant a_1 = \frac{1}{5}$ from Theorem 3.2 we infer that

$$\int_{B_r(y)} e^{10\pi w_k^2}\, dx \leqslant \int_{B_r(y)} e^{2\pi w_k^2 / \|\nabla w_k\|^2_{L^2(B_r(y))}}\, dx \leqslant C.$$

Hence, estimating $|v_k|^2 = |w_k + c_k|^2 \leqslant \frac{9}{8}|w_k|^2 + C$ and observing that

$$\varphi_k(v_k) + \log v_k \leqslant \max_{0 \leqslant s \leqslant s_0} \varphi_k(s) + v_k(1 + \varphi_k'(s_0)) + 4\pi v_k^2 \leqslant 5\pi\,|w_k|^2 + C,$$

we find that

$$\int_{B_r(y)} |f_k(v_k)|^2\,dx = \int_{B_r(y)} v_k^2 e^{2\varphi_k(v_k)}\,dx \leqslant C \int_{B_r(y)} e^{10\pi w_k^2}\,dx \leqslant C,$$

uniformly in k.

Given $\psi \in C_0^\infty(B_r(y))$, upon multiplying Eq. (14) by $\psi(v_k - v_0) \in H_0^1(\Omega_k)$, then we obtain

$$\int_{B_r(y)} \nabla v_k\,\nabla(\psi(v_k - v_0))\,dx = r_k^2 \int_{B_r(y)} f_k(v_k)(v_k - v_0)\,\psi\,dx \to 0$$

as $k \to \infty$. On the other hand,

$$\int_{B_r(y)} \nabla v_k\,\nabla(\psi(v_k - v_0))\,dx = \int_{B_r(y)} |\nabla(v_k - v_0)|^2\,\psi\,dx + o(1),$$

where $o(1) \to 0$ as $k \to \infty$. Letting $k \to \infty$, we deduce that

$$\int_{B_r(y)} |\nabla(v_k - v_0)|^2\,\psi\,dx \to 0 \qquad (k \to \infty)$$

for any $\psi \in C_0^\infty(B_r(y))$; that is, $v_k \to v_0$ in $H^1_{\mathrm{loc}}(B_r(y))$. ∎

LEMMA 4.4. *For any $y \in \mathbb{R}^2$, any $r > 0$ there holds $c_k(y, r) \to \infty$ as $k \to \infty$. In particular, the sequence (Ω_k) exhausts $\mathbb{R}^2$.*

Proof. Let

$$A = \{y \in \mathbb{R}^2;\ \liminf_{k \to \infty} |c_k(y, r)| < \infty \text{ for some } r > 0\}.$$

By (18), either $A = \emptyset$, or $A = \mathbb{R}^2$; moreover, $y \in A$ if and only if $(c_k(y, 1/2))$ is bounded. Also observe that in the case that the sequence (Ω_k) only exhausts a half-space $\mathbb{R}^2_+$ any point $y \notin \overline{\mathbb{R}^2_+}$ satisfies $c_k(y, r) = 0$ for $r < \operatorname{dist}(y, \mathbb{R}^2_+)$ and sufficiently large k. Hence in this case necessarily $A = \mathbb{R}^2$.

We now show that $A = \emptyset$ is the only possibility compatible with the normalization (15). In particular then, the sequence (Ω_k) will exhaust all of $\mathbb{R}^2$.

Indeed, suppose by contradiction that $A = \mathbb{R}^2$. Then by Lemma 4.3, applied on a cover of $\mathbb{R}^2$ by balls of radius 1/2, a subsequence $v_k \to v_0$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$. In particular,

$$\int_{B_1(0)} r_k^2 f_k(v_k)\, v_k\, dx \to 0,$$

contradicting (15).

Thus, $A = \emptyset$ and $c_k(y, r) \to \infty$ as $k \to \infty$ for any $y \in \mathbb{R}^2$, any $r > 0$. ∎

Express

$$r_k^2 f_k(v_k) = \exp(\varphi_k(v_k) + \log v_k + 2 \log r_k).$$

Fix $y \in \mathbb{R}^2$, $r > 0$, and decompose $v_k = w_k + c_k$ on $B_r(y)$ as above. Note that v_k and w_k are superharmonic on $B_r(y)$ on account of Eq. (14). Thus, by the mean value theorem and Lemma 4.2, we conclude that

$$v_k(x) \geqslant c_k\left(x, \frac{r}{2}\right) \geqslant c_k(y, r) - C$$

for all $x \in B_{r/2}(y)$ with a constant C independent of y, r, and k. Hence also

$$w_k(x) = v_k(x) - c_k(y, r) \geqslant -C.$$

Therefore, and since $c_k \to \infty$ $(k \to \infty)$, we can uniformly bound

$$\frac{w_k}{c_k} \geqslant -\frac{1}{2}, \qquad v_k \geqslant c_k - C \geqslant s_0 \tag{19}$$

on $B_{r/2}(y)$ for large k. Then

$$\log v_k = \log c_k + \log\left(1 + \frac{w_k}{c_k}\right)$$

and

$$\varphi_k(v_k) = \varphi_k(c_k) + \varphi_k'(c_k)\, w_k + R_k(c_k, w_k),$$

where

$$0 \leqslant R_k(c_k, w_k) \leqslant 4\pi w_k^2$$

on $B_{r/2}(y)$ for sufficiently large k.

Tentatively define

$$\eta_k = \varphi_k(c_k) + \varphi_k'(c_k)\, w_k + 2 \log(c_k r_k).$$

Then, η_k satisfies the equation

$$-\Delta\eta_k = -\varphi_k'(c_k)\,\Delta w_k = -\varphi_k'(c_k)\,\Delta v_k = \varphi_k'(c_k)\, r_k^2 f_k(v_k) = V_k e^{\eta_k} \quad (20)$$

on $B_r(y)$, where

$$V_k = \frac{\varphi_k'(c_k)}{c_k} \exp\left(R_k(c_k, w_k) + \log\left(1 + \frac{w_k}{c_k}\right)\right) \geqslant 0.$$

Moreover, we have

$$r_k^2 f_k(v_k)\, v_k = W_k e^{\eta_k},$$

where

$$W_k = \exp\left(R_k(c_k, w_k) + 2\log\left(1 + \frac{w_k}{c_k}\right)\right) \geqslant \left(1 + \frac{w_k}{c_k}\right)^2.$$

LEMMA 4.5. $\limsup_{k\to\infty} \int_{B_{r/2}(y)} e^{\eta_k}\, dx \leqslant 8.$

Proof. By (19) we can estimate $W_k \geqslant \frac{1}{4}$ on $B_{r/2}(y)$ for large k. Hence, from (13) we deduce that with error $o(1) \to 0$ as $k \to \infty$ there holds

$$\begin{aligned}\int_{B_{r/2}(y)} e^{\eta_k}\, dx &\leqslant 4\int_{B_{r/2}(y)} W_k e^{\eta_k}\, dx = 4\int_{B_{r/2}(y)} r_k^2 f_k(v_k)\, v_k\, dx \\ &\leqslant 4\int_{\Omega_k} r_k^2 f_k(v_k)\, v_k\, dx = 4\int_{\Omega} f_k(u_k)\, u_k\, dx \\ &= 8\beta + o(1) \leqslant 8\end{aligned}$$

for large k. ∎

LEMMA 4.6. $\sup_{B_{r/4}(y)} \eta_k \leqslant C$, *uniformly for sufficiently large* k.

Proof. It suffices to consider $r = 1$. The general case then follows by a covering argument. We verify that the hypotheses of [5, Corollary 4] are satisfied for Eq. (20). Fix $p > 1$, say $p = 4$, with dual exponent $p' = \frac{4}{3}$. Then, as in the proof of Lemma 4.3, for $0 < a \leqslant a_1(p)$ the family

$$0 \leqslant V_k \leqslant \frac{\varphi_k'(c_k)}{c_k}\, e^{Cw_k^2} \leqslant 8\pi e^{Cw_k^2}, \qquad k \geqslant k_0, \quad (21)$$

is bounded in L^p on $B_{1/2}(y)$.

Moreover, denoting $s^+ = \max\{s, 0\}$ for $s \in \mathbb{R}$, from Jensen's inequality and Lemma 4.5 we have

$$\fint_{B_{1/2}(y)} \eta_k^+ \, dx \leqslant \log\left(1 + \fint_{B_{1/2}(y)} e^{\eta_k} \, dx\right) \leqslant C.$$

Finally, by (19) for sufficiently large k there holds

$$\varphi_k'(c_k) \leqslant 8\pi c_k \leqslant 8\pi(v_k + C) \leqslant C v_k.$$

Hence, we can estimate

$$\begin{aligned}\int_{B_{1/2}(y)} V_k e^{\eta_k} \, dx &\leqslant C \int_{B_{1/2}(y)} W_k e^{\eta_k} \, dx \\ &= C \int_{B_{1/2}(y)} r_k^2 v_k f_k(v_k) \, dx \leqslant Ca < \frac{4\pi}{p'},\end{aligned}$$

if $0 < a \leqslant a_1(p)$ is chosen sufficiently small. The desired conclusion now follows from [5, Corollary 4]. ∎

In the following we assume that $0 < a < a_1$, where $a_1 = a_1(p)$ has been chosen as in the proof of Lemma 4.6 to guarantee that V_k is bounded in $L^p(B_r(y))$ for some $p > 2$, say, $p = 4$.

