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Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem

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Abstract

In this paper we deal with three types of problems concerning the Hardy–Rellich's embedding for a bi-Laplacian operator. First we obtain the Hardy–Rellich inequalities in the critical dimension $n = 4$. Then we derive a maximum principle for fourth order operators with singular terms. Then we study the existence, non-existence, simplicity and asymptotic behavior of the first eigenvalue of the Hardy–Rellich operator $\Delta^2 - \frac{n^2(n-4)^2}{16}\frac{q(x)}{|x|^4}$ under various assumptions on the perturbation q.

Keywords: Biharmonic equation; Hardy–Rellich's inequality; Maximum principle; Perturbed eigenvalue problem; Boggio's principle; Dirichlet and Navier boundary conditions

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $0 \in \Omega$. Let us recall that the Hardy–Rellich's inequality states that for all $u \in H_0^2(\Omega)$

$$\int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4} \geqslant 0, \quad n \geqslant 5, \tag{1.1}$$

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where $\frac{n^2(n-4)^2}{16}$ is the best constant in (1.1) and it is never achieved in any domain $\Omega \subset \mathbb{R}^n$. This inequality was firstly proved by F. Rellich [14] for $u \in H_0^2(\Omega)$ and it was extended to functions in $H^2(\Omega) \cap H_0^1(\Omega)$ by Dold et al. [9]. On the lines of improving Hardy–Sobolev inequality for functions in $H_0^1(\Omega)$ (see [1,3,6,8,10]) there has been a considerable interest in improving (1.1). Recently, Gazzola et al. [11] proved that for $n \geqslant 5$ there exist $C, C_1 > 0$ such that for all $u \in H_0^2(\Omega)$ or $u \in H^2(\Omega) \cap H_0^1(\Omega)$, the following inequality holds,

$$\int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \geqslant C \int_\Omega \frac{u^2}{|x|^2} + C_1 \int_\Omega u^2, \quad n \geqslant 5. \tag{1.2}$$

Furthermore, Tertikas and Zographopolous [17] have improved this by obtaining an optimal inequality,

$$\int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \geqslant \frac{n(n-4)}{8} \int_\Omega \frac{u^2}{|x|^4(\ln R/|x|)^2}, \quad n \geqslant 5, \tag{1.3}$$

which holds for every $u \in H_0^2(\Omega)$ or $u \in H^2(\Omega) \cap H_0^1(\Omega)$ where $R > e\,\mathrm{diam}(\Omega)$. For the sake of completeness we give the proof of the generalized inequality in Appendix A (Theorem A.1) so that (1.3) follows as a consequence of this theorem.

In this paper we consider the following three problems:

- Optimal Hardy–Rellich inequality in $n = 4$.
- Maximum principle for the bi-Laplacian equation with singular potential.
- Existence and non-existence of the perturbed Hardy–Rellich operator.

Surprisingly optimal Hardy–Rellich inequality for $n = 4$ turn out to be different compared to $n \geqslant 5$ and this will be dealt in Section 3.

Secondly, for $n \geqslant 5$ the best constant

$$\lambda = \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \colon \int_\Omega u^2 = 1 \right\} \tag{1.4}$$

is never attained in any domain Ω and hence as in [4] we look to the perturbed problem

$$\lambda(q) = \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{qu^2}{|x|^4} \colon \int_\Omega u^2 = 1 \right\}, \tag{1.5}$$

where $q \in C^0(\overline{\Omega})$ be such that $0 \leqslant q(x) \leqslant 1$. In Section 5 we give a necessary and sufficient condition on q for which $\lambda(q)$ is achieved as in [4] for the Hardy–Sobolev inequality. Unlike in the Laplacian case, the bi-Laplacian does not satisfy the maximum principle which is a main ingredient in obtaining the results. Therefore in Section 4 we prove a weak type maximum principle for bi-Laplacian with singular potential using continuation method which will be used to prove the existence and non-existence of minimizers for (1.5). Finally in Appendix A we prove some technical lemmas and we give some new Hardy–Rellich's inequalities and $W^{1,p}(\Omega)$ estimates.

2. Main results

Let $R > 0$, $B(R)$ denote the open ball of radius R with center at the origin and B denotes the ball with radius one. Let $0 \in \Omega \subset \mathbb{R}^n$ be a domain with smooth boundary and define

$$C^k_{0,r}(\Omega) = \bigl\{v \in C^k_0(\Omega)\colon\ v \text{ is radial and } \operatorname{supp} v \subset \Omega\bigr\},$$
$$H^m_{0,r}(\Omega) = \bigl\{v \in H^m_0(\Omega)\colon\ v \text{ is radial}\bigr\}.$$

Let $-\Delta_{\mathbb{S}^3}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^3$. Then the spectrum of $-\Delta_{\mathbb{S}^3} = -\Delta_\sigma$ is discrete with eigenvalues given by $\{i(i+2)\colon\ i \in \mathbb{N} \cup \{0\}\}$. Let V_i be the eigenspace corresponding to the eigenvalue $i(i+2)$. Let $P_i : L^2(\mathbb{R}^4) \to V_i$ be the orthogonal projection given by

$$P_i(f) = \sum_{j=1}^{k_i} \biggl(\int_{\mathbb{S}^3} f(r\omega)\varphi_{i,j}(\omega)\, d\sigma(\omega) \biggr) \varphi_{i,j}, \tag{2.1}$$

where $\{\varphi_{i,j}\}_{1 \leqslant j \leqslant k_i}$ is a complete orthonormal set for V_i and $r = |x|$.

For $t \in (0, 1]$, define the functions $\{Y_i(t)\}_{i \in \mathbb{N}}$ inductively as follows:

$$Y_1(t) := (1 - \ln t)^{-1}, \quad t \in (0, 1],$$
$$Y_i(t) := Y_{i-1}\bigl(Y_1(t)\bigr), \quad t \in (0, 1),\ i = 2, 3, 4, \ldots,$$
$$Y_i(0) = 0, \qquad Y_i(1) = 1,$$
$$0 \leqslant Y_i(t) \leqslant 1.$$

Note in the case of bi-Laplacian there are two types of Hardy–Rellich's inequality that is interaction between Δu with u and Δu with ∇u.

Theorem 2.1. (a) *Let $0 \in \Omega \subset B(R)$ be a bounded domain in $\mathbb{R}^4$, $R > 0$, $R_1 > eR$. Then $\forall u \in H^2_0(\Omega)$ or $\forall u \in H^2(\Omega) \cap H^1_0(\Omega)$ we have*

$$\int_\Omega |\Delta u|^2 - \int_\Omega \frac{u^2}{|x|^4 (\ln \frac{R_1}{|x|})^2} \geqslant \sum_{i=2}^{\infty} \int_\Omega \frac{u^2}{|x|^4 (\ln \frac{R_1}{|x|})^2} X_2^2 \cdots X_i^2, \tag{2.2}$$

where

$$X_i(x) := Y_i\biggl(\frac{|x|}{R}\biggr), \quad i = 1, 2, 3, 4, \ldots.$$

The constants -1 (the coefficient of $\int_\Omega \frac{u^2}{|x|^4 (\ln R_1/|x|)^2}$) is the best constant and is never achieved by any nontrivial function $u \in H^2_0(\Omega)$ or $\forall u \in H^2(\Omega) \cap H^1_0(\Omega)$.

(b) *Let* $0 < R \leqslant \infty$, *and* $0 \in \Omega \subset B(R) \subset \mathbb{R}^4$. *Suppose* $u \in H_0^2(\Omega)$, *then*

$$\int_\Omega |\Delta u|^2 - 4\int_\Omega \frac{|\nabla u|^2}{|x|^2} \geqslant -3\int_\Omega \frac{(P_1 u)^2}{|x|^4}, \tag{2.3}$$

$$\int_\Omega |\Delta u|^2 - 4\int_\Omega \frac{|\nabla u|^2}{|x|^2} \geqslant -3\int_\Omega \frac{(P_1 u)^2}{|x|^4} + \frac{3}{4}\int_\Omega \frac{|P_1 u|^2}{|x|^4(\ln R/|x|)^2} + \frac{9}{32}\int_\Omega \frac{|P_1 u|^2}{|x|^4(\ln R/|x|)^4}, \tag{2.4}$$

$-4, -3, \frac{3}{4}, \frac{9}{32}$ *are the best constants and equality holds iff* $u \equiv 0$.

Next we study the eigenvalue problems associated with the perturbed Hardy–Rellich operator. Let $n \geqslant 4$ and $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $R_1 > e\,\mathrm{diam}(\Omega)$. Let $q \in C^0(\overline{\Omega})$ be such that $0 \leqslant q(x) \leqslant 1$ for $u \in H^2(\Omega)$

$$I_q(u) = \begin{cases} \int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{qu^2}{|x|^4}, & n \geqslant 5, \\ \int_\Omega |\Delta u|^2 - \int_\Omega \frac{qu^2}{|x|^4(\ln R/|x|)^2}, & n = 4, \end{cases} \tag{2.5}$$

and

$$\lambda_D(q) = \inf_{u \in H_0^2(\Omega)} \left\{ I_q(u)\colon \int_\Omega u^2 = 1 \right\}, \tag{2.6}$$

$$\lambda_N(q) = \inf_{u \in H^2(\Omega) \cap H_0^1(\Omega)} \left\{ I_q(u)\colon \int_\Omega u^2 = 1 \right\}. \tag{2.7}$$

Now define the associated Hardy–Rellich operator

$$L_q u = \begin{cases} \Delta^2 u - \frac{n^2(n-4)^2}{16}\frac{qu}{|x|^4}, & n \geqslant 5, \\ \Delta^2 u - \frac{qu}{|x|^4(\ln R/|x|)^2}, & n = 4. \end{cases} \tag{2.8}$$

Then if u is a minimizer in any one of (2.6) and (2.7), then u satisfies

$$L_q u = \lambda u \quad \text{in } \Omega \tag{2.9}$$

with the following boundary conditions:

(i) In the case of (2.6), $\lambda = \lambda_D(q)$ satisfies the Dirichlet boundary condition

$$\mathbb{P}_D\colon\ u = \frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \partial\Omega. \tag{2.10}$$

(ii) In the case of (2.7), $\lambda = \lambda_N(q)$ satisfies the Navier boundary condition

$$\mathbb{P}_N\colon\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \tag{2.11}$$

Now we observe that in the case of Navier boundary condition ($\mathbb{P}_N$), maximum principle holds and hence if the minimizers exists, then we can expect a non-negative solution.

In the Dirichlet case ($\mathbb{P}_D$), no maximum principle holds and since $u \in H^2(\Omega)$ need not imply $|u| \in H^2(\Omega)$, we cannot expect a non-negative minimizer. Therefore obtaining the a priori estimates is difficult. In view of this we develop a weak maximum principle with singular potential in Section 4 which will be used to prove the following theorems. Here we give a necessary and sufficient condition on the perturbation q in order to get a minimizer.

Theorem 2.2. *Let $n \geqslant 5$.*

(i) *If q satisfies*

$$\liminf_{x \to 0} \left(\ln \frac{1}{|x|} \right)^2 \big(1 - q(x)\big) > \frac{6(n^2 - 4n + 8)}{n^2(n-4)^2} \tag{2.12}$$

then $\lambda_D(q)$ and $\lambda_N(q)$ are achieved by u. Moreover, in the case of Navier boundary condition, we can choose $u > 0$.

(ii) *Let $0 < R < 1$. Assume that q satisfies*

$$\sup_{0<x<R} \left(\ln \frac{1}{|x|} \right)^2 \big(1 - q(x)\big) \leqslant \frac{6(n^2 - 4n + 8)}{n^2(n-4)^2}. \tag{2.13}$$

Then,
(a) *$\lambda_N(q)$ is not achieved*;
(b) *if $\Omega = B$, then $\lambda_D(q)$ is not achieved by any non-negative function.*

(iii) *Let $1 \leqslant p < 2$, then there exists $u \in W^{2,p}(\Omega)$ satisfying in the sense of distribution,*

$$L_q(u) = \lambda u \quad \text{in } \Omega, \tag{2.14}$$

where $\lambda \in \{\lambda_D(q), \lambda_N(q)\}$:
(a) *if $\lambda = \lambda_N(q)$, then $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $u \geqslant 0$;*
(b) *if $\lambda = \lambda_D(q)$, then $u \in W_0^{2,p}(\Omega)$.*

Next we consider the case $n = 4$. In view of Theorem 2.1 we have the following theorem.

Theorem 2.3. *Let $n = 4$.*

(i) *If q satisfies*

$$\liminf_{x \to 0} \left(\ln\left(\ln \frac{R}{|x|} \right) \right)^2 \big(1 - q(x)\big) > 3 \tag{2.15}$$

then $\lambda_D(q)$ and $\lambda_N(q)$ are achieved by u.

(ii) *Let $0 < R_1 < 1$. Assume that q satisfies*

$$\sup_{0<x<R_1} \left(\ln\left(\ln \frac{R}{|x|} \right) \right)^2 \big(1 - q(x)\big) \leqslant 3. \tag{2.16}$$

Then,
(a) $\lambda_N(q)$ *is not achieved*;
(b) *if* $\Omega = B$*, then* $\lambda_D(q)$ *is not achieved by any non-negative function.*

(iii) *Let* $1 \leqslant p < 2$*, then there exists* $u \in W^{2,p}(\Omega)$ *satisfying in the sense of distribution*

$$L_q(u) = \lambda u \quad \text{in } \Omega, \tag{2.17}$$

where $\lambda \in \{\lambda_D(q), \lambda_N(q)\}$*. Moreover,*
(a) *if* $\lambda = \lambda_N(q)$*, then* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *with* $u \geqslant 0$;
(b) *if* $\lambda = \lambda_D(q)$*, then* $u \in W_0^{2,p}(\Omega)$.

Next we take $q(x) = \nu$ a constant such that $0 < \nu < 1$ and we study the behavior of minimizers at the origin as $\nu \to 1$. To do this let $0 < \beta(\nu) < 1/2$ be the unique zero of the polynomial

$$g(\beta) = \beta(1-\beta)\big(2 + \beta(n-4)\big)\big(n - 2 - \beta(n-4)\big) - \frac{n^2}{16}\nu. \tag{2.18}$$

Clearly $\beta(\nu) \to 1/2$ as $\nu \to 1$. Then we have the following theorem.

Theorem 2.4. *Let* $n \geqslant 5$*,* $q(x) = \nu$ *and* $0 < \nu < 1$*. Let* $\lambda > 0$*,* $u_\nu \in H^2(\Omega)$ *be a non-negative solution of*

$$L_\nu u = \lambda u \quad \text{in } \Omega, \tag{2.19}$$

where $\lambda \in \{\lambda_D(\nu), \lambda_N(\nu)\}$*, then there exist* $C_1 > 0$*,* $C_2 > 0$ *such that*

$$C_1 \leqslant \liminf_{x \to 0} \left(\ln \frac{1}{|x|}\right)^{(n-4)\beta(\nu)} u_\nu(x) \leqslant \limsup_{x \to 0} \left(\ln \frac{1}{|x|}\right)^{(n-4)\beta(\nu)} u_\nu(x) \leqslant C_2.$$

Theorem 2.5. *Let* $n = 4$*,* $q(x) = \nu$ *and* $0 < \nu < 1$*. Let* $\lambda > 0$*,* $u_\nu \in H^2(\Omega)$ *be a non-negative solution of*

$$L_\nu u = \lambda u \quad \text{in } \Omega, \tag{2.20}$$

where $\lambda \in \{\lambda_D(\nu), \lambda_N(\nu)\}$*, then there exist* $C_1 > 0$*,* $C_2 > 0$ *such that*

$$C_1 \leqslant \liminf_{x \to 0} \left(\ln \frac{R}{|x|}\right)^{\frac{-1+\sqrt{1-\nu}}{2}} u_\nu(x) \leqslant \limsup_{x \to 0} \left(\ln \frac{R}{|x|}\right)^{\frac{-1+\sqrt{1-\nu}}{2}} u_\nu(x) \leqslant C_2.$$

Theorem 2.6. *Let* B *be the unit ball centered at origin and* $0 \leqslant q(x) \leqslant 1$*. Moreover, let* $u_1, u_2 \in H_0^2(B)$ *be two non-negative minimizers for* $\lambda_D(q)$*. Then* $u_1 = m u_2$ *for some* $m > 0$.

3. Hardy–Rellich inequalities

Lemma 3.1. *Let $n \geqslant 4$ and B be the unit ball centered at zero. Then $\forall u \in H^2_{0,r}(B)$ or $\forall u \in H^2_r(B) \cap H^1_{0,r}(B)$*

$$\int_B |\Delta u|^2 \geqslant \frac{n^2}{4} \int_B \frac{|\nabla u|^2}{|x|^2} \tag{3.1}$$

and equality holds iff $u \equiv 0$.

Proof. *Proof of the first part.* Let us assume that $u \in C_0^\infty(B)$. Note that

$$\int_B |\Delta u|^2 - \frac{n^2}{4} \int_B \frac{|\nabla u|^2}{|x|^2} = \omega_n \left\{ \int_0^1 u_{rr}^2 r^{n-1} - \frac{(n-2)^2}{4} \int_0^1 \frac{u_r^2}{r^2} r^{n-1} \right\}.$$

Setting $u_r = v$ and using the classical Hardy inequality in $H_0^1(B)$, we get

$$\omega_n \left\{ \int_0^1 u_{rr}^2 r^{n-1} - \frac{(n-2)^2}{4} \int_0^1 \frac{u_r^2}{r^2} r^{n-1} \right\} = \int_B |\nabla v|^2 - \frac{(n-2)^2}{4} \int_B \frac{v^2}{|x|^2} \geqslant 0$$

and equality holds iff $v = 0$ and hence $u = 0$.

Proof of the second part. Let $u \in C_r^2(\overline{B}) \cap C^1_{0,r}(B)$ then

$$\int_B |\Delta u|^2 - \frac{n^2}{4} \int_B \frac{|\nabla u|^2}{|x|^2} = \omega_n \left[\int_0^1 (u_{rr})^2 r^{n-1} - \frac{(n-2)^2}{4} \int_0^1 \frac{(u_r)^2}{r^2} r^{n-1} + (n-1)\big(u_r(1)\big)^2 \right],$$

$$\int_B |\Delta u|^2 - \frac{n^2}{4} \int_B \frac{|\nabla u|^2}{|x|^2} = \omega_n \left[\int_0^1 (v_r)^2 r^{n-1} - \frac{(n-2)^2}{4} \int_0^1 v^2 r^{n-3} + (n-1) v^2(1) \right], \tag{3.2}$$

where $u_r = v$. Putting $z = r^{\frac{n-2}{2}} v$ and integrating by parts the right-hand side of (3.2) becomes

$$\omega_n \left[\int_0^1 (v_r)^2 r^{n-1} - \frac{(n-2)^2}{4} \int_0^1 v^2 r^{n-3} + (n-1) v^2(1) \right]$$

$$= \omega_n \left[\int_0^1 (z_r)^2 r - \frac{(n-2)}{2} z^2(1) + (n-1) z^2(1) \right] = \omega_n \left[\int_0^1 (z_r)^2 r + \frac{n}{2} z^2(1) \right] \geqslant 0.$$

Again the inequality holds iff $z(1) = 0$, $z_r = 0$ and hence $u = 0$. Hence we are done. □