Lemma 4.7. *$w_k \to 0$ in $C^1(B_{r/5}(y))$ as $k \to \infty$.*

Proof. By Lemma 4.1 there holds $w_k \rightharpoonup 0$ weakly in $H^1(B_r(y))$. Moreover, the uniform bound on η_k from Lemma 4.6 and the uniform L^p-bound for V_k imply that, as $k \to \infty$,

$$-\Delta w_k = \frac{-\Delta \eta_k}{\varphi_k'(c_k)} \to 0 \qquad \text{in } L^p(B_{r/4}(y)).$$

For any $\psi \in C_0^\infty(B_{r/4}(y))$ then the function $w_k \psi$ satisfies

$$-\Delta(w_k \psi) = -(\Delta w_k)\, \psi - 2\nabla w_k \, \nabla \psi - w_k \, \Delta \psi \rightharpoonup 0 \tag{22}$$

weakly in $L^2(B_{r/4}(y))$. Hence $w_k \psi \rightharpoonup 0$ weakly in $H^{2,2}(B_{r/4}(y))$ and therefore strongly in $W^{1,p}(B_{r/4}(y))$. Since ψ is arbitrary, we conclude that $w_k \to 0$ strongly in $W^{1,p}_{\mathrm{loc}}(B_{r/4}(y))$.

Going back to (22), then $-\Delta(w_k \psi) \to 0$ in $L^p_{\mathrm{loc}}(B_{r/4}(y))$, and it follows that $w_k \to 0$ in $W^{2,p}_{\mathrm{loc}}(B_{r/4}(y))$. Since $W^{2,p} \hookrightarrow C^1$, the claim follows. ∎

In particular, Lemma 4.7 implies that for arbitrary fixed $x_0 \in \mathbb{R}^2$ and any $r > 0$, $y \in \mathbb{R}^2$ we have

$$\begin{aligned}\eta_k &= \varphi_k(c_k) + \varphi_k'(c_k)\, w_k + 2\log(c_k r_k)\\ &= \varphi_k(v_k) + 2\log(v_k(x_0)) + 2\log r_k + o(1)\end{aligned}$$

on $B_{r/5}(y)$, where $o(1) \to 0$ in $C^1(B_{r/5}(y))$ as $k \to \infty$.

Indeed, letting $R = 5(|y - x_0| + r)$ so that $B_r(y) \subset B_{R/5}(x_0)$ and letting

$$v_k = w_k + c_k(y, r) = w_k^0 + c_k(x_0, R)$$

be the respective decompositions of v_k, from Lemma 4.7 we conclude that

$$\begin{aligned}|v_k(x_0) - c_k(y, r)| &\leqslant |v_k(x_0) - v_k(y)| + |v_k(y) - c_k(y, r)|\\ &= |w_k^0(x_0) - w_k^0(y)| + |w_k(y)| \to 0\end{aligned}$$

as $k \to \infty$.

Fixing any point $x_0 \in \mathbb{R}^2$, and letting η_k be defined relative to the decomposition (16) of v_k on $B_R(0)$ for a suitable sequence $R = R^{(k)} \to \infty$, we then obtain a sequence (η_k) which is well-defined on any domain $D \subset\subset \mathbb{R}^2$ for sufficiently large k and differs from the function

$$\eta_k^{(1)} = \varphi_k(v_k) + 2\log(v_k(x_0)) + 2\log r_k$$

by an error $o(1) \to 0$ in $C^1(D)$ as $k \to \infty$.

More concisely, we may represent η_k as

$$\eta_k = \varphi(v_k) + 2\log(r_k v_k) + o(1).$$

with error $o(1) \to 0$ locally C^1-uniformly on $\mathbb{R}^2$.

Moreover, we may achieve that

$$V_k \to V = \lim_{k \to \infty} \frac{\varphi_k'(c_k)}{c_k} \equiv 8\pi \in \mathbb{R}$$

and $W_k \to 1$ locally uniformly on $\mathbb{R}^2$ as $k \to \infty$ for this choice of radii $R^{(k)}$.

In view of (20) and Lemma 4.5 we can now invoke the result [5, Theorem 3] and its improvement [11, Theorem, p. 1256] to conclude that one of the following must occur. As $k \to \infty$, either

(a) $\eta_k \to -\infty$ locally uniformly on $\mathbb{R}^2$; or

(b) there are points $x_1, ..., x_L \in \mathbb{R}^2$ and numbers $m_1, ..., m_L \in \mathbb{N}$ such that $\eta_k \to -\infty$ locally uniformly on $\mathbb{R}^2 \setminus \{x_1, ..., x_L\}$ and $V_k e^{\eta_k}\, dx \rightharpoonup \sum_{l=1}^{L} 8\pi m_l \delta_{x_l}$ weakly in the sense of measures; or

(c) $\eta_k \to \eta$ locally uniformly in $C^{1,\alpha}$ for any $\alpha < 1$.

But, in view of our normalization (15) we have

$$a \geqslant \int_{B_1(0)} W_k e^{\eta_k}\, dx = \int_{B_1(0)} r_k^2 v_k f_k(v_k)\, dx \geqslant \frac{a}{2}.$$

Therefore, and since $W_k \to 1$ locally uniformly as $k \to 0$, Case (a) is ruled out. Case (b) is impossible in view of Lemma 4.6. Thus, only possibility (c) remains; that is, $\eta_k \to \eta$ in $C^{1,\alpha}$, and hence also $\eta_k^{(1)} \to \eta$ locally C^1-uniformly as $k \to \infty$, where η solves the Liouville [13] equation

$$-\Delta\eta = V_0 e^\eta \qquad \text{on } \mathbb{R}^2 \tag{23}$$

with $V_0 \equiv 8\pi$. Moreover, $e^\eta \in L^1(\mathbb{R}^2)$; in fact, for any $L \in \mathbb{N}$ there holds

$$\int_{B_L(0)} e^\eta\, dx = \lim_{k\to\infty} \int_{B_L(0)} W_k e^{\eta_k}\, dx = \lim_{k\to\infty} \int_{B_{Lr_k}(x_k)} u_k f_k(u_k)\, dx \leqslant 2\beta.$$

By a result of Chen and Li [8, Lemma 1], then $\eta(x) \to -\infty$ as $|x| \to \infty$. Choose $x_0 \in \mathbb{R}^2$, $r_0 > 0$ such that $\eta(x_0) = \sup_{\mathbb{R}^2} \eta = -2 \log r_0$. The scaled and shifted function

$$\tilde{\eta}(x) = \eta(x_0 + r_0 x/\sqrt{V_0}) + 2 \log r_0$$

then satisfies the equation

$$-\Delta\tilde{\eta} = e^{\tilde{\eta}} \qquad \text{on } \mathbb{R}^2$$

with $\tilde{\eta}(x) \leqslant \tilde{\eta}(0) = 0$ and $\int_{\mathbb{R}^2} e^{\tilde{\eta}}\, dx < \infty$.

Hence by the result of Chen and Li [8] it follows that

$$\tilde{\eta}(x) = \log \frac{1}{(1+|x|^2/8)^2};$$

in particular,

$$\int_{\mathbb{R}^2} e^{\tilde{\eta}}\, dx = V_0 \int_{\mathbb{R}^2} e^\eta\, dx = 8\pi$$

and thus

$$\int_{\mathbb{R}^2} e^\eta\, dx = \lim_{L\to\infty} \lim_{k\to\infty} \int_{B_{Lr_k}(x_k)} u_k f_k(u_k)\, dx = 1. \tag{24}$$

Redefining

$$\tilde{x}_k = x_k + r_k x_0, \qquad \tilde{r}_k = r_k r_0/\sqrt{V_0},$$

we then obtain that

$$\varphi_k(u_k(\tilde{x}_k + \tilde{r}_k x)) + 2\log(u_k(\tilde{x}_k)) + 2\log \tilde{r}_k + \log 8\pi$$
$$= \eta_k^{(1)}(x_0 + r_0 x/\sqrt{V_0}) + 2\log r_0 \to \tilde{\eta}(x) = \log \frac{1}{(1 + |x|^2/8)^2}$$

locally C^1-uniformly on $\mathbb{R}^2$, proving the last assertion in Theorem 1.1.

5. ENERGY ESTIMATE

It remains to show that $\beta = \frac{1}{2}$.

The argument is particularly elegant in the radially symmetric case. Indeed, if $\Omega = B_R(0)$, by a result of Gidas *et al.* [10] the function u_k is radially symmetric and radially non-increasing, and, in particular, $x_k = 0$ for any k. The proof of Lemma 5.1 below then does not require the Fubini-type analysis that we use to estimate the oscillation of u_k on suitable circles $\partial B_r(x_k)$, and also the auxiliary Lemma 5.2 is not needed to obtain the improved, final form of that result, Lemma 5.3. Moreover, Lemma 5.4 and Lemma 5.5 are superfluous, as the functions Φ_k and $\bar{\Phi}_k$ introduced below are identical in the radial case, and we may conclude as in Lemma 5.6.

In the case of a general domain Ω the argument is slightly more technical. Let x_k, r_k be determined as above such that

$$\eta_k^{(1)}(x) = \varphi_k(u_k(x_k + r_k x)) + 2\log(r_k u_k(x_k + r_k x)) \to \eta = \eta^{(1)}$$

locally C^1-uniformly, where

$$\int_{\mathbb{R}^2} e^{\eta}\, dx = 1.$$

Observe that now for convenience we use the alternative representation of $\eta_k^{(1)}$ and the original x_k, r_k rather than $\tilde{x}_k$, $\tilde{r}_k$.

LEMMA 5.1. *There exists radii $t_k > 0$, $k \in \mathbb{N}$, such that with error $o(1) \to 0$ as $k \to \infty$ there holds $t_k \to 0$, $r_k/t_k \to 0$,*

$$\int_{\Omega \setminus B_{t_k}(x_k)} u_k f_k(u_k)\, dx \to 0,$$

$$\int_{\Omega \setminus B_{t_k/2}(x_k)} |\nabla u_k|^2\, dx \geqslant 1 - o(1)$$

while

$$\inf_{B_{t_k}(x_k)} u_k \to \infty.$$

Moreover, $\operatorname{dist}(x_k, \partial\Omega) \geqslant 2t_k$.