Proof of Theorem 2.1(a). Let us first assume that $u \in H^2_{0,r}(B)$ where B is a unit ball centered at origin. It follows easily that Y_i satisfies the following identities. For $t \in [0, 1)$ we have

$$\frac{dY_n}{dt} = \frac{Y_n(t)^2 Y_{n-1}(t)^2 \cdots Y_1(t)^2}{t}.$$

For $t = |x|/R$ and $X_i(x) = Y_i(|x|/R)$

$$\frac{\nabla X_n}{X_n} = \frac{x}{|x|^2} X_1(x) \cdots X_n(x).$$

Let $v_1 = X_1^{1/2} u$. Then $v_1(0) = 0$ and

$$\frac{\nabla u}{u} = -\frac{1}{2}\frac{\nabla X_1}{X_1} + \frac{\nabla v_1}{v_1},$$

$$\begin{aligned}
\int_B \frac{|\nabla u|^2}{|x|^2} &= \frac{1}{4}\int_B \frac{u^2}{|x|^4} X_1^2 + \int_B \frac{|\nabla v_1|^2}{v_1^2}\frac{u^2}{|x|^2} - \int_B \left\langle \frac{\nabla X_1}{X_1}, \frac{\nabla v_1}{v_1}\right\rangle \frac{u^2}{|x|^2} \\
&= \frac{1}{4}\int_B \frac{u^2}{|x|^4} X_1^2 + \int_B \frac{|\nabla v_1|^2}{|x|^2} X_1 - \frac{1}{2}\int_B \left\langle \frac{x}{|x|^4}, \nabla v_1^2 \right\rangle \\
&= \frac{1}{4}\int_B \frac{u^2}{|x|^4} X_1^2 + \int_B \frac{|\nabla v_1|^2}{|x|^2} X_1 - C v_1^2(0) \\
&= \frac{1}{4}\int_B \frac{u^2}{|x|^4} X_1^2 + \int_B \frac{|\nabla v_1|^2}{|x|^2} X_1.
\end{aligned}$$

Let, for $i \geqslant 2$, $v_i(x) = X_i^{1/2} v_{i-1}(x)$. Then

$$\begin{aligned}
\int_B \frac{|\nabla v_1|^2}{|x|^2} X_1 &= \frac{1}{4}\int_B \frac{v_1^2}{|x|^2}\frac{|\nabla X_2|^2}{|X_2|^2} X_1 - \frac{1}{2}\left\langle \frac{x}{|x|^4}, \nabla v_1^2 \right\rangle + \int_B \frac{|\nabla v_2|^2}{|x|^2} X_1 X_2 \\
&= \frac{1}{4}\int_B \frac{u^2}{|x|^4} X_1^2 X_2^2 + C v_1^2(0) + \int_B \frac{|\nabla v_2|^2}{|x|^2} X_1 X_2.
\end{aligned}$$

Hence by induction we have

$$\int_B \frac{|\nabla u|^2}{|x|^2} \geqslant \frac{1}{4}\sum_{i=1}^{\infty}\int_B \frac{u^2}{|x|^4} X_1^2 X_2^2 \cdots X_i^2.$$

Therefore, from Lemma 3.1 we have if u is radial,

$$\int_B |\Delta u|^2 \geqslant 4\int_B \frac{|\nabla u|^2}{|x|^2} \geqslant \sum_{i=1}^{\infty}\int_B \frac{u^2}{|x|^4} X_1^2 X_2^2 \cdots X_i^2.$$

Let $R_1 \geqslant eR$. Then for $|x| \leqslant R$, we have $\ln R_1 - \ln|x| \geqslant \ln e + \ln R - \ln|x|$ which implies that

$$\left(\ln \frac{R_1}{|x|}\right)^2 \geqslant \left(1 - \ln\frac{|x|}{R}\right)^2 = X_1^{-2}.$$

Hence we have

$$\begin{aligned}\int_B |\Delta u|^2 &\geqslant \int_B \frac{u^2}{|x|^4} X_1^2 + \sum_{i=2}^{\infty}\int_B \frac{u^2}{|x|^4(\ln\frac{R_1}{|x|})^2} X_2^2 \cdots X_i^2\\ &\geqslant \int_B \frac{u^2}{|x|^4(\ln\frac{R_1}{|x|})^2} + \sum_{i=2}^{\infty}\int_B \frac{u^2}{|x|^4(\ln\frac{R_1}{|x|})^2} X_2^2 \cdots X_i^2.\end{aligned}$$

This proves the inequality (2.2) for $u \in H^2_{0,r}(B)$.

Let $u \in H_0^2(\Omega)$, we apply the idea of [15]. Consider $|\Omega| = |B|$. Then we may restrict ourselves to $\Omega = B$ and superharmonic radial function u. Define $f = -\Delta u$.

$$\begin{cases} -\Delta w = f^* & \text{in } B,\\ w = 0 & \text{on } \partial B,\end{cases} \tag{3.3}$$

where f^* denotes the Schwarz symmetrization of f. Then $w \in H_r^2(B) \cap H^1_{0,r}(B)$. By [16] we have $w \geqslant u^* \geqslant 0$. Hence

$$\int_B |\Delta w|^2\,dx = \int_B (f^*)^2\,dx = \int_\Omega |f|^2\,dx = \int_\Omega |\Delta u|^2\,dx,$$

$$\int_B \frac{w^2}{|x|^4(\ln\frac{R_1}{|x|})^2}\,dx \geqslant \int_B \frac{u^{*2}}{|x|^4(\ln\frac{R_1}{|x|})^2}\,dx \geqslant \int_\Omega \frac{|u|^2}{|x|^4(\ln\frac{R_1}{|x|})^2}\,dx.$$

Similarly we get

$$\sum_{i=2}^{\infty}\int_B \frac{w^2}{|x|^4(\ln\frac{R_1}{|x|})^2} X_2^2 \cdots X_i^2 \geqslant \sum_{i=2}^{\infty}\int_\Omega \frac{u^2}{|x|^4(\ln\frac{R_1}{|x|})^2} X_2^2 \cdots X_i^2.$$

Hence the inequality (2.5) holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u \in H_0^2(\Omega)$.

Now we prove the sharpness of the previous inequality, i.e., we show the existence of a family of radial functions ψ_δ such that

$$\lim_{\delta\to 0}\frac{\int_\Omega |\Delta\psi_\delta|^2}{\int_\Omega \frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}=1.$$

Let $B(1)\subset\Omega$ and $\varphi\in C_0^\infty(\Omega)$ be radial such that

$$\varphi(x)=\begin{cases}1 & \text{in } B(\frac{1}{2}),\\ 0 & \text{on } \Omega\setminus B(1).\end{cases}$$

Define

$$\psi_\delta(x)=\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\varphi(x),$$

$$\Delta\psi_\delta(x)=\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\Delta\varphi+\Delta\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\varphi+2\left\langle\nabla\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta},\nabla\varphi\right\rangle.$$

Then

$$\begin{aligned}\left|\Delta\psi_\delta(x)\right|^2&=\left(\ln\frac{R_1}{|x|}\right)^{1-2\delta}(\Delta\varphi)^2+\left(\Delta\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\right)^2\varphi^2+4\left(\left\langle\nabla\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta},\nabla\varphi\right\rangle\right)^2\\&\quad+2\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\Delta\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\varphi\Delta\varphi+4\Delta\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\varphi\left\langle\nabla\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta},\nabla\varphi\right\rangle\\&\quad+4\Delta\varphi\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta}\left\langle\nabla\left(\ln\frac{R_1}{|x|}\right)^{\frac{1}{2}-\delta},\nabla\varphi\right\rangle.\end{aligned}$$

Hence we have

$$\frac{\int_\Omega|\Delta\psi_\delta|^2}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}=\frac{\int_\Omega(\Delta(\ln\frac{R_1}{|x|})^{\frac{1}{2}-\delta})^2\phi^2}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}+\frac{O(1)}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}.$$

This implies

$$\begin{aligned}\frac{\int_\Omega|\Delta\psi_\delta|^2}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}&=4\left(\frac{1}{2}-\delta\right)^2+4\left(\frac{1}{2}-\delta\right)^2\left(\frac{1}{2}+\delta\right)\frac{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^3}}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}\\&\quad+\left(\frac{1}{4}-\delta^2\right)^2\frac{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^3}}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^4}}+\frac{O(1)}{\int_\Omega\frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}}.\end{aligned}$$

Taking the limit as $\delta \to 0$ and noting that

$$\lim_{\delta\to 0}\int_{\Omega} \frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2} = \infty,$$

we have

$$\lim_{\delta\to 0} \frac{\int_\Omega |\Delta\psi_\delta|^2}{\int_\Omega \frac{\psi_\delta^2}{|x|^4(\ln\frac{R_1}{|x|})^2}} = 1.$$

Hence 1 is the best constant in (2.2) and it is never achieved in any bounded domain (this is clear from (2.2)). □

Proof of Theorem 2.1(b). The restriction of Δ to the unit sphere $\mathbb{S}^{n-1}$ will be denoted by Δ_σ, the Laplace–Beltrami operator. Then the Laplacian operator in $\mathbb{R}^n$ can be written in (r,σ) as

$$\Delta = \Delta_r + \frac{1}{r^2}\Delta_\sigma,$$

where Δ_r is the radial Laplacian. For $u \in H_0^2(\Omega) \subset H_0^2(B(R))$, let

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{k_m} u_{i,m}(r)\phi_{i,m}(\sigma),$$

where $\phi_{i,m}$ are the complete orthonormal basis of eigenfunctions of the Laplace–Beltrami operator with eigenvalues $c_m = m(m+2)$, $m \geqslant 0$. Then

$$\int_{\mathbb{R}^4} \frac{|\nabla u|^2}{|x|^2} = \sum_{m=0}^{\infty}\sum_{i=1}^{k_m}\left\{\int_{\mathbb{R}^4} \frac{|\nabla u_{i,m}|^2}{|x|^2} + c_m\int_{\mathbb{R}^4}\frac{u_{i,m}^2}{|x|^4}\right\}.$$

Note that $u_{i,m}(0) = 0$ for $m \geqslant 1$ and hence $\int_{\mathbb{R}^4}\frac{u_{i,m}^2}{|x|^4} < \infty$. Moreover, $\phi_{0,m}(\sigma) = 1$, $u_{0,0}(r)$ is the radial part of u. Now by Euler's theorem $\langle x, \nabla f\rangle = 0$ if f is homogeneous of degree zero and hence $\langle \nabla u_{i,m}, \nabla\phi_{i,m}\rangle = \frac{(u_{i,m})_r}{r}\langle x, \nabla\phi_{i,m}\rangle = 0$. Since $-\Delta_\sigma\phi_{i,m} = c_m\phi_{i,m}$ by direct calculation we have

$$\begin{aligned}
&\int_{\mathbb{R}^4}(\Delta u)^2 - 4\int_{\mathbb{R}^4}\frac{|\nabla u|^2}{|x|^2} \\
&\quad= \sum_{m=1}^{\infty}\sum_{i=1}^{k_m}\left\{\int_{\mathbb{R}^4}|\Delta u_{i,m}|^2 + \left(c_m^2 - 4c_m\right)\int_{\mathbb{R}^4}\frac{u_{i,m}^2}{|x|^4} + (2c_m - 4)\int_{\mathbb{R}^4}\frac{|\nabla u_{i,m}|^2}{|x|^2}\right\} \\
&\qquad+ \left\{\int_{\mathbb{R}^4}|\Delta u_0|^2 - 4\int_{\mathbb{R}^4}\frac{|\nabla u_0|^2}{|x|^2}\right\}.
\end{aligned}$$

Now for $m \geqslant 2$, $(c_m^2 - 4c_m) > 0$, $(2c_m - 4) > 0$, hence from Lemma 3.1, we have

$$\int_{\mathbb{R}^4} |\Delta u|^2 - 4\int_{\mathbb{R}^4} \frac{|\nabla u|^2}{|x|^2} \geqslant \sum_{i=1}^{k_1} \left\{ \int_{\mathbb{R}^4} |\Delta u_{i,1}|^2 + 2\int_{\mathbb{R}^4} \frac{|\nabla u_{i,1}|^2}{|x|^2} - 3\int_{\mathbb{R}^4} \frac{|u_{i,1}|^2}{|x|^4} \right\}$$

$$\geqslant -3\sum_{i=1}^{k_1} \int_{\mathbb{R}^4} \frac{|u_{i,1}|^2}{|x|^4} = -3\int_{\mathbb{R}^4} \frac{|P_1 u|^2}{|x|^4}. \tag{3.4}$$

This proves the required result.

Next let us show that -3 is the best constant.

Claim. *Let $R > 0$ and define $X = \{v \in C_r^2(\overline{B(R)})\colon\ v(0) = v(R) = 0\}$. Moreover, set*

$$\lambda\big(B(R)\big) = \inf_{v \in X} \left\{ \int_{B(R)} (\Delta v)^2 + 2\int_{B(R)} \frac{|\nabla v|^2}{|x|^2} \colon \int_{B(R)} \frac{v^2}{|x|^4} = 1 \right\}.$$

We claim that $\lambda(B(R)) = 0$.

Since v is radial, now by change of variables $|x| = r = 2e^{-t/2}$ we obtain $v(x) = y(t)$, $R = 2e^{-T/2}$, $y(\infty) = y_t(\infty) = 0$, $y(T) = 0$ and y satisfies

$$2\int_{B(R)} \frac{|v|^2}{|x|^4} = \omega_4 \int_T^\infty y^2\,dt,$$

$$\int_{B(R)} (\Delta v)^2 + 2\int_{B(R)} \frac{|\nabla v|^2}{|x|^2} = \omega_4\left[8\int_T^\infty (y_{tt} - y_t)^2\,dt + 4\int_T^\infty y_t^2\,dt\right]$$

$$= \omega_4\left[8\int_T^\infty y_{tt}^2 + 12\int_T^\infty y_t^2\,dt + 8y(T)^2\right]$$

$$= \omega_4\left[8\int_T^\infty y_{tt}^2 + 12\int_T^\infty y_t^2\,dt\right]. \tag{3.5}$$

Hence we have

$$8\int_T^\infty y_{tt}^2 + 12\int_T^\infty y_t^2\,dt \geqslant \lambda\big(B(R)\big)\int_T^\infty y^2\,dt.$$

Setting $\theta = t - T$ and $y(t) = z(\theta)$ the above inequality gives

$$8\int_0^\infty z_{\theta\theta}^2 + 12\int_0^\infty z_\theta^2\,dt \geqslant \lambda\big(B(R)\big)\int_0^\infty z^2\,d\theta. \tag{3.6}$$

Let $\alpha > 0$, $0 \leqslant \varphi \in C_0^2(0,\infty)$ and $z_\alpha(\theta) = \varphi(\alpha\theta)$, then (3.6) gives

$$8\alpha^4 \int_0^\infty \varphi_{\theta\theta}^2 + 12\alpha^2 \int_0^\infty \varphi_\theta^2\, dt \geqslant \lambda\big(B(R)\big) \int_0^\infty \varphi^2\, d\theta$$

and letting $\alpha \to 0$ we get $\lambda(B(R)) = 0$. This proves the claim.

Suppose now that -3 is not the best constant. Then from (3.6) it follows that $\lambda(B(R)) > 0$ which contradicts our claim. Now we prove (2.4).

Let $z \in C^2(0,\infty)$; $z(0) = z_t(0) = z(\infty) = z_t(\infty) = 0$. Hence $z(t) = \int_0^t z_t(\theta)\, d\theta$ and by Hardy's inequality

$$\int_0^\infty \frac{z^2}{\theta^2} \leqslant 4 \int_0^\infty z_\theta^2, \qquad \int_0^\infty \frac{z_\theta^2}{\theta^2} \leqslant 4 \int_0^\infty z_{\theta\theta}^2.$$

So we have

$$\int_0^\infty \frac{z z_\theta}{\theta^3} = \frac{1}{2} \int_0^\infty \frac{(z^2)_\theta}{\theta^3} = \frac{3}{2} \int_0^\infty \frac{z^2}{\theta^4}$$

and by Hölder inequality

$$\int_0^\infty \frac{z^2}{\theta^4} \leqslant \frac{4}{9} \int_0^\infty \frac{z_\theta^2}{\theta^2}.$$

Then

$$\int_0^\infty \frac{z^2}{\theta^4} \leqslant \frac{16}{9} \int_0^\infty z_{\theta\theta}^2 \quad \text{and}$$

$$8 \int_0^\infty z_{\theta\theta}^2 + 12 \int_0^\infty z_\theta^2 \geqslant \frac{9}{2} \int_0^\infty \frac{z^2}{\theta^4} + 3 \int_0^\infty \frac{z^2}{\theta^2}.$$

Going back to (3.5), we obtain

$$8 \int_T^\infty y_{tt}^2 + 12 \int_T^\infty y_t^2 \geqslant \frac{9}{2} \int_0^\infty \frac{y^2}{(t-T)^4} + 3 \int_0^\infty \frac{y^2}{(t-T)^2}.$$

Now $t - T = 2\ln\frac{R}{r}$ and by taking $v = u_{i,1}$ we have from (3.5)

$$\int_{B(R)} \left(|\Delta v|^2 + 2\frac{|\nabla v|^2}{|x|^2} + 3\frac{|v|^2}{|x|^4} \right) \geqslant \frac{3}{4} \int_\Omega \frac{|v|^2}{|x|^4 (\ln\frac{R}{|x|})^2} + \frac{9}{32} \int_\Omega \frac{|v|^2}{|x|^4 (\ln\frac{R}{|x|})^4}.$$

Substituting this in (3.4) we obtain

$$\int_\Omega |\Delta u|^2 - 4\int_\Omega \frac{|\nabla u|^2}{|x|^2} \geqslant -3\int_\Omega \frac{(P_1 u)^2}{|x|^4} + \frac{3}{4}\int_\Omega \frac{|P_1 u|^2}{|x|^4(\ln\frac{R}{|x|})^2} + \frac{9}{32}\int_\Omega \frac{|P_1 u|^2}{|x|^4(\ln\frac{R}{|x|})^4},$$

where -4, -3, $\frac{3}{4}$, $\frac{9}{32}$ are best constants which are never achieved. This proves the theorem. □

Remark 3.1. We are able to generalize Hardy–Rellich type of inequality for p-biharmonic operators where $n = 2p$ in [5]. In that paper we have completely characterized the Hardy–Rellich inequalities in the critical dimension but in the radial case. Note that the method of Szegő cannot be used in higher order Sobolev spaces in $H_0^m(\Omega)$ where $m > 2$.

4. A maximum principle

Here we prove the maximum principle using the continuation method for the Navier boundary condition which is good enough for our purpose.

The operator $\Delta^2 - V$ is said to be coercive on $H^2(\Omega) \cap H_0^1(\Omega)$ if

$$\int_\Omega (\Delta u)^2 - \int_\Omega V u^2 \geqslant C\int_\Omega (\Delta u)^2$$

for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and for some $C > 0$.

Main lemma. *Let $V \in L^\infty(\Omega)$, $V \geqslant 0$ and the operator $\Delta^2 - V$ be coercive on $H^2(\Omega) \cap H_0^1(\Omega)$. Let $f \in L^2(\Omega)$, $\phi \in H^{5/2}(\partial\Omega)$, $\psi \in H^{3/2}(\partial\Omega)$ such that $f \geqslant 0$, $\phi \geqslant 0$, $\psi \geqslant 0$. Let $u \in H^2(\Omega)$ be a solution of*

$$\text{(A)}\quad \begin{cases} \Delta^2 u - V u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ -\Delta u = \psi & \text{on } \partial\Omega. \end{cases}$$

Then $u \geqslant 0$ in Ω.

Proof. Since Δ^2 satisfies weak maximum principle with respect to nonzero Navier data, we have that

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ -\Delta u = \psi & \text{on } \partial\Omega \end{cases}$$

implies that $u \geqslant 0$ in Ω. Hence the solution u can be written in terms of the Green operator as

$$u = G(f) + G_1(\phi) + G_2(\psi),$$

where G, G_1, G_2 are the integral operators with positive kernels. Let $0 < \epsilon \leqslant 1$ and consider the perturbation of the above problem with the same boundary conditions

$$\Delta^2 u = \epsilon V u + f.$$

Since the operator is coercive there exists a unique solution given by

$$u = G(\epsilon V u + f) + G_1(\phi) + G_2(\psi).$$

Hence

$$(I - \epsilon G_V)(u) = G(f) + G_1(\phi) + G_2(\psi),$$

where $G_V(u) = G(Vu)$. Since $0 \leqslant V \leqslant C$ we have that $G_V(u)$ is a bounded operator. This implies that for small $\epsilon > 0$, it exists $(I - \epsilon G_V(u))^{-1}$ and it is an integral operator with non-negative kernels. This implies

$$u = (I - \epsilon G_V)^{-1}\{G(f) + G_1(\phi) + G_2(\psi)\} \geqslant 0.$$

Let

$$\mathcal{A} = \{t \in [0,1]\colon \forall \epsilon \in [0,t],\ (\Delta^2 - \epsilon V) \text{ satisfies weak maximum principle with nonzero Navier data}\}.$$

Then we have $\mathcal{A} \neq \emptyset$. We claim that $\sup \mathcal{A} = 1$.