Proof. For any number $\lambda > 0$ such that $2\beta - 1 < \lambda < 1$ choose $t_k = t_k^\lambda$ such that

$$\int_{\Omega \setminus B_{t_k}(x_k)} |\nabla u_k|^2 \, dx = \lambda$$

for each k. Note that Lemma 3.3 implies $t_k \to 0$ as $k \to \infty$. Moreover, $r_k/t_k \to 0$ $(k \to \infty)$. Else there exists $L \in \mathbb{N}$ such that $t_k \leqslant Lr_k$ for a sequence $k \to \infty$. Hence, by Lemma 4.7, with error $o(1) \to 0$ as $k \to \infty$ we obtain

$$\begin{aligned} 2\beta + o(1) &= \int_\Omega |\nabla u_k|^2 \, dx \\ &\leqslant \int_{B_{Lr_k}(x_k)} |\nabla u_k|^2 \, dx + \int_{\Omega \setminus B_{t_k}(x_k)} |\nabla u_k|^2 \, dx = \lambda + o(1), \end{aligned}$$

yielding the contradiction $1 \leqslant 2\beta \leqslant \lambda < 1$.

By Fubini's theorem, for any $R > 0$ there holds

$$\begin{aligned} &\log 2 \cdot \inf_{R \leqslant r \leqslant 2R} \left(r \int_{\partial B_r(x_k)} |\nabla u_k|^2 \, do \right) \\ &\quad \leqslant \int_R^{2R} \left(r \int_{\partial B_r(x_k)} |\nabla u_k|^2 \, do \right) \frac{dr}{r} \leqslant \int_\Omega |\nabla u_k|^2 \, dx \leqslant 2\beta + o(1), \end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$. Hence there exist $t'_k \in [t_k/2, t_k]$, $t''_k \in [t_k, 2t_k]$ such that for $r = t'_k$ or $r = t''_k$ we have

$$r \int_{\partial B_r(x_k)} |\nabla u_k|^2 \, do \leqslant \frac{2\beta}{\log 2} + o(1) \leqslant 4$$

for large k. Taking account of the estimate

$$\begin{aligned} \left(\sup_{\partial B_r(x_k)} u_k - \inf_{\partial B_r(x_k)} u_k \right)^2 &\leqslant \left(\int_{\partial B_r(x_k)} |\nabla u_k| \, do \right)^2 \\ &\leqslant 2\pi r \int_{\partial B_r(x_k)} |\nabla u_k|^2 \, do, \end{aligned}$$

we conclude that

$$\operatorname{osc}_{\partial B_r(x_k)} u_k = \sup_{\partial B_r(x_k)} u_k - \inf_{\partial B_r(x_k)} u_k \leqslant \sqrt{8\pi} =: C_0$$

for $r = t'_k$ or $r = t''_k$ and sufficiently large k.

Also observe that the estimate $t'_k \leqslant t_k \leqslant t''_k$ and our choice of t_k imply that

$$\int_{\Omega \setminus B_{t''_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant \lambda \leqslant \int_{\Omega \setminus B_{t'_k}(x_k)} |\nabla u_k|^2 \, dx$$

for all k. In particular,

$$\int_{B_{t'_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant \int_{\Omega} |\nabla u_k|^2 \, dx - \lambda = 2\beta - \lambda + o(1) \leqslant \gamma < 1$$

for large k.

This implies that $\sup_{\partial B_{t'_k}(x_k)} u_k \to \infty$ as $k \to \infty$. Indeed, arguing by contradiction, suppose that

$$\sup_{\partial B_{t'_k}(x_k)} u_k \leqslant C_1$$

for all k. Then the sequence

$$\tilde{u}_k = \max\{0, u_k - C_1\} \in H^1_0(B_{t'_k}(x_k))$$

satisfies $u_k \leqslant \tilde{u}_k + C_1$. Hence there holds

$$u_k f_k(u_k) \leqslant C_2 e^{4\pi\gamma^{-1}\tilde{u}_k^2}$$

with a uniform constant C_2. Moreover, we have

$$\int_{B_{t'_k}(x_k)} |\nabla \tilde{u}_k|^2 \, dx \leqslant \int_{B_{t'_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant \gamma$$

for large k. The Moser–Trudinger inequality then implies that

$$\int_{B_{t'_k/2}(x_k)} u_k f_k(u_k) \, dx \leqslant C_2 \int_{B_{t'_k}(x_k)} e^{4\pi\gamma^{-1}\tilde{u}_k^2} \, dx \leqslant C t'^2_k \to 0$$

as $k \to \infty$, whereas (24) gives

$$1 = \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{Lr_k}(x_k)} u_k f_k(u_k) \, dx \leqslant \liminf_{k \to \infty} \int_{B_{t'_k/2}(x_k)} u_k f_k(u_k) \, dx.$$

We now claim that also $\inf_{\partial B_{t''_k}(x_k)} u_k \to \infty$ as $k \to \infty$. Indeed, we have

$$\begin{aligned}\sup_{\partial B_{t'_k}(x_k)} u_k - \inf_{\partial B_{t''_k}(x_k)} u_k &\leqslant \operatorname{osc}_{\partial B_{t'_k}(x_k)} u_k + \operatorname{osc}_{\partial B_{t''_k}(x_k)} u_k \\ &\quad + \inf_{\partial B_{t'_k}(x_k)} u_k - \sup_{\partial B_{t''_k}(x_k)} u_k \\ &\leqslant 2C_0 + \inf_{\partial B_{t'_k}(x_k)} u_k - \sup_{\partial B_{t''_k}(x_k)} u_k,\end{aligned}$$

and since $t''_k \leqslant 2t_k \leqslant 4t'_k$ the latter can be estimated

$$\begin{aligned}&\inf_{\partial B_{t'_k}(x_k)} u_k - \sup_{\partial B_{t''_k}(x_k)} u_k \\ &\quad\leqslant \frac{1}{2\pi} \int_{\partial B_1(0)} (u_k(x_k + t'_k \xi) - u_k(x_k + t''_k \xi))\, do(\xi) \\ &\quad\leqslant C \int_{\partial B_1(0)} \int_{t'_k}^{t''_k} |\nabla u_k(x_k + r\xi)|\, dr\, do(\xi) \\ &\quad\leqslant C \left(\int_\Omega |\nabla u_k|^2\, dx \right)^{1/2} \leqslant C_3,\end{aligned}$$

uniformly in k. Hence $\inf_{\partial B_{t''_k}(x_k)} u_k \to \infty$ and therefore, since $f_k(u_k) \geqslant 0$, also

$$\inf_{B_{t''_k}(x_k)} u_k \to \infty$$

by the maximum principle.

Repeating the above argument with $T_k = \operatorname{dist}(x_k, \partial\Omega)$ instead of t_k and suitable numbers T'_k, T''_k satisfying $T_k/4 \leqslant T'_k \leqslant T_k/2 \leqslant T_k \leqslant T''_k \leqslant 2T_k$, we obtain the estimate

$$\sup_{\partial B_{T'_k}(x_k)} u_k \leqslant \inf_{\partial B_{T''_k}(x_k)} u_k + C_4 = C_4.$$

Hence it follows that $t''_k \leqslant T'_k$ for large k, and therefore

$$2t''_k \leqslant 2T'_k \leqslant T_k = \operatorname{dist}(x_k, \partial\Omega).$$

Finally, since $\operatorname{osc}_{\partial B_{t''_k}(x_k)} u_k \leqslant C_0$ it also follows that

$$1 \leqslant \alpha_k := \sup_{\partial B_{t''_k}(x_k)} u_k \Big/ \inf_{\partial B_{t''_k}(x_k)} u_k \to 1$$

as $k \to \infty$. Thus, for any k the function u'_k, given by

$$u'_k = \sup_{\partial B_{t''_k}(x_k)} u_k \qquad \text{in } B_{t''_k}(x_k)$$

and

$$u'_k = \max\{u_k, \min\{\alpha_k u_k, \sup_{\partial B_{t''_k}(x_k)} u_k\}\} \qquad \text{in } \Omega \backslash B_{t''_k}(x_k),$$

belongs to $H^1_0(\Omega)$ and

$$\int_\Omega |\nabla u'_k|^2 \, dx \leqslant \alpha_k^2 \int_{\Omega \backslash B_{t''_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant \alpha_k^2 \lambda \leqslant C < 1$$

for sufficiently large k. Clearly, $u'_k \rightharpoonup 0$ weakly in $H^1_0(\Omega)$ and $u_k \leqslant u'_k$ in $\Omega \backslash B_{t''_k}(x_k)$. Thus, by the Moser–Trudinger inequality, observing that our functions $f_k = f_k(s)$ are increasing for $s \geqslant s_1$ with $s_1 \geqslant 0$ independent of k, with error $o(1) \to 0$ as $k \to \infty$ there holds

$$\int_{\Omega \backslash B_{t''_k}(x_k)} u_k f_k(u_k) \, dx \leqslant \int_\Omega u'_k f_k(u'_k) \, dx + o(1) \to 0$$

as $k \to \infty$.

Letting $\lambda = \lambda_k \to 1$ suitably and replacing t_k by t''_k, we thus obtain the assertion of the lemma. ∎

Lemma 5.2. *For any sequence of radii $s_k > 0$ such that $s_k / r_k \to \infty$ there holds*

$$\lim_{k \to \infty} \int_{B_{2s_k} \backslash B_{s_k}(x_k)} u_k f_k(u_k) \, dx = 0.$$

Proof. By Lemma 5.1 it suffices to consider $s_k \leqslant t_k$. We argue by contradiction. Then for any sufficiently small number $0 < a < a_1$, with a_1 as determined in Section 4, and any $L \in \mathbb{N}$ there exist points $y_k \in B_{t_k}(x_k)$ such that $2s_k = |x_k - y_k| \geqslant L r_k$ and

$$\int_{B_{s_k}(y_k)} u_k f_k(u_k) \, dx \geqslant a.$$

Decreasing s_k further, and possibly choosing new points $y_k \in B_{t_k} \backslash B_{Lr_k}(x_k)$, still satisfying $|x_k - y_k| \geqslant 2s_k$ for the new s_k, we can achieve that

$$\frac{a}{2} \leqslant \int_{B_{s_k}(y_k)} u_k f_k(u_k)\, dx \leqslant \sup_{s,\, y} \int_{B_s(y)} u_k f_k(u_k)\, dx \leqslant a,$$

where the supremum is taken over all $y \in B_{t_k} \backslash B_{Lr_k}(x_k)$ and $s \leqslant \min\{|x_k - y|/2, s_k\}$.