Suppose $\sup \mathcal{A} = t_0 < 1$. Then $(\Delta^2 - t_0 V)$ is coercive and by continuity satisfies the weak maximum principle with respect to nonzero Navier data. Hence by the above argument we can find an $\epsilon_0 > 0$ such that $\forall 0 < \epsilon \leqslant \epsilon_0$, $(\Delta^2 - (t_0 + \epsilon)V)$ satisfies the weak maximum principle with nonzero Navier data which implies a contradiction. Hence $\sup \mathcal{A} = 1$. Thus if u is a solution of (A), then $u \geqslant 0$ in Ω. □

Corollary 4.1. *Let $n \geqslant 5$, $V \in L^\infty(\Omega)$ and $V \geqslant 0$. Let us suppose that the operator $\Delta^2 - \frac{V}{|x|^4}$ is coercive on $H^2(\Omega) \cap H_0^1(\Omega)$. Let $f \in L^2(\Omega)$, $\phi \in H^{5/2}(\partial\Omega)$, $\psi \in H^{3/2}(\partial\Omega)$ such that $f \geqslant 0$, $\phi \geqslant 0$, $\psi \geqslant 0$. Let $u \in H^2(\Omega)$ be a solution of*

$$\text{(B)} \quad \begin{cases} \Delta^2 u - \frac{V}{|x|^4} u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ -\Delta u = \psi & \text{on } \partial\Omega. \end{cases}$$

Then $u \geqslant 0$ in Ω.

Proof. Let $B \subset \Omega$. Choose $\varphi \in C_c^\infty(\Omega)$ and $0 \leqslant \varphi \leqslant 1$ such that

$$\varphi(x) = \begin{cases} 1 & \text{in } B(\frac{1}{2}), \\ 0 & \text{on } \Omega \setminus B. \end{cases}$$

Choose $\delta > 0$ and $\varphi_\delta(x) = \varphi(\frac{x}{\delta})$. Define $V_\delta = \frac{1-\varphi_\delta(x)}{|x|^4} V$. Then $V_\delta \in L^\infty(\Omega)$.

Now consider the problem

$$\text{(C)}\quad \begin{cases} \Delta^2 u - V_\delta u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ -\Delta u = \psi & \text{on } \partial\Omega. \end{cases}$$

Then

$$\begin{aligned} \langle \Delta^2 u - V_\delta u, u \rangle_{L^2(\Omega)} &= \left\langle \Delta^2 u - \frac{V}{|x|^4} u, u \right\rangle_{L^2(\Omega)} + \left\langle \frac{\varphi_\delta(x)}{|x|^4} V u, u \right\rangle_{L^2(\Omega)} \\ &\geqslant \left\langle \Delta^2 u - \frac{V}{|x|^4} u, u \right\rangle_{L^2(\Omega)}. \end{aligned}$$

Hence $(\Delta^2 - V_\delta)$ is coercive in $H^2(\Omega) \cap H_0^1(\Omega)$. Note that the coercivity is independent of the choice of δ. Hence by the previous theorem if u_δ is a solution to (C) then $u_\delta \geqslant 0$. Now we claim that $u_\delta(x) \to u(x)$ as $\delta \to 0$.

Set $w_\delta = u_\delta - u$. Then we have that w_δ satisfies

$$\begin{cases} \Delta^2 w_\delta - V_\delta u_\delta + \frac{V}{|x|^4} u = 0 & \text{in } \Omega, \\ w_\delta = 0 & \text{on } \partial\Omega, \\ -\Delta w_\delta = 0 & \text{on } \partial\Omega. \end{cases}$$

We have

$$\left\langle \Delta^2 w_\delta - V_\delta w_\delta - \left(V_\delta - \frac{V}{|x|^4} \right) u, w_\delta \right\rangle_{L^2(\Omega)} = 0.$$

Hence,

$$\langle \Delta^2 w_\delta - V_\delta w_\delta, w_\delta \rangle_{L^2(\Omega)} + \int_\Omega \frac{V u w_\delta \varphi_\delta}{|x|^4} = 0.$$

Then

$$\begin{aligned} \|w_\delta\|^2_{H^2(\Omega)\cap H_0^1(\Omega)} &\leqslant C_1 \int_\Omega \frac{V|u w_\delta|\varphi_\delta}{|x|^4} \leqslant C_1 \left(\int_\Omega \frac{w_\delta^2}{|x|^4} \right)^{1/2} \left(\int_\Omega \frac{u^2}{|x|^4} \varphi_\delta^2 \right)^{1/2} \\ &\leqslant C_1 \|w_\delta\|_{H^2(\Omega)\cap H_0^1(\Omega)} \left(\int_\Omega \frac{u^2}{|x|^4} \varphi_\delta^2 \right)^{1/2}. \end{aligned}$$

Thus we have

$$\|w_\delta\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leqslant C_1 \left(\int_\Omega \frac{u^2}{|x|^4} \varphi_\delta^2 \right)^{1/2}.$$

Hence by dominated convergence theorem we have that $w_\delta \to 0$ in $H^2(\Omega) \cap H_0^1(\Omega)$. As $u_\delta \geqslant 0$ we also have $u \geqslant 0$ and u is a solution of (B). □

The same proof of the previous theorem gives the following result.

Corollary 4.2. *Let $n = 4$, $V \in L^\infty(\Omega)$, $V(x) \geqslant 0$ and suppose that the operator*

$$\Delta^2 - \frac{V}{|x|^4 (\ln \frac{R}{|x|})^2}$$

is coercive on $H^2(\Omega) \cap H_0^1(\Omega)$. Let $f \in L^2(\Omega)$, $\phi \in H^{5/2}(\partial\Omega)$, $\psi \in H^{3/2}(\partial\Omega)$ such that $f \geqslant 0$, $\phi \geqslant 0$, $\psi \geqslant 0$. Let $u \in H^2(\Omega)$ be a solution of

$$\text{(D)} \qquad \begin{cases} \Delta^2 u - \frac{V}{|x|^4 (\ln \frac{R}{|x|})^2} u = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ -\Delta u = \psi & \text{on } \partial\Omega. \end{cases}$$

Then $u \geqslant 0$ in Ω.

We end this section by stating a general maximum principle for differential operator of even order.

Corollary 4.3. *Let us assume that the L is a differential operator of even order $4k$ and*:

(i) *$V \in L^\infty(\Omega)$ and $V \geqslant 0$*;
(ii) *$L - V$ is coercive on $H^{2k}(\Omega) \cap H_0^k(\Omega)$ and self-adjoint*;
(iii) *$L - V$ satisfies weak maximum principle with respect to nonzero Navier data.*

Let us consider the problem

$$\text{(E)} \qquad \begin{cases} Lu - Vu = f & \text{in } \Omega, \\ (-\Delta)^{i-1} u = \psi_{i-1} & \text{on } \partial\Omega. \end{cases}$$

Then $f \geqslant 0$, $\psi_{i-1} \geqslant 0$ for $i \in \{1, 2, \ldots, 2k\}$ and $\psi_{i-1} \in C^\infty(\partial\Omega)$ implies that $u \geqslant 0$ in Ω.

5. Proof of the theorems

In view of the lack of maximum principle for the Dirichlet boundary condition we will only prove the theorems in this case (the case of Navier boundary conditions follows in a similar way). In order to follow the same proofs as in [4], we need some test functions and their main properties will be proved in Appendix A. Let us recall some known results for biharmonic operator:

Boggio's Principle. Consider the biharmonic equation

$$\text{(F)} \qquad \begin{cases} \Delta^2 u = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on } \partial B, \end{cases}$$

where $B = \{x \in \mathbb{R}^n\colon |x| < 1\}$ and γ is the outer normal at the boundary of B. Then Boggio's principle [7] states that the Green function associated to the biharmonic problem with zero Dirichlet data in a ball is strictly positive. Hence if $f \geqslant 0$ a.e. then $u > 0$ in B. If f has enough regularity such that $u \in C^2(\overline{B})$ then we have an analogue of Hopf maximum principle, i.e., $\frac{\partial^2 u}{\partial \gamma^2} > 0$ on ∂B (this was proved by Grunau and Sweers in [13]).

Remark 5.1. Consider the problem

$$\begin{cases} \Delta^2 u \geqslant 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on } \partial B. \end{cases}$$

If $u \in C^2(\overline{B})$, then Δu changes sign. Suppose Δu has a definite sign. Without loss of generality suppose $-\Delta u > 0$ in B, then Hopf maximum principle says that $\frac{\partial u}{\partial \gamma} < 0$ on ∂B which contradicts the second boundary condition.

Theorem 5.1. *Consider the problem*

$$\begin{cases} \Delta^2 u - \frac{n^2(n-4)^2}{16} q(x) \frac{u}{|x|^4} = \lambda u & \text{in } B, \\ u \neq 0 & \text{in } B, \\ u \in H_0^2(B), \end{cases} \tag{5.1}$$

where B is the unit ball centered at origin. If (5.1) *admits a solution u for some $\lambda = \lambda(q)$, then u does not change sign in B.*

Proof. Note that proving existence of positive solutions is quite hard in the sense that $u^+, u^- \notin H_0^2(B)$, which played a crucial role in second order equations. Suppose $u \in H_0^2(B)$ solves the above problem with $\lambda = \lambda(q)$ with (2.9) and u changes sign. Define

$$K := \{v \in H_0^2(B)\colon\ v \geqslant 0 \text{ a.e.}\}.$$

Let

$$a(u, v) = \langle u, v\rangle_{H_0^2(B)} = \int_B \Delta u \Delta v, \quad \forall u, v \in H_0^2(B).$$

Note that K is a closed convex cone. Hence there exists a projection $P : H_0^2(B) \to K$ such that for all $u \in H_0^2(B)$, $\forall w \in K$

$$a\big(u - P(u), w - P(u)\big) \leqslant 0. \tag{5.2}$$

Since K is a cone we can replace w by tw for $t > 0$ and letting $t \to \infty$ to obtain

$$a\big(u - P(u), w\big) \leqslant \lim_{t\to\infty} \frac{1}{t} a\big(u - P(u), P(u)\big)$$

which implies that $\Delta^2(u - P(u)) \leqslant 0$ and by Boggio's principle, $u - P(u) \leqslant 0$.

Now replacing w by $tP(u)$ for $t>0$ in (5.2) we have

$$(t-1)a\big(u-P(u),P(u)\big)\leqslant 0$$

and hence $a(u-P(u),P(u))=0$.

Hence we can write $u=u_1+u_2$, $u_1=P(u)\in K$, $u_2=u-P(u)$, $u_1\perp u_2$ and $u_2\leqslant 0$. Since u changes sign we have that $u_1\not\equiv 0$ and $u_2\not\equiv 0$. Therefore we have,

$$\frac{\int_B|\Delta(u_1-u_2)|^2-\frac{n^2(n-4)^2}{16}\int_B\frac{q(u_1-u_2)^2}{|x|^4}}{\int_B(u_1-u_2)^2}<\frac{\int_B|\Delta(u_1+u_2)|^2-\frac{n^2(n-4)^2}{16}\int_B\frac{q(u_1+u_2)^2}{|x|^4}}{\int_B(u_1+u_2)^2}$$

which contradicts (2.6). Then u does not change sign and noting that the Green function is strictly positive we have either $u>0$ or $u<0$ in B. □

Proof of Theorem 2.2(i) *(Existence).* Let q be as in the assumption, and $0<s<1$ and

$$\lambda_s(q):=\inf_{u\in H_0^2(\Omega)}\left\{\int_\Omega|\Delta u|^2-\frac{n^2(n-4)^2}{16}s\int_\Omega\frac{qu^2}{|x|^4}\colon\int_\Omega u^2=1\right\}.$$

From (1.1) the operator $\Delta^2-\frac{n^2(n-4)^2}{16}(\frac{sq}{|x|^4})$ defined on $H_0^2(\Omega)$ is coercive for $0<s<1$. Hence there exists a $u_s\in H_0^2(\Omega)$ satisfying

$$\begin{cases}\Delta^2u_s-\frac{n^2(n-4)^2}{16}sq(x)\frac{u_s}{|x|^4}=\lambda_s(q)u_s & \text{in }\Omega,\\ u_s\neq 0 & \text{in }\Omega,\\ u_s\in H_0^2(\Omega),\\ \|u_s\|_{H_0^2(\Omega)}=\int_\Omega|\Delta u_s|^2=1.\end{cases}$$

Then we have $u_s\rightharpoonup u_1$ in $H_0^2(\Omega)$, $u_s\to u_1$ in $L^2(\Omega)$ and $u_s\to u_1$ a.e. Let

$$u(x)=|x|^{-\frac{n-4}{2}}\left(\ln\frac{1}{|x|}\right)^{-\delta}$$

with $\delta>\frac12$. From Lemma A.1, we have

$$\begin{aligned}&\Delta^2u-\frac{n^2(n-4)^2}{16}q(x)\frac{u}{|x|^4}-\lambda_s(q)u\\ &\quad=\frac{n^2(n-4)^2}{16}\frac{u}{|x|^4(\ln\frac{1}{|x|})^2}\left[\big(1-q(x)\big)\left(\ln\frac{1}{|x|}\right)^2-\frac{8\delta(\delta+1)(n^2-4n+8)}{n^2(n-4)^2}\right.\\ &\qquad\left.+\frac{16}{n^2(n-4)^2}\frac{\delta(\delta+1)(\delta+2)(\delta+3)}{(\ln\frac{1}{|x|})^2}-\frac{16}{n^2(n-4)^2}\lambda_s(q)|x|^4\left(\ln\frac{1}{|x|}\right)^2\right].\end{aligned}$$

Choose $\delta > \frac{1}{2}$ such that

$$\liminf_{x\to 0}\left(\ln\frac{1}{|x|}\right)^2\big(1-q(x)\big) > \frac{8\delta(\delta+1)(n^2-4n+8)}{n^2(n-4)^2}$$

(this is possible by assumption (2.12)). From (2.12) we can find an $R>0$ such that $\overline{B(R)}\subset\Omega$,

$$\begin{cases}\Delta^2 u - \frac{n^2(n-4)^2}{16}q(x)\frac{u}{|x|^4} - \lambda_s(q)u \geqslant 0 & \text{in } \overline{B(R)},\\ \Delta u < 0 & \text{in } \partial B(R).\end{cases}$$

Now using standard elliptic estimates, we can find $M>0$ such that

$$\begin{cases}u_s \leqslant Mu & \text{on } \partial B(R),\\ -\Delta u_s \leqslant -M\Delta u & \text{on } \partial B(R).\end{cases}$$

Let $w_s := u_s - Mu$. Then

$$\begin{cases}w_s \leqslant 0 & \text{on } \partial B(R),\\ -\Delta w_s \leqslant 0 & \text{on } \partial B(R).\end{cases}$$

We are required to show that $w_s \leqslant 0$ in $B(R)$.

Now,

$$\Delta^2 w_s - \frac{n^2(n-4)^2}{16}sq(x)\frac{w_s}{|x|^4} - \lambda_s(q)w_s \leqslant -M\frac{n^2(n-4)^2}{16|x|^4}(1-s)qu.$$

Thus we have

$$\Delta^2 w_s - \frac{n^2(n-4)^2}{16}sq(x)\frac{w_s}{|x|^4} - \lambda_s(q)w_s \leqslant 0 \quad \text{in } B(R).$$

So, we are in the case

$$\begin{cases}\Delta^2 w_s - \frac{n^2(n-4)^2}{16}sq(x)\frac{w_s}{|x|^4} - \lambda_s(q)w_s \leqslant 0 & \text{in } B(R),\\ w_s \leqslant 0 & \text{on } \partial B(R),\\ -\Delta w_s \leqslant 0 & \text{on } \partial B(R).\end{cases}$$

Claim. *For $R>0$ sufficiently small, the operator*

$$\Delta^2 - \frac{n^2(n-4)^2}{16}sq(x)\frac{1}{|x|^4} - \lambda_s(q)$$

is coercive in $H^2(B(R))\cap H_0^1(B(R))$.

Let $q_R(x) = q(Rx)$ for $|x| \leqslant 1$, and

$$\mu(R, sq) = \inf_{H^2(B(R))\cap H_0^1(B(R))} \left\{ \int_{B(R)} |\Delta v|^2 - \frac{n^2(n-4)^2}{16} s \int_{B(R)} \frac{qv^2}{|x|^4} \colon \int_{B(R)} v^2 = 1 \right\}.$$

Then

$$\mu(R, sq) = \frac{1}{R^4}\mu(1, sq) \geqslant \frac{1}{R^4}\mu(1, 1).$$

Hence $\mu(R, sq) \to \infty$ as $R \to 0$, uniformly in s and q. Since $\{\lambda_s(q)\}$ is bounded for $0 < s < 1$ and hence for $v \in H^2(B(R)) \cap H_0^1(B(R))$

$$\begin{aligned}
&\int_{B(R)} |\Delta v|^2 - \frac{n^2(n-4)^2}{16} s \int_{B(R)} \frac{qv^2}{|x|^4} - \lambda_s(q) \int_{B(R)} v^2 \\
&\quad = (1-s)\left\{ \int_{B(R)} |\Delta v|^2 - \lambda_s(q) \int_{B(R)} v^2 \right\} \\
&\qquad + s\left\{ \int_{B(R)} |\Delta v|^2 - \frac{n^2(n-4)^2}{16} \int_{B(R)} \frac{qv^2}{|x|^4} - \lambda_s(q) \int_{B(R)} v^2 \right\} \\
&\quad \geqslant (1-s)\left(1 - \frac{\lambda_s(q)}{\mu(R,0)}\right) \int_{B(R)} |\Delta v|^2 + s\bigl(\mu(R,q) - \lambda_s(q)\bigr) \int_{B(R)} v^2 \\
&\quad \geqslant \frac{1-s}{2} \int_{B(R)} |\Delta v|^2.
\end{aligned}$$

This proves the claim.

Therefore from Corollary 4.1, we have $w_s \leqslant 0$ in $B(R)$. This implies that $u_s \leqslant Mu$ in $B(R)$. Since $-u_s$ is also a solution we have that $|u_s| \leqslant Mu$ in $B(R)$. Hence we have

$$\frac{{u_s}^2}{|x|^4} \leqslant M^2 \frac{u^2}{|x|^4}.$$

Since $\int_{B(R)} \frac{u^2}{|x|^4} < \infty$ by dominated convergence theorem we have

$$\int_\Omega \frac{qu_s^2}{|x|^4} \to \int_\Omega \frac{qu_1^2}{|x|^4}, \qquad 1 = \int_\Omega u_s^2 \to \int_\Omega u_1^2$$

as $s \to 1$. Therefore from the weak-lower semicontinuity of $H_0^2(\Omega)$ norm and the fact that $\lambda_s(q) \to \lambda(q)$ as $s \to 1$ we have

$$I_q(u_1) \leqslant \liminf_{s\to 1} I_q(u_s).$$

This implies that $\|u_s\|_{H_0^2(\Omega)} \to \|u_1\|_{H_0^2(\Omega)}$ and hence $u_s \to u_1$. Hence $\lambda_D(q)$ is attained by u_1. Note that $u_1 > 0$ if Ω is a ball (see Theorem 5.1). □

Proof of Theorem 2.2(ii) *(Non-existence).* We argue by contradiction. Suppose that $\lambda_D(q)$ is attained at $u_1 \in H_0^2(B)$. We claim that there exist $m > 0$, $R > 0$ such that

$$u_1 \geqslant m|x|^{-\frac{n-4}{2}}\left(\ln\frac{1}{|x|}\right)^{-1/2} \quad \text{in } B(R).$$

This will lead to a contradiction, since by Hardy–Rellich's inequality

$$\frac{u_1^2}{|x|^4} \in L^1\big(B(R)\big) \quad \text{but} \quad \int\limits_{B(R)} |x|^{-n}\left(\ln\frac{1}{|x|}\right)^{-1} = \infty.$$

In order to prove this we have the following

Claim 1. *There exists* $0 < R_1 < 1$ *such that* $-\Delta u_1(x) > 0$ *in* $B(R_1)$.