Letting $L \to \infty$, we then pass to a diagonal subsequence satisfying the above for $k \geqslant k_0(L)$. Finally, remark that Lemma 5.1 implies that $s_k \leqslant |x_k - y_k| \leqslant t_k \to 0$ as $k \to \infty$.

Now we distinguish two cases.

Case 1. Suppose there holds

$$\frac{|x_k - y_k|}{s_k} \to \infty \qquad (k \to \infty).$$

Then, given $L \in \mathbb{N}$, it follows that $B_{Lr_k}(x_k) \cap B_{Ls_k}(y_k) = \emptyset$ for sufficiently large k and our previous argument may be applied to show that a sequence

$$\eta_k^{(2)}(y) = \varphi_k(u_k(y_k + s_k y)) + 2\log(s_k u_k(y_k)) \to \eta^{(2)}$$

locally C^1-uniformly on $\mathbb{R}^2$ as $k \to \infty$, where $\eta^{(2)}$ solves the equation

$$-\Delta \eta^{(2)} = 8\pi e^{\eta^{(2)}} \qquad \text{in } \mathbb{R}^2$$

with

$$1 = \int_{\mathbb{R}^2} e^{\eta^{(2)}}\, dx = \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{Ls_k}(y_k)} u_k f_k(u_k)\, dx.$$

Thus, with error $o(1) \to 0$ as $k \to \infty$, for any $L \in \mathbb{N}$ we have

$$\begin{aligned} 2\beta &= \int_\Omega u_k f_k(u_k)\, dx + o(1) \\ &\geqslant \int_{B_{Lr_k}(x_k)} \cdots + \int_{B_{Ls_k}(y_k)} \cdots + o(1) = \int_{B_L(0)} (e^{\eta^{(1)}} + e^{\eta^{(2)}})\, dx + o(1). \end{aligned}$$

Upon letting $L \to \infty$, we conclude that $\beta \geqslant 1$ contrary to assumption.

Thus we are left with

Case 2. There exists a constant C such that

$$|x_k - y_k| \leqslant C s_k, \qquad k \in \mathbb{N}.$$

In this case we may scale with s_k around x_k to obtain the sequence

$$v_k(x) = v_k^{(3)}(x) = u_k(x_k + s_k x), \qquad k \in \mathbb{N}.$$

Note that $r_k/s_k \to 0$ as $k \to \infty$. Thus for any $y \in \mathbb{R}^2 \setminus \{0\}$ and any $r \leqslant \min\{1, |y|/2\}$ we have

$$\int_{B_r(y)} s_k^2 v_k f_k(v_k)\, dx \leqslant a$$

if $k \geqslant k_0(y)$. Splitting

$$v_k = w_k + c_k$$

on $B_r(y)$ as before, where

$$c_k = c_k(y, r) = \fint_{B_r(y)} v_k(x)\, dx,$$

by Lemma 4.1 for any $r < \min\{1, |y|/2\}$ we have

$$\limsup_{k \to \infty} \int_{B_r(y)} |\nabla w_k|^2\, dx \leqslant a.$$

Moreover, since $s_k \leqslant t_k$, from Lemma 5.1 we deduce that

$$c_k(0, 1) = \fint_{B_{s_k}(x_k)} u_k\, dx \geqslant \inf_{B_{t_k}(x_k)} u_k \to \infty$$

and hence from Lemma 4.2 that

$$c_k(y, r) \to \infty \qquad \text{as} \quad k \to \infty,$$

locally uniformly in $y \in \mathbb{R}^2$, $r > 0$.

In particular, the rescaled domains

$$\Omega_k = \Omega_k^{(3)} = \{x \in \mathbb{R}^2;\ x_k + s_k x\}$$

exhaust $\mathbb{R}^2$ as $k \to \infty$.

As before, let

$$\eta_k = \eta_k^{(3)} = \varphi_k(c_k) + \varphi_k'(c_k)\, w_k + 2 \log(c_k s_k) \qquad \text{on } B_r(y).$$

Then we have

$$-\Delta \eta_k = V_k e^{\eta_k},$$

where

$$V_k = \frac{\varphi_k'(c_k)}{c_k}\left(1+\frac{w_k}{c_k}\right) e^{R_k(c_k,\, w_k)}.$$

For $y \in \mathbb{R}^2 \setminus \{0\}$ and $r < \min\{1, |y|/2\}$, Lemmas 4.5, 4.6, and 4.7 then may be carried over unchanged from our previous construction to conclude that $w_k \to 0$ in $C^1(B_{r/5}(y))$ as $k \to \infty$ on any such ball $B_r(y)$.

It follows that

$$\eta_k^{(3)} = \varphi_k(v_k) + 2\log(v_k s_k) + o(1),$$

where $o(1) \to 0$ locally C^1-uniformly on $\mathbb{R}^2 \setminus \{0\}$, and our initial assumption implies that

$$\liminf_{k \to \infty} \int_{B_1((y_k - x_k)/s_k)} e^{\eta_k^{(3)}}\, dx = \liminf_{k \to \infty} \int_{B_{s_k}(y_k)} u_k f_k(u_k)\, dx \geqslant \frac{a}{2} > 0.$$

From the results in [5, 11], and using our normalization, we then deduce that $\eta_k^{(3)} \to \eta^{(3)}$ locally C^1-uniformly on $\mathbb{R}^2 \setminus \{0\}$ where $\eta^{(3)}$ solves

$$-\Delta \eta^{(3)} = 8\pi e^{\eta^{(3)}} \qquad \text{in} \quad \mathbb{R}^2 \setminus \{0\} \tag{25}$$

with $e^{\eta^{(3)}} \in L^1(\mathbb{R}^2 \setminus \{0\})$.

We claim that $\eta^{(3)}$ may be extended as a distribution solution $\eta^{(3)} \in \bigcap_{q<2} W^{1,q}(B_1(0))$ of the differential inequality

$$-\Delta \eta^{(3)} \geqslant 8\pi(e^{\eta^{(3)}} + \delta_0) \qquad \text{on } B_1(0).$$

Indeed, let $g_k \in H_0^1(B_1(0))$ solve

$$-\Delta g_k = V_k e^{\eta_k} \qquad \text{in } B_1(0).$$

Observe that with uniform constants C for large k in view of (19) we have

$$V_k = \frac{\varphi_k'(c_k)}{v_k} W_k \leqslant \frac{\varphi_k'(c_k)}{c_k - C} W_k \leqslant C W_k.$$

Hence from (13) we conclude that

$$\int_{B_1(0)} V_k e^{\eta_k}\, dx \leqslant C \int_{B_1(0)} W_k e^{\eta_k}\, dx = C \int_{B_{s_k}(x_k)} u_k f_k(u_k)\, dx \leqslant C,$$

and (g_k) is bounded in $W_0^{1,q}(B_1(0))$ for any $q < 2$. But $h_k = \eta_k - g_k$ is harmonic on $B_1(0)$ with boundary data $h_k = \eta_k \to \eta^{(3)}$ in C^1 on $\partial B_1(0)$. Thus

$\eta_k = g_k + h_k \rightharpoonup \eta^{(3)}$ weakly in $W^{1,q}(B_1(0))$ for any $q < 2$. Moreover, for any testing function $0 \leqslant \psi \in C_0^\infty(B_1(0))$ we find

$$\begin{aligned}
\int_{B_1(0)} \eta^{(3)}(-\Delta\psi)\,dx
&= \lim_{k\to\infty} \int_{B_1(0)} V_k e^{\eta_k}\psi\,dx \\
&= \lim_{\varepsilon\to 0}\left(\lim_{k\to\infty} \int_{B_\varepsilon(0)} V_k e^{\eta_k}\psi\,dx + 8\pi \int_{B_1\setminus B_\varepsilon(0)} e^{\eta^{(3)}}\psi\,dx\right) \\
&= \psi(0) \lim_{\varepsilon\to 0}\lim_{k\to\infty} \int_{B_\varepsilon(0)} V_k e^{\eta_k}\,dx + 8\pi \int_{B_1(0)} e^{\eta^{(3)}}\psi\,dx.
\end{aligned}$$

Finally, for any $\varepsilon > 0$ by (24) we have

$$\lim_{k\to\infty} \int_{B_\varepsilon(0)} V_k e^{\eta_k}\,dx \geqslant 8\pi \lim_{L\to\infty}\lim_{k\to\infty} \int_{B_{Lr_k}(x_k)} u_k f_k(u_k)\,dx = 8\pi.$$

Thus $-\Delta\eta^{(3)} \geqslant 8\pi(e^{\eta^{(3)}} + \delta_0)$, as claimed.

In particular, $-\Delta\eta^{(3)} \geqslant 8\pi\delta_0$ on $B_1(0)$. Therefore, with $C_1 = \inf_{\partial B_1(0)} \eta^{(3)}$, we conclude that

$$\eta^{(3)}(x) \geqslant 4 \log \frac{1}{|x|} + C_1,$$

which yields the contradiction

$$\infty > \int_{B_1(0)} e^{\eta^{(3)}}\,dx \geqslant C \int_{B_1(0)} |x|^{-4}\,dx = \infty.$$

Thus, also Case 2 is ruled out and the proof is complete. ∎

In particular, as a consequence of Lemma 5.2 we may sharpen Lemma 5.1, as follows.

LEMMA 5.3. *There exist radii $t_k > 0$, $k \in \mathbb{N}$, such that with error $o(1) \to 0$ as $k \to \infty$ there holds $t_k \to 0$, $r_k/t_k \to 0$,*

$$\int_{\Omega\setminus B_{t_k}(x_k)} u_k f_k(u_k)\,dx = o(1),$$

$$\int_{\Omega\setminus B_{t_k}(x_k)} |\nabla u_k|^2\,dx \geqslant 1 + o(1),$$

while

$$\inf_{B_{t_k}(x_k)} u_k \to \infty.$$

Moreover, $\mathrm{dist}(x_k, \partial\Omega) \geqslant 2t_k$.

Proof. Repeat the construction in the proof of Lemma 5.1 but replace t_k by t'_k instead of t''_k at the end. Since $t'_k \leqslant t''_k$, the condition $\mathrm{dist}(x_k, \partial\Omega) \geqslant 2t'_k$ is immediate. Observing that $t'_k \geqslant t''_k/4$, from Lemma 5.2 we deduce that as $k \to \infty$

$$\int_{B_{t''_k} \setminus B_{t'_k}(x_k)} u_k f_k(u_k)\, dx \to 0$$

and hence that

$$\int_{\Omega \setminus B_{t'_k}(x_k)} u_k f_k(u_k)\, dx \to 0;$$

moreover, by construction

$$\int_{\Omega \setminus B_{t'_k}(x_k)} |\nabla u_k|^2\, dx \geqslant \lambda_k \to 1.$$

Finally, we have

$$\inf_{B_{t'_k}(x_k)} u_k \geqslant \inf_{B_{t''_k}(x_k)} u_k \to \infty. \qquad \blacksquare$$

Consider now the sequence

$$\eta_k = \eta_k^{(0)} = \varphi_k(u_k) + 2 \log u_k, \qquad k \in \mathbb{N},$$

in the original coordinates. For $r > 0$, $y \in \Omega$ also decompose

$$u_k = w_k^{(0)} + c_k \qquad \text{on } B_r(y),$$

where

$$c_k = c_k(y, r) = \fint_{B_r(y)} y_k\, dx = c_k^{(0)}(y, r).$$

LEMMA 5.4. *For any* $\varepsilon > 0$ *there exists a constant* $C_1 = C_1(\varepsilon)$ *depending on the sequence* (φ_k) *such that*

$$\lim_{L\to\infty} \limsup_{k\to\infty} \sup_{y} (s^2 e^{\eta_k^{(0)}(y)}) \leqslant \varepsilon$$

$$\lim_{L\to\infty} \limsup_{k\to\infty} \sup_{y} \|(|w_k^{(0)}| + s\, |\nabla w_k^{(0)}|)\|_{L^\infty(B_{s/10}(y))} \leqslant \varepsilon,$$

where the supremum is taken with respect to $y \in \Omega \backslash B_{Lr_k}(x_k)$ *such that* $u_k(y) \geqslant C_1$, *with* $s = |x_k - y|/2$ *and with* $w_k^{(0)} = u_k - c_k(y, s)$.

Proof. (i) Under the slightly stronger assumption that $c_k(y, s) \geqslant C_1$ for some sufficiently large number C_1, the assertion follows from the blow-up argument used in the proof of Lemma 5.2.

Indeed, suppose by contradiction that there is $\varepsilon > 0$ and points $y_k \in \Omega$ such that for any $L \in \mathbb{N}$ there holds $2s_k = |y_k - x_k| \geqslant Lr_k$ for $k \geqslant k_0(L)$, $c_k(y_k, s_k) \to \infty$ as $k \to \infty$, and such that either

$$s_k^2 e^{\eta_k^{(0)}(y_k)} \geqslant \varepsilon$$

or

$$\|(|w_k^{(0)}| + s_k\, |\nabla w_k^{(0)}|)\|_{L^\infty(B_{s_k/10}(y_k))} \geqslant \varepsilon.$$

Observe that the assumption $c_k(y_k, s_k) \to \infty$ in view of weak convergence $u_k \rightharpoonup 0$ in $H_0^1(\Omega)$ implies that $s_k \to 0$ as $k \to \infty$.

For the rescaled sequence

$$v_k(x) = v_k^{(4)}(x) = u_k(x_k + s_k x)$$

with associated sequences $w_k^{(4)}$ and $\eta_k^{(4)} = \varphi_k(v_k^{(4)}) + 2\log(s_k v_k^{(4)})$ as in the analysis of Case 2 in the proof of Lemma 5.2, thereby using the result of Lemma 5.2, we then obtain that

$$\|w_k^{(4)}\|_{C^1(B_{r/5}(y))} \to 0 \qquad \text{as} \quad k \to \infty$$

for any $y \in \mathbb{R}^2 \backslash \{0\}$, $r < \min\{1, |y|/2\}$, and hence, in particular, that

$$\begin{aligned} &\|(|w_k^{(0)}| + s_k\, |\nabla w_k^{(0)}|)\|_{L^\infty(B_{s_k/10}(y_k))} \\ &\quad = \|w_k^{(4)}\|_{C^1(B_{1/10}((y_k - x_k)/s_k))} \to 0 \qquad \text{as} \quad k \to \infty. \end{aligned}$$

In addition, from the Brezis–Merle result and Lemma 5.2, we obtain that $\eta_k^{(4)} \to -\infty$ locally uniformly on $\mathbb{R}^2 \setminus \{0\}$; in particular, as $k \to \infty$,

$$\eta_k^{(0)}(y_k) + 2\log s_k = \varphi_k(u_k(y_k)) + 2\log(s_k u_k(y_k))$$
$$= \eta_k^{(4)}\left(\frac{y_k - x_k}{s_k}\right) \to -\infty,$$

and we achieve the desired contradiction.

(ii) We now show that $c_k(y_k, s_k) \to \infty$ whenever $u_k(y_k) \to \infty$ as $k \to \infty$. Arguing indirectly, suppose that $c_k(y_k, s_k) \leqslant C$ uniformly in k.

Observe that Lemma 3.3 implies that $y_k \to x_0 = \lim_{k\to\infty} x_k$ and hence $s_k \to 0$ as $k \to \infty$. Indeed, for any $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \leqslant \psi \leqslant 1$ and $\psi \equiv 0$ near x_0, by Lemma 3.3 and the Moser–Trudinger inequality there holds

$$f_k(\psi u_k) \to 0 \qquad \text{in } L^2(\Omega)$$

and hence for any such ψ there holds

$$-\Delta(u_k\psi) = \psi f_k(u_k) - 2\nabla u_k \nabla\psi - u_k\,\Delta\psi \to 0 \qquad \text{in } L^2(\Omega)$$

as $k \to \infty$. Thus, for any such ψ we have $u_k\psi \to 0$ in $H^2 \cap H_0^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$ as $k \to \infty$, and it follows $y_k \to x_0 (k \to \infty)$, as claimed.

Scale

$$v_k(x) = v_k^{(5)}(x) = u_k(y_k + s_k x)$$

and decompose

$$v_k = w_k + c_k, \qquad c_k = c_k^{(5)} = c_k(y, r)$$

on $B_r(y)$, as usual. Observe that our assumption that $c_k^{(0)}(y_k, s_k) \leqslant C$ translates into the uniform bound $c_k(0, 1) \leqslant C$. Hence by Lemmas 4.1, 4.2, 4.3, and Lemma 5.2 we obtain that $v_k \to v_0$, $w_k \to 0$ in $H^1(B_{3/2}(0))$ as $k \to \infty$.

Thus, if $B_{3/2}(0)$ lies within our rescaled domain $\Omega_k^{(5)}$, upon decomposing $v_k = w_k + c_k$ on $B_{3/2}(0)$ and choosing a cut-off function $\psi \in C_0^\infty(B_{3/2}(0))$ such that $0 \leqslant \psi \leqslant 1$ and $\psi \equiv 1$ on $B_1(0)$, we obtain that, as $k \to \infty$.

$$-\Delta(w_k\psi) = \psi s_k^2 f_k(v_k) - 2\nabla w_k \nabla\psi - w_k\,\Delta\psi \to 0 \qquad \text{in } L^2(B_{3/2}(0))$$

and hence $w_k\psi \to 0$ in $H^2 \cap H_0^1(B_{3/2}(0)) \hookrightarrow C^0(\overline{B_{3/2}(0)})$.

If $B_{3/2}(0) \cap \partial\Omega^{(5)} \neq \emptyset$, it follows that $v_0 \equiv 0$ and we may argue in the same way for the function $v_k\psi$ instead of $w_k\psi$.

In particular, we conclude that

$$|v_k(0) - c_k(0, 1)| = |u_k(y_k) - c_k(y_k, s_k)| \to 0$$

and hence that

$$c_k(y_k, s_k) \to \infty$$

as $k \to \infty$, contradicting our initial assumption.
The proof is complete. ∎

Introducing polar coordinates (r, θ) around x_k, we next let

$$\bar{u}_k(r) = \fint_{\partial B_r(x_k)} u_k \, dv$$

denote the spherical mean of u_k, etc. We also write $\bar{u}_k(x) = \bar{u}_k(r)$ for $x \in \partial B_r(x_k)$ and denote

$$w_k = u_k - \bar{u}_k.$$

Expanding $\eta_k = \eta_k^{(0)}$ around $\bar{u}_k$ on $B_{t_k}(x_k)$, we find

$$\begin{aligned}\eta_k &= \varphi_k(u_k) + 2 \log u_k \\ &= \varphi_k(\bar{u}_k) + 2 \log \bar{u}_k + \left(\varphi_k'(\bar{u}_k) + \frac{2}{\bar{u}_k} \right) w_k + R_k(\bar{u}_k, w_k),\end{aligned}$$

where

$$(4\pi - o(1))\, w_k^2 \leqslant R_k(\bar{u}_k, w_k) \leqslant 4\pi w_k^2, \qquad \frac{\varphi_k'(\bar{u}_k)}{\bar{u}_k} + \frac{2}{\bar{u}_k^2} = 8\pi + o(1)$$

with error $o(1) \to 0$ uniformly as $k \to \infty$. Hence we can represent

$$\begin{aligned}\eta_k - \bar{\eta}_k &= \left(\varphi_k'(\bar{u}_k) + \frac{2}{\bar{u}_k} \right) w_k + R_k(\bar{u}_k, w_k) - \overline{R_k(\bar{u}_k, w_k)} \\ &= (8\pi + o(1))\, \bar{u}_k w_k + O(w_k^2 + \overline{w_k^2}).\end{aligned}$$