From Corollary 4.1 we can assume that $u_1 > 0$ in B. So we have $\Delta^2 u_1 = h > 0$ in B. Let $G(x, y)$ denote the Green function associated with the biharmonic operator with zero Dirichlet condition in the unit ball. Then by [12],

$$G(x, y) = c_n\left([xy]^{4-n} - [XY]^{4-n} - \frac{n-4}{2}\big(1-|x|^2\big)\big(1-|y|^2\big)[XY]^{2-n}\right),$$

where

$$c_n > 0, \qquad [xy] = |x-y|, \qquad [XY] = \left||x|y - \frac{x}{|x|}\right|.$$

Then

$$\Delta_x G(0, y) = c_n(n-4)\left(-\frac{2}{|y|^{n-2}} + (2-n)|y|^2 + n\right).$$

Hence $-\Delta_x G(0, y) > 0$ for all $y \in B$. By continuity we have that for all $\epsilon > 0$ there exists $R_1(\epsilon) < 1$ such that for $|y| \leqslant (1-\epsilon)$, it holds $-\Delta_x G(x, y) > 0$, $\forall x \in B(R_1(\epsilon))$. Suppose the claim is not true. Then there exists a sequence $x_k \neq 0$ such that $x_k \to 0$ as $k \to \infty$ and $-\Delta u_1(x_k) \leqslant 0$. Hence for large k and by Fatou's lemma

$$\begin{aligned}
0 &\geqslant \lim_{x_k \to 0} -\Delta u_1(x_k) = \lim_{x_k \to 0}\left\{\int\limits_{B(1-\epsilon)} \big(-\Delta G(x_k, y)h(y)\big)\,dy + \int\limits_{B\setminus B(1-\epsilon)} \big(-\Delta G(x_k, y)h(y)\big)\,dy\right\} \\
&\geqslant \left\{\int\limits_{B(1-\epsilon)} \liminf_{x_k \to 0}\big(-\Delta G(x_k, y)\big)h(y)\,dy + \int\limits_{B\setminus B(1-\epsilon)} \big(-\Delta G(0, y)h(y)\big)\,dy\right\} \\
&= -\int\limits_B \Delta G(0, y)h(y)\,dy > 0
\end{aligned}$$

which is a contradiction. This proves the claim.

Since the operator $\Delta^2 - \frac{n^2(n-4)^2}{16}\frac{q}{|x|^4}$ is non-negative but need not be coercive. Hence we cannot apply Corollary 4.1 to obtain lower bound for u_1. For this we have the following

Claim 2. *Consider the problem*

$$(\mathrm{G})\qquad \begin{cases} \Delta^2 u - \frac{n^2(n-4)^2}{16}\frac{qu}{|x|^4} = f & \text{in } B,\\ u = \phi & \text{on } \partial B,\\ -\Delta u = \psi & \text{on } \partial B. \end{cases}$$

Let $f \in L^2(B)$, $\phi \in H^{52}(\partial B)$, $\psi \in H^{3/2}(\partial B)$ such that $f \geqslant 0$, $\phi \geqslant 0$, $\psi \geqslant 0$ and is a solution of (G). Then $u \geqslant 0$.

For $s \in (0,1)$ as the operator $\Delta^2 - \frac{n^2(n-4)^2}{16}s\frac{q}{|x|^4}$ is coercive on $H^2(B) \cap H_0^1(B)$ by Corollary 4.1, there exists a unique solution $u_s \geqslant 0$ in B satisfying

$$(\mathrm{H})\qquad \begin{cases} \Delta^2 u_s - \frac{n^2(n-4)^2}{16}s\frac{qu_s}{|x|^4} = f & \text{in } B,\\ u_s = \phi & \text{on } \partial B,\\ -\Delta u_s = \psi & \text{on } \partial B. \end{cases}$$

Subtracting (G) from (H) and $v_s := u_s - u$ we have

$$\int_B (\Delta v_s)^2 - \frac{n^2(n-4)^2}{16}s\int_B \frac{qv_s^2}{|x|^4} = \frac{n^2(n-4)^2}{16}(1-s)\int_B \frac{quv_s}{|x|^4}. \tag{5.3}$$

Hence from (5.3) we have

$$\left\{\int_B (\Delta v_s)^2 - \frac{n^2(n-4)^2}{16}\int_B \frac{qv_s^2}{|x|^4}\right\} + \frac{n^2(n-4)^2}{16}(1-s)\int_B \frac{qv_s^2}{|x|^4} = \frac{n^2(n-4)^2}{16}(1-s)\int_B \frac{quv_s}{|x|^4}.$$

As the term in the curly bracket is non-negative we have

$$\frac{n^2(n-4)^2}{16}(1-s)\int_B \frac{q{v_s}^2}{|x|^4} \leqslant \frac{n^2(n-4)^2}{16}(1-s)\int_B \frac{quv_s}{|x|^4}.$$

By the Hölder inequality,

$$\int_B \frac{qv_s^2}{|x|^4} \leqslant C.$$

Therefore from (5.3) and $W^{1,p}(B)$ estimates (A.17) we have

$$\|v_s\|^2_{W_0^{1,p}(B)} \leqslant C_1 \frac{n^2(n-4)^2}{16}(1-s)\int_B \frac{quv_s}{|x|^4}$$
$$\leqslant C_1(1-s)\left(\int_B \frac{qu^2}{|x|^4}\right)^{1/2}\left(\int_B \frac{qv_s^2}{|x|^4}\right)^{1/2} \leqslant C_2(1-s).$$

This implies that $\|v_s\|^2_{W_0^{1,p}(B)} \to 0$ as $s \to 1$. As $u_s \geqslant 0$ we have $u \geqslant 0$ in B. This proves the claim.

Let

$$\varphi_s(x) := |x|^{-\frac{n-4}{2}}\left(\ln \frac{1}{|x|}\right)^{-s/2},$$

where $s > 1$. Note that $\varphi_s \in H^2(B(R))$ and from Lemma A.1 φ_s satisfies

$$\Delta^2\varphi_s - \frac{n^2(n-4)^2}{16} q \frac{\varphi_s}{|x|^4} = \frac{n^2(n-4)^2}{16} \frac{\varphi_s}{|x|^4(\ln \frac{1}{|x|})^2}\left[\big(1-q(x)\big)\left(\ln \frac{1}{|x|}\right)^2 - \frac{2s(s+2)(n^2-4n+8)}{n^2(n-4)^2} + \frac{s(s+2)(s+4)(s+6)}{n^2(n-4)^2} \frac{1}{(\ln \frac{1}{|x|})^2}\right].$$

Hence from (2.13) we can choose $R_2 > 0$ such that

$$\Delta^2\varphi_s - \frac{n^2(n-4)^2}{16} \frac{q\varphi_s}{|x|^4} \leqslant 0 \quad \text{in } B(R_2).$$

Let $0 < R < \min\{R_1, R_2\}$. Then from Claim 1 and Corollary 4.1 $u_1 > 0$, $-\Delta u_1 > 0$ in $B(R)$ and so we can choose $m > 0$ independent of s such that

$$\begin{cases} u_1 \geqslant m\varphi_s & \text{on } \partial B(R), \\ -\Delta u_1 \geqslant -m\Delta\varphi_s & \text{on } \partial B(R). \end{cases}$$

Define $w_s := u_1 - m\varphi_s$. Then we have

$$\begin{cases} \Delta^2 w_s - \frac{n^2(n-4)^2}{16}\frac{qw_s}{|x|^4} \geqslant 0 & \text{in } B(R), \\ w_s \geqslant 0 & \text{on } \partial B(R), \\ -\Delta w_s \geqslant 0 & \text{on } \partial B(R). \end{cases}$$

This implies that $w_s \geqslant 0$ in $B(R)$ by Claim 2. This proves 2.2(ii). □

Proof of Theorem 2.2(iii) *(Existence of $W_0^{2,p}$ solution).* Let $0 < \nu < 1$ and u_ν satisfy $L_{\nu q} u_\nu = \lambda u_\nu$ with $\int_B u_\nu^2 = 1$ and $\lambda = \lambda_\nu(q)$. Note that the existence of u_ν is assured by the fact that $\Delta^2 - \frac{n^2(n-4)^2}{16}\nu q \frac{1}{|x|^4}$ is coercive on $H_0^2(\Omega)$. We will show that $u_\nu \to u_1$ in $W_0^{2,p}(\Omega)$ for all $p < 2$.

Let $\xi_\nu(x) = |x|^{-\frac{(n-4)}{2}\nu}(\ln\frac{1}{|x|})^\delta$ and $\delta \in \mathbb{R}$. Then from Lemma A.1, we have the following:

$$\Delta^2\xi_\nu = \frac{\xi_\nu}{|x|^4}\Bigg[\frac{(n-4)^2}{16}\big(4+\nu(n-4)\big)(2-\nu)\big(2(n-2)-(n-4)\nu\big) - \frac{n-4}{2}\frac{\{\nu^2(n-4)^2 - 2\nu(n-4)^2 - 4(n-2)\}(1-\nu)\delta}{(\ln\frac{1}{|x|})} - \frac{\delta(1-\delta)}{2}\frac{\{3\nu^2(n-4)^2 - 6\nu(n-4)^2 + 2(n(n-10)+20)\}}{(\ln\frac{1}{|x|})^2} - \frac{2\delta(1-\delta)(n-4)(1-\nu)}{(\ln\frac{1}{|x|})^3} - \frac{\delta(1-\delta)(2-\delta)(3-\delta)}{(\ln\frac{1}{|x|})^4}\Bigg].$$

Note that for $0 < R < 1$, $\xi_\nu \in H^2(B(R))$ iff $\nu < 1$. Fix $0 < \delta < 1$. Therefore there exists $R > 0$ such that

$$\Delta^2\xi_\nu - \frac{n^2(n-4)^2}{16}\nu\frac{q\xi_\nu}{|x|^4} - \lambda_\nu(q)\xi_\nu \geqslant 0 \quad \text{in } B(R).$$

Since $\xi_\nu > 0$ and $-\Delta\xi_\nu > 0$ in a small ball, using standard elliptic estimates there exists $M_1 > 0$ such that

$$\begin{cases} u_\nu \leqslant M_1\xi_\nu & \text{on } \partial B(R),\\ -\Delta u_\nu \leqslant -M_1\Delta\xi_\nu & \text{on } \partial B(R).\end{cases}$$