By Jensen's inequality, $e^{\bar{\eta}_k} \leqslant \overline{e^{\eta_k}}$. Expanding to second order around $\bar{\eta}_k$, observing that

$$\fint_{\partial B_r(x_k)} (\eta_k - \bar{\eta}_k)\, e^{\bar{\eta}_k} \, do = 0,$$

for $0<r<t_k$ then we obtain

$$0 \leqslant \int_{\partial B_r(x_k)} (e^{\eta_k} - e_k^{\eta})\, do \leqslant \tfrac{1}{2} \int_{\partial B_r(x_k)} |\eta_k - \bar{\eta}_k|^2 \max\{e^{\eta_k}, e^{\bar{\eta}_k}\}\, do$$

$$\leqslant C \int_{\partial B_r(x_k)} (\bar{u}_k^2 w_k^2 + |R_k(\bar{u}_k, w_k) - \overline{R_k(\bar{u}_k, w_k)}|^2) \max_{\partial B_r(x_k)} e^{\eta_k}\, do.$$

Observe that Lemmas 5.3 and 5.4 imply that $|w_k| \leqslant \bar{u}_k$ on $B_{t_k}(x_k)$ and hence

$$\int_{\partial B_r(x_k)} |R_k(\bar{u}_k, w_k) - \overline{R_k(\bar{u}_k, w_k)}|^2\, do$$

$$\leqslant C \int_{\partial B_r(x_k)} (w_k^4 + (\overline{w_k^2})^2)\, do \leqslant C \int_{\partial B_r(x_k)} \bar{u}_k^2 w_k^2\, do$$

for $r \leqslant t_k$ and all sufficiently large k.

By Lemma 5.4 and Poincaré's inequality then it follows that

$$0 \leqslant \int_{\partial B_r(x_k)} (e^{\eta_k} - e_k^{\eta})\, do \leqslant C \max_{\partial B_r(x_k)} (e^{\eta_k} r^2)\, \bar{u}_k^2 \left(r^{-2} \int_{\partial B_r(x_k)} w_k^2\, do \right)$$

$$\leqslant o(1) \int_{\partial B_r(x_k)} \bar{u}_k^2\, |\nabla w_k|^2\, do, \tag{26}$$

where $o(1) \to 0$ uniformly for $r \in [Lr_k, t_k]$ as $k_0(L) \leqslant k \to \infty$ and $L \to \infty$.

LEMMA 5.5. $\limsup_{k\to\infty} \int_{B_{t_k}(x_k)} \bar{u}_k^2\, |\nabla w_k|^2\, dx \leqslant C.$

Proof. Let $T_k = \operatorname{dist}(x_k, \partial\Omega) \geqslant 2t_k$. By a Fubini-type argument as in the proof of Lemma 5.1 there exist radii T'_k with $T_k/2 \leqslant T'_k \leqslant T_k$ such that

$$\sup_{\partial B_{T'_k}(x_k)} u_k^2 + T'_k \int_{\partial B_{T'_k}(x_k)} |\nabla u_k|^2\, dx \leqslant C,$$

uniformly in k. For ease of notation we now replace T_k by T'_k. It suffices to bound

$$\int_{B_{T_k}(x_k)} \bar{u}_k^2\, |\nabla w_k|^2\, dx.$$

Compute

$$-\Delta \frac{\bar{u}_k^2 w_k^2}{2} + \bar{u}_k^2 \, |\nabla w_k|^2 = \bar{u}_k w_k^2(-\Delta \bar{u}_k) + \bar{u}_k^2 w_k(-\Delta w_k)$$

$$- |\nabla \bar{u}_k|^2 \, w_k^2 - 4\bar{u}_k w_k \, \nabla \bar{u}_k \, \nabla w_k. \tag{27}$$

The Laplace operator commutes with the spherical mean. Thus

$$\begin{aligned}
&\bar{u}_k w_k^2(-\Delta \bar{u}_k) + \bar{u}_k^2 w_k(-\Delta w_k) \\
&\quad = \bar{u}_k w_k^2 \overline{f_k(u_k)} + \bar{u}_k^2 w_k(f_k(u_k) - \overline{f_k(u_k)}) \\
&\quad = \bar{u}_k w_k u_k f_k(u_k) - \bar{u}_k w_k^2(f_k(u_k) - \overline{f_k(u_k)}) - \bar{u}_k^2 w_k \overline{f_k(u_k)}.
\end{aligned}$$

Upon integrating over $\partial B_r(x_k)$ the last term vanishes. By a similar observation, for the contribution from the first term on the right we obtain

$$\begin{aligned}
\int_{\partial B_r(x_k)} \bar{u}_k w_k u_k f_k(u_k) \, do &= \int_{\partial B_r(x_k)} \bar{u}_k w_k e^{\eta_k} \, do \\
&= \int_{\partial B_r(x_k)} \bar{u}_k w_k (e^{\eta_k} - e^{\bar{\eta}_k}) \, do.
\end{aligned}$$

Estimating

$$\frac{e^{\eta_k} - e^{\bar{\eta}_k}}{\eta_k - \bar{\eta}_k} \leqslant \max\{e^{\eta_k}, e^{\bar{\eta}_k}\} \leqslant \max_{\partial B_r(x_k)} e^{\eta_k},$$

from Lemma 5.4 for any $\varepsilon > 0$ we can either bound the latter integrand by a uniform constant $C_2 = C_2(\varepsilon)$ or $\bar{u}_k \geqslant C_1 = C_1(\varepsilon)$ and,

$$\max_{\partial B_r(x_k)} e^{\eta_k + 2 \log r} \leqslant \varepsilon + o(1)$$

with error $o(1) \to 0$ uniformly for $r \geqslant L r_k$ as $k_0(L) \leqslant k \to \infty$ and $L \to \infty$. Thus, we obtain the estimate

$$\bar{u}_k w_k (e^{\eta_k} - e^{\bar{\eta}_k}) \leqslant C_2(\varepsilon) + (\varepsilon + o(1)) \, r^{-2} \bar{u}_k^2 w_k^2.$$

Again the Poincaré inequality gives

$$\int_{\partial B_r(x_k)} w_k^2 \, do \leqslant C r^2 \int_{\partial B_r(x_k)} |\nabla w_k|^2 \, do.$$

Hence we find that

$$\int_{\partial B_r(x_k)} \bar{u}_k w_k u_k f_k(u_k)\, dx \leqslant C(\varepsilon) + (\varepsilon + o(1)) \int_{\partial B_r(x_k)} \bar{u}_k^2\, |\nabla w_k|^2\, do$$

with $o(1) \to 0$ uniformly for $r \geqslant Lr_k$ as $k_0(L) \leqslant k \to \infty$, $L \to \infty$.

If $r \leqslant Lr_k$, for $k \geqslant k_0(L)$ we have

$$\max_{\partial B_r(x_k)} e^{\eta_k + 2\log r} \leqslant \max_{B_L(0)} e^{\eta_k^{(1)}(x) + 2\log|x|} \leqslant C$$

and therefore

$$\begin{aligned}\int_{\partial B_r(x_k)} \bar{u}_k w_k u_k f_k(u_k)\, dx &\leqslant Cr^{-2} \int_{\partial B_r(x_k)} \bar{u}_k^2 w_k^2\, do \\ &\leqslant C \int_{\partial B_r(x_k)} \bar{u}_k^2\, |\nabla w_k|^2\, do.\end{aligned}$$

Finally, we use positivity of u_k, $f_k(u_k)$ and the monotonicity of f_k for $s \geqslant s_1$ to bound

$$\begin{aligned}&\int_{\partial B_r(x_k)} \bar{u}_k w_k^2 (\overline{f_k(u_k)} - f_k(u_k))\, do \\ &\leqslant \int_{\partial B_r(x_k)} w_k^2 \bar{u}_k \overline{f_k(u_k)}\, do \\ &= \fint_{\partial B_r(x_k)} w_k^2\, do \int_{\partial B_r(x_k)} (e^{\eta_k} - (u_k - \bar{u}_k)(f_k(u_k) - f_k(\bar{u}_k)))\, do \\ &\leqslant \max_{\partial B_r(x_k)} w_k^2 \int_{\partial B_r(x_k)} (C + e^{\eta_k})\, do \\ &\leqslant C(\varepsilon) + (\varepsilon + o(1)) \int_{\partial B_r(x_k)} e^{\eta_k}\, do,\end{aligned}$$

where $o(1) \to 0$ uniformly for $r \leqslant T_k$ as $k \to \infty$ on account of Lemmas 4.7 and 5.4.

Splitting

$$4\, |\bar{u}_k w_k\, \nabla \bar{u}_k\, \nabla w_k| \leqslant \tfrac{1}{4}\, \bar{u}_k^2\, |\nabla w_k|^2 + 16 w_k^2\, |\nabla \bar{u}_k|^2,$$

and choosing $\varepsilon = \frac{1}{4}$, upon integrating (27) over $B_{T_k}(x_k)$ we then obtain that

$$\begin{aligned}\int_{B_{T_k}(x_k)} \bar{u}_k^2 \, |\nabla w_k|^2 \, dx \\ \leqslant \int_{\partial B_{T_k}(x_k)} \partial_n(\bar{u}_k^2 w_k^2) \, dx + C \int_{B_{T_k}(x_k)} w_k^2 \, |\nabla \bar{u}_k|^2 \, dx \\ + C \int_{B_{Lr_k}(x_k)} \bar{u}_k^2 \, |\nabla w_k|^2 \, dx + o(1) \int_{B_{T_k}(x_k)} \bar{u}_k^2 \, |\nabla w_k|^2 \, dx + C\end{aligned}$$

for any L, where $o(1) \to 0$ as $k \to \infty$, $L \to \infty$. But by choice of T_k, observing that $|w_k| \leqslant u_k + \bar{u}_k$, $|\nabla w_k| \leqslant |\nabla u_k| + |\nabla \bar{u}_k|$, and, finally, that

$$\begin{aligned}\left(\int_{\partial B_{T_k}(x_k)} |\nabla \bar{u}| \, do \right)^2 &\leqslant 2\pi T_k \int_{\partial B_{T_k}(x_k)} |\nabla \bar{u}|^2 \, do \\ &\leqslant 2\pi T_k \int_{\partial B_{T_k}(x_k)} |\nabla u|^2 \, do \leqslant C,\end{aligned}$$

we can estimate the first term on the right by a uniform constant. Similarly, using Lemma 5.4, we find

$$\int_{B_{T_k}(x_k)} w_k^2 \, |\nabla \bar{u}_k|^2 \, dx \leqslant C \int_{B_{T_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant C.$$

Moreover, on $B_{Lr_k}(x_k)$ we have

$$\begin{aligned}\int_{B_{Lr_k}(x_k)} \bar{u}_k^2 \, |\nabla w_k|^2 \, dx &\leqslant C \int_{B_{Lr_k}(x_k)} |\nabla(\eta_k - \bar{\eta}_k)|^2 \, dx \\ &\leqslant C \int_{B_L(0)} |\nabla(\eta_k^{(1)} - \bar{\eta}_k^{(1)})|^2 \, dx \\ &\leqslant CL^2 \, \|\eta_k^{(1)} - \eta^{(1)}\|_{C^1(B_L(0))} \to 0\end{aligned}$$

as $k \to \infty$. Thus, choosing $L \in \mathbb{N}$ sufficiently large, we find that

$$\limsup_{k \to \infty} \int_{B_{T_k}(x_k)} \bar{u}_k^2 \, |\nabla w_k|^2 \, dx \leqslant C,$$

as desired. ∎

Letting $\eta_k = \eta_k^{(0)}$ as above, setting

$$\Phi_k(r) = \int_{B_r(x_k)} u_k f_k(u_k)\, dx = \int_{B_r(x_k)} e^{\eta_k}\, dx,$$

$$\Psi_k(r) = \int_{B_r(x_k)} |\nabla u_k|^2\, dx,$$

and defining $\bar{\eta}_k = \overline{\eta_k^{(0)}}$ as well as

$$\bar{\Phi}_k(r) = \int_{B_r(x_k)} e^{\bar{\eta}_k}\, dx,$$

from Lemma 4.7, the estimate (26), and Lemma 5.5 we conclude that

$$\sup_r\, (\Phi_k(r) - \bar{\Phi}_k(r)) \to 0 \qquad \text{as} \quad k \to \infty.$$

We can now identify the blow-up energy level.

LEMMA 5.6. *There holds* $\beta = \frac{1}{2}$.

Proof. Rewriting $\bar{\Phi}_k$ as

$$\bar{\Phi}_k(r) = \int_{B_1(0)} r^2 e^{\bar{\eta}_k(x_k + rx)}\, dx,$$

and shifting x_k to 0 for convenience, we compute

$$\begin{aligned} r\bar{\Phi}_k'(r) &= \int_{B_1(0)} e^{\bar{\eta}_k(rx)} (2r^2 + r^3 x \cdot \nabla \bar{\eta}_k)\, dx \\ &= 2\bar{\Phi}_k(r) + \int_0^r \left(\frac{1}{2\pi} \int_{\partial B_\rho(0)} e^{\bar{\eta}_k}\, do \int_{\partial B_\rho(0)} \partial_n \bar{\eta}_k\, do \right) d\rho. \end{aligned}$$

Observe that for $r \leqslant t_k$ with error $o(1) \to 0$ as $k \to \infty$ we have

$$\begin{aligned} -\Delta \eta_k = -\Delta \eta_k^{(0)} &= \left(\varphi_k'(u_k) + \frac{2}{u_k} \right) f_k(u_k) - \left(\varphi_k''(u_k) - \frac{2}{u_k^2} \right) |\nabla u_k|^2 \\ &= (8\pi + o(1))\, e^{\eta_k} - (8\pi + o(1))\, |\nabla u_k|^2. \end{aligned}$$

Thus

$$-\int_{\partial B_r(0)} \partial_n \bar{\eta}_k \, do$$

$$= -\int_{\partial B_r(0)} \partial_n \eta_k \, do = \int_{B_r(0)} -\Delta \eta_k \, dx$$

$$= (8\pi + o(1)) \int_{B_r(0)} e^{\eta_k} \, dx - (8\pi + o(1)) \int_{B_r(0)} |\nabla u_k|^2 \, dx$$

$$= 8\pi(\Phi_k(r) - \Psi_k(r)) + o(1) \geqslant 8\pi(\bar{\Phi}_k(r) - \Psi_k(r)) + o(1).$$

Hence for $r \leqslant t_k$ we obtain

$$0 \leqslant r\bar{\Phi}_k'(r) \leqslant 2\bar{\Phi}_k(r) - 4\int_0^r \bar{\Phi}_k'(\rho)(\bar{\Phi}_k(\rho) - \Psi_k(\rho)) \, d\rho + o(1)$$

$$= 2\bar{\Phi}_k(r) - 2\bar{\Phi}_k^2(r) + 4\int_0^r \bar{\Phi}_k'(\rho) \, \Psi_k(\rho) \, d\rho + o(1)$$

$$= 2\bar{\Phi}_k(r)(1 - \bar{\Phi}_k(r)) + 4\int_0^r \Psi_k'(\rho)(\bar{\Phi}_k(r) - \bar{\Phi}(\rho)) \, d\rho + o(1).$$

Splitting the last integral

$$I_k = \int_0^r \Psi_k'(\rho)(\bar{\Phi}_k(r) - \bar{\Phi}_k(\rho)) \, d\rho = \int_0^{Lr_k} \cdots + \int_{Lr_k}^r \cdots$$

and observing that $\Psi_k(Lr_k) = o(1)$, $\Phi_k(Lr_k) = \bar{\Phi}_k(Lr_k) + o(1) = 1 + o(1)$ with error $o(1) \to 0$ if first $k \to \infty$ and then $L \to \infty$, we may estimate

$$I_k \leqslant \Psi_k(r)(\bar{\Phi}_k(r) - 1) + o(1)$$

to obtain the inequality

$$0 \leqslant 2(\bar{\Phi}_k(r) - 2\Psi_k(r))(1 - \bar{\Phi}_k(r)) + o(1)$$

for any $r \leqslant t_k$. In view of Lemma 5.3, at $r = t_k$ with error $o(1) \to 0$ as $k \to \infty$ we have $\bar{\Phi}_k(r) = \Phi_k(r) + o(1) = 2\beta + o(1)$, $\Psi_k(r) \leqslant 2\beta - 1 + o(1)$, yielding the inequality

$$0 \leqslant 4(1 - \beta)(1 - 2\beta) + o(1).$$

Since $\frac{1}{2} \leqslant \beta < 1$, the claim follows. ∎

APPENDIX A

Here we show that the family $(u_\rho)_{\rho>0}$ defined in Section 2 indeed is a Palais–Smale sequence at level $\beta=\frac{1}{2}$, that is, satisfies

$$\|dE(u_\rho)\|_{H^1(\Omega)} = \sup_{v\in H_0^1(\Omega);\ \|v\|_{H_0^1(\Omega)}\leqslant 1} \langle dE(u_\rho), v\rangle \to 0 \qquad \text{as} \quad \rho\to 0.$$

For $\Omega\subset B_{R_0}(0)$ we may regard $u_\rho\in H_0^1(B_{R_0}(0))$ and it will suffice to show that

$$\|dE(u_\rho)\|_{H^{-1}(B_{R_0}(0))} \to 0 \qquad \text{as} \quad \rho\to 0. \tag{28}$$

Given $v\in H_0^1(B_{R_0}(0))$, for $0\leqslant r\leqslant R_0$ denote as

$$\bar{v}(r) = \fint_{\partial B_r(0)} v\, do$$

the spherical mean. Then $\bar{v}\in H_0^1(B_{R_0}(0))$, and by radial symmetry of u_ρ we have

$$\langle dE(u_\rho), v\rangle = \langle dE(u_\rho), \bar{v}\rangle.$$

Since, moreover, there also holds $\|\bar{v}\|_{H_0^1(B_{R_0}(0))}\leqslant \|v\|_{H_0^1(B_{R_0}(0))}$, in order to estimate $dE(u_\rho)$ in the dual norm it suffices to consider radially symmetric testing functions $v=\bar{v}\in H_0^1(B_{R_0}(0))$.

Let $v'=\frac{d}{dr}v$, etc. Then in view of (8), we may further assume

$$\begin{aligned} 0 &= \int_{B_{R_0}(0)} \nabla u_\rho \nabla v\, dx = 2\pi \int_0^R u'_\rho v' r\, dr \\ &= -\frac{\sqrt{2\pi}\, a_\rho}{\sqrt{\log(R/\rho)}} \int_\rho^R v'(r)\, dr = \frac{\sqrt{2\pi}\, a_\rho}{\sqrt{\log(R/\rho)}} (v(\rho)-v(R)); \end{aligned} \tag{29}$$

that is,

$$v(\rho) = v(R).$$

Let $G_\rho\colon H_0^1(B_\rho(0))\to\mathbb{R}$, $H_\rho\colon H_0^1(B_R\backslash B_\rho(0))\to\mathbb{R}$ be given by, respectively,

$$G_\rho(w) = \int_{B_\rho(0)} wf(u_\rho)\, dx, \qquad H_\rho(w) = \int_{B_R\backslash B_\rho(0)} wf(u_\rho)\, dx.$$

Then from (29) it follows that

$$
\begin{aligned}
-\langle dE(u_\rho), v\rangle &= \int_{B_R(0)} vf(u_\rho)\,dx \\
&= v(R)\int_{B_R(0)} f(u_\rho)\,dx + G_\rho(v-v(R)) + H_\rho(v-v(R)) \\
&= I + II + III.
\end{aligned}
$$

Hölder's inequality and the bound $\|v\|_{H_0^1(B_{R_0}(0))} \leqslant 1$ imply the uniform estimate

$$
\begin{aligned}
|v(R)| = |v(R_0)-v(R)| &\leqslant \int_R^{R_0} |v'|\,dr \\
&\leqslant \left(\frac{1}{2\pi}\log\left(\frac{R_0}{R}\right)\cdot 2\pi\int_R^{R_0}|v'|^2\,r\,dr\right)^{1/2} \leqslant \left(\frac{1}{2\pi}\log\left(\frac{R_0}{R}\right)\right)^{1/2}.
\end{aligned}
$$

Since $f(u_\rho)\to 0$ in $L^1(B_R(0))$ as $\rho\to 0$, we conclude that

$$
|I| \leqslant \sup_{v=\bar v\in H_0^1(B_{R_0}(0));\ \|v\|_{H_0^1(B_{R_0}(0))}\leqslant 1} v(R)\int_{B_R(0)} f(u_\rho)\,dx \to 0. \tag{30}
$$

Let $v=v_\rho(r)\in H_0^1(B_\rho(0))$ maximize G_ρ subject to the constraint $\|v\|_{H_0^1(B_\rho(0))}\leqslant 1$. Then $\|v_\rho\|_{H_0^1(B_\rho(0))}=1$, and there exists $\lambda_\rho>0$ such that

$$
-\Delta v_\rho = -\frac{1}{r}(rv_\rho')' = \lambda_\rho f(u_\rho) \qquad \text{in } B_\rho(0). \tag{31}
$$

Multiplying (31) by v_ρ and integrating by parts, we infer that

$$
1 = \int_{B_\rho(0)} |\nabla v_\rho|^2\,dx = \lambda_\rho G_\rho(v_\rho). \tag{32}
$$

Since u_ρ is a constant on $B_\rho(0)$, we may also integrate (31) directly from $r=0$ to $s<\rho$ to find

$$
-v_\rho'(s) = \lambda_\rho s^{-1}\int_0^s f(u_\rho)\,r\,dr = \tfrac{1}{2}\lambda_\rho f(u_\rho)\,s
$$

and hence, for $0<r<\rho$, that

$$
v_\rho(r) = -\int_r^\rho v_\rho'(s)\,ds = \tfrac{1}{4}\lambda_\rho f(u_\rho)(\rho^2-r^2).
$$

Again using that $u_\rho \equiv u_\rho(0)$, we deduce that

$$G_\rho(v_\rho) = 2\pi \int_0^\rho v_\rho f(u_\rho)\, r\, dr = \frac{\pi}{8}\lambda_\rho f^2(u_\rho)\, \rho^4$$
$$= \frac{\lambda_\rho}{8\pi u_\rho^2(0)} \left(\int_{B_\rho(0)} u_\rho f(u_\rho)\, dx \right)^2 \leqslant \frac{\lambda_\rho a_\rho^2}{8\pi u_\rho^2(0)}.$$

By (32) therefore

$$1 = \lambda_\rho G_\rho(v_\rho) \leqslant \frac{\lambda_\rho^2 a_\rho^2}{8\pi u_\rho^2(u)},$$

and it follows that, as $\rho \to 0$,

$$\lambda_\rho^2 \geqslant 8\pi u_\rho^2(0)/a_\rho^2 \to \infty$$

and thus

$$|II| \leqslant \sup_{v \in H_0^1(B_\rho(0));\ \|v\|_{H_0^1(B_\rho(0))}} G_\rho(v) \leqslant G_\rho(v_\rho) = \frac{1}{\lambda_\rho} \to 0. \tag{33}$$

Similarly, let $w = w_\rho(r) \in H_0^1(B_R \backslash B_\rho(0))$ maximize H_ρ subject to the constraint $\|w\|_{H_0^1(B_R \backslash B_\rho(0))} \leqslant 1$. Then again $\|w_\rho\|_{H_0^1(B_R \backslash B_\rho(0))} = 1$, and there exists $\mu_\rho > 0$ such that

$$-\Delta w_\rho = -\frac{1}{r}(r w_\rho')' = \mu_\rho f(u_\rho) \qquad \text{in } B_R \backslash B_\rho(0), \tag{34}$$

while $w_\rho(R) = w_\rho(\rho) = 0$.

Multiplying (34) by w_ρ and integrating, we obtain that

$$1 = \int_{B_R \backslash B_\rho(0)} |\nabla w_\rho|^2\, dx = \mu_\rho H_\rho(w_\rho), \tag{35}$$

similar to (32). Moreover, observe that (34) implies that $r \mapsto r w_\rho'(r)$ is non-increasing. Therefore, for any $s > \rho$ either $w_\rho'(s) \leqslant 0$ or we can estimate

$$1 \geqslant 2\pi \int_\rho^s (w_\rho'(r))^2\, r\, dr \geqslant 2\pi \int_\rho^s (w_\rho'(s))^2\, s^2 \frac{dr}{r} = 2\pi \log\left(\frac{s}{\rho}\right) (w_\rho'(s))^2\, s^2.$$

In particular, if we choose $s = \rho^\alpha$ for some $\frac{1}{2} \leqslant \alpha < 1$, it follows that either $w_\rho'(\rho^\alpha) \leqslant 0$ or

$$w_\rho'(\rho^\alpha)\, \rho^\alpha \leqslant \frac{1}{\sqrt{2\pi(1-\alpha)\log(1/\rho)}} \to 0 \tag{36}$$

as $\rho \to 0$. Next, we integrate (34) from ρ to ρ^α and use monotonicity of u_ρ to bound

$$\begin{aligned}\rho w'_\rho(\rho) - \rho^\alpha w'_\rho(\rho^\alpha) &= \mu_\rho \int_\rho^{\rho^\alpha} f(u_\rho)\, r\, dr \\ &\leqslant \mu_\rho \int_\rho^{\rho^\alpha} u_\rho f(u_\rho)\, r\, dr / u_\rho(\rho^\alpha) \\ &\leqslant \frac{\mu_\rho a_\rho^2}{2\pi u_\rho(\rho^\alpha)} = \frac{\mu_\rho a_\rho \sqrt{\log(R/\rho)}}{\sqrt{2\pi}\log(R/\rho^\alpha)}.\end{aligned}$$

Using also (36) and the fact that $a_\rho \to 1$ as $\rho \to 0$, we conclude that for sufficiently small $\rho > 0$ and all $s \in [\rho, \rho^\alpha]$ there holds

$$s w'_\rho(s) \leqslant \rho w'_\rho(\rho) \leqslant C(1+\mu_\rho) \sqrt{\log\left(\frac{1}{\rho}\right)}^{-1}$$

with a uniform constant C.

Thus, for $\rho \leqslant r \leqslant \rho^\alpha$, $\frac{1}{2} \leqslant \alpha < 1$ we can now estimate

$$\begin{aligned}w_\rho(r) &\leqslant \int_\rho^r w'_\rho(s)\, ds \leqslant \rho w'_\rho(\rho) \int_\rho^r \frac{ds}{s} \\ &\leqslant C(1+\mu_\rho) \frac{\log(r/\rho)}{\sqrt{\log(1/\rho)}} \leqslant C(1+\mu_\rho)(1-\alpha) \sqrt{\log\left(\frac{1}{\rho}\right)} \\ &\leqslant C(1+\mu_\rho)(1-\alpha)\, m_{\rho,R}(\rho^\alpha) \leqslant C(1+\mu_\rho)(1-\alpha)\, u_\rho(r)\end{aligned}$$

and it follows that

$$\begin{aligned}\int_\rho^{\rho^\alpha} w_\rho f(u_\rho)\, r\, dr &\leqslant C(1+\mu_\rho)(1-\alpha) \int_\rho^{\rho^\alpha} u_\rho f(u_\rho)\, r\, dr \\ &\leqslant C(1+\mu_\rho)(1-\alpha).\end{aligned}$$

Finally, the truncated function

$$\tilde{u}_\rho(r) = \min\{u_\rho(r), u_\rho(\rho^\alpha)\}$$

satisfies

$$\int_{B_R(0)} |\nabla \tilde{u}_\rho|^2\, dx = \frac{a_\rho^2}{\log(R/\rho)} \int_{\rho^\alpha}^R \frac{dr}{r} = a_\rho^2 \frac{\log(R/\rho^\alpha)}{\log(R/\rho)} \leqslant C < 1$$

for any $\alpha < 1$ and sufficiently small $\rho > 0$.

Hence $(f(\tilde{u}_\rho))_{\rho>0}$ is bounded in $L^p(B_R(0))$ for some $p=p(\alpha)>1$, and thus also $(f(u_\rho)_{|B_R\backslash B_{\rho^\alpha}(0)})_{\rho>0}$. Since $H_0^1(B_R\backslash B_\rho(0)) \hookrightarrow H_0^1(B_R(0)) \hookrightarrow L^q(B_R(0))$ for any $q<\infty$, it follows that for any fixed $\alpha<1$, as $\rho\to 0$ there holds

$$\sup_{w\in H_0^1(B_R\backslash B_\rho(0));\ \|w\|_{H_0^1(B_R\backslash B_\rho(0))}} \int_{\rho^\alpha}^R wf(u_\rho)\, r\, dr \to 0.$$

We conclude that

$$\begin{aligned}\limsup_{\rho\to 0} H_\rho(w_\rho) &\leqslant 2\pi \limsup_{\rho\to 0} \int_\rho^{\rho^\alpha} w_\rho f(u_\rho)\, r\, dr \\ &\leqslant C(1-\alpha) \limsup_{\rho\to 0}\, (1+\mu_\rho).\end{aligned}$$

Thus, as $\rho\to 0$, either $\mu_\rho\to\infty$ and $H_\rho(w_\rho)\to 0$ by (35), or $\mu_\rho\leqslant C<\infty$ and we may let $\alpha\to 1$ to conclude that $H_\rho(w_\rho)\to 0$ and therefore, in view of (30), (33), that

$$\limsup_{\rho\to 0} \|dE(u_\rho)\|_{H^{-1}(\Omega)} \leqslant \limsup_{\rho\to 0} H_\rho(w_\rho)=0,$$

as desired.

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