Define $w_\nu = u_\nu - M_1\xi_\nu$. Then w_ν satisfies

$$\begin{cases} \Delta^2 w_\nu - \frac{n^2(n-4)^2}{16}\nu\frac{qw_\nu}{|x|^4} - \lambda_\nu(q)w_\nu \leqslant 0 & \text{in } B(R),\\ w_\nu \leqslant 0 & \text{on } \partial B(R),\\ -\Delta w_\nu \leqslant 0 & \text{on } \partial B(R).\end{cases}$$

Taking R small enough (if necessary) and proceeding as in the claim of the proof of Theorem 2.2(i) we have

$$\begin{cases} w_\nu \leqslant 0 & \text{in } B(R),\\ -\Delta w_\nu \leqslant 0 & \text{in } B(R),\end{cases}$$

and then as $-u_\nu$ is also a solution, we have

$$\begin{cases} |u_\nu| \leqslant M_1|x|^{-\frac{n-4}{2}\nu}\big(\ln\frac{1}{|x|}\big)^\delta & \text{in } B(R),\\ |\Delta u_\nu| \leqslant -M_1\Delta\xi_\nu & \text{in } B(R).\end{cases} \tag{5.4}$$

Now we claim that there exist $M_1, M_2, M_3, R > 0$ such that for $\nu \in (\frac{1}{2}, 1)$

$$\begin{cases} |u_\nu| \leqslant M_1 |x|^{-\frac{n-4}{2}\nu}\left(\ln \frac{1}{|x|}\right)^\delta & \text{in } B(R), \\ |\nabla u_\nu| \leqslant M_2 |x|^{-(\frac{n-4}{2}\nu+1)}\left(\ln \frac{1}{|x|}\right)^\delta & \text{in } B(R), \\ |\nabla^2 u_\nu| \leqslant M_3 |x|^{-(\frac{n-4}{2}\nu+2)}\left(\ln \frac{1}{|x|}\right)^\delta & \text{in } B(R). \end{cases} \tag{5.5}$$

To prove the above estimates on $|\nabla u_\nu|$, $|\nabla^2 u_\nu|$ we proceed as in [8].

Let $x \in B(\frac{R}{2})$ where R is chosen as above. Let $r = \frac{1}{2}|x|$ and define $\tilde{u}_\nu(y) = u_\nu(x + ry)$ for $y \in B(1)$. Then $\tilde{u}_\nu$ satisfies

$$\Delta^2 \tilde{u}_\nu(y) = \tilde{c}_\nu(y)\tilde{u}_\nu(y),$$

where $|\tilde{c}_\nu(y)| \leqslant C$ and $\nu \in (\frac{1}{2}, 1)$. Then by Green formula we have for all $x \in B(1)$

$$\begin{aligned} \left|\nabla^2 \tilde{u}_\nu(0)\right| + \left|\nabla \tilde{u}_\nu(0)\right| &\leqslant C_1\left(\|\tilde{u}_\nu\|_{L^\infty(B(1))} + \|\Delta \tilde{u}_\nu\|_{L^\infty(B(1))} + \left\|\Delta^2 \tilde{u}_\nu\right\|_{L^\infty(B(1))}\right) \\ &\leqslant 2CC_1\left(\|\tilde{u}_\nu\|_{L^\infty(B(1))} + \|\Delta \tilde{u}_\nu\|_{L^\infty(B(1))}\right). \end{aligned}$$

For $|y| \leqslant 1$ we have $|x + ry| \geqslant |x| - r \geqslant \frac{|x|}{2}$ and hence from (5.4) there exist some $M_1 > 0$, $M_2 > 0$,

$$\begin{aligned} \left|\tilde{u}_\nu(y)\right| + \left|\Delta \tilde{u}_\nu(y)\right| &\leqslant M_1 |x + ry|^{-\frac{n-4}{2}\nu}\left(\ln \frac{1}{|x + ry|}\right)^\delta \left(\frac{r^2}{|x + ry|^2} + 1\right) \\ &\leqslant M_2 |x|^{-\frac{n-4}{2}\nu}\left(\ln \frac{1}{|x|}\right)^\delta. \end{aligned}$$

These estimates proves (5.5). Hence we can find a subsequence $u_\nu \rightharpoonup u_1$ (say) in $W_0^{2,p}(\Omega)$ as $\nu \to 1$. Then $u_\nu \to u_1$ a.e. and $1 = \int_\Omega u_\nu^2 = \int_\Omega u_1^2$. Hence $u_1 \neq 0$ satisfies (2.14) in $\mathcal{D}'(\Omega)$. Furthermore, from Theorem 5.1, $u_\nu > 0$ if $\Omega = B$ and then we have that $u_1 \geqslant 0$ if $\Omega = B$. Note that $u_1 \geqslant 0$ on a set of positive measure in B. This proves the theorem. □

Proof of Theorem 2.3(i) *(Existence).* Let

$$\lambda_s(q) := \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega |\Delta u|^2 - s \int_\Omega \frac{q u^2}{|x|^4 (\ln \frac{R}{|x|})^2} \colon \int_\Omega u^2 = 1 \right\}.$$

Since the operator

$$\Delta^2 - \frac{sq}{|x|^4 (\ln \frac{R}{|x|})^2}$$

defined on $H_0^2(\Omega)$ is coercive for $0 < s < 1$ (Theorem 2.1) and hence there exists $u_s \in H_0^2(\Omega)$ satisfying

$$\begin{cases} \Delta^2 u_s - sq(x)\frac{u_s}{|x|^4(\ln R/|x|)^2} = \lambda_s(q)u_s & \text{in } \Omega, \\ u_s \neq 0 & \text{in } \Omega, \\ u_s \in H_0^2(\Omega), \\ \|u_s\|_{H_0^2(\Omega)} = \int_\Omega |\Delta u_s|^2 = 1. \end{cases}$$

We will prove the existence of a solution to (2.9) by showing that u_s converges in $H_0^2(\Omega)$ to u_1 (say) and u_1 satisfies (2.9).

Let $u(x) = (\ln \frac{R}{|x|})^{1/2}(\ln(\ln \frac{R}{|x|}))^\delta$, where δ is chosen so that $\delta > \frac{1}{2}$ and

$$\liminf_{x\to 0}\left(\ln\left(\ln \frac{R}{|x|}\right)\right)^2\big(1-q(x)\big) > 4\delta(\delta+1).$$

Then from Lemma A.2 and (2.15), for small $R_1 > 0$, we obtain

$$\begin{aligned} &\Delta^2 u - q(x)\frac{u}{|x|^4(\ln \frac{R}{|x|})^2} - \lambda_s(q)u \\ &\quad = \frac{u}{|x|^4(\ln \frac{R}{|x|})^2(\ln(\ln \frac{R}{|x|}))^2}\left[(1-q)\left(\ln\left(\ln \frac{R}{|x|}\right)\right)^2 - 4\delta(\delta+1) + o(1)\right] \\ &\quad \geqslant 0. \end{aligned}$$

Now proceeding exactly as in case of Theorem 2.2(i) we obtain, for all $s \in (0, 1)$,

$$|u_s| \leqslant Mu \quad \text{in } B(R) \tag{5.6}$$

for some $M > 0$. Using the boundedness of u_s in $H_0^2(\Omega)$, we can find $u_1 \in H_0^2(\Omega)$ such that $u_s \rightharpoonup u_1$ (along a subsequence if necessary), strongly in $L^2(\Omega)$ and almost everywhere in Ω as $s \to 1$. Hence by dominated convergence theorem we obtain

$$\int_\Omega \frac{qu_s^2}{|x|^4(\ln \frac{R}{|x|})^2} \to \int_\Omega \frac{qu_1^2}{|x|^4(\ln \frac{R}{|x|})^2} \tag{5.7}$$

and

$$\int_\Omega u_s^2 \to \int_\Omega u_1^2. \tag{5.8}$$

Hence proceeding as in Theorem 2.2(i), we have $u_s \to u_1$ in $H_0^2(\Omega)$. Hence there exists a solution to (2.9) for $\lambda = \lambda(q)$ by passing to the limit as $s \to 1$. □

Proof of Theorem 2.3(ii) *(Non-existence).* By contradiction let us suppose that (2.9) has a non-negative radial solution $u_1 \in H_0^2(B)$ for some $\lambda \geqslant 0$. We claim that there exist $m > 0$, $R_1 > 0$ such that

$$u_1 \geqslant m\left(\ln \frac{R}{|x|}\right)^{1/2}\left(\ln\left(\ln \frac{R}{|x|}\right)\right)^{-1/2} \quad \text{in } B(R_1).$$

This will lead to a contradiction, since by Hardy–Rellich's inequality

$$\frac{u_1^2}{|x|^4(\ln \frac{R}{|x|})^2} \in L^1\big(B(R_1)\big) \quad \text{but} \quad \int\limits_{B(R_1)} |x|^{-4}\left(\ln \frac{R}{|x|}\right)\left(\ln\left(\ln \frac{R}{|x|}\right)\right)^{-1} = \infty.$$

Define

$$\varphi_s(x) := \left(\ln \frac{R}{|x|}\right)^{1/2}\left(\ln\left(\ln \frac{R}{|x|}\right)\right)^{-s/2},$$

where $s > 1$. Note that $\varphi_s \in H_r^2(B(R_1))$ and that $u_1 > 0$ in B by Boggio's principle. So we have $\Delta^2 u_1 = h > 0$ in B. Let $G(x, y)$ denote the Green function associated with the biharmonic operator with zero Dirichlet condition in the unit ball. Then by [12]

$$G(x, y) = c_n\big(2\ln[xy] - 2\ln[XY] - \big(1 - |x|^2\big)\big(1 - |y|^2\big)[XY]^{-2}\big),$$

where

$$c_n > 0, \qquad [xy] = |x - y|, \qquad [XY] = \left||x|y - \frac{x}{|x|}\right|.$$

Then

$$\Delta_x G(0, y) = c_n\left(-\frac{4}{|y|^2} + 8 - 4|y|^2\right).$$

Hence $-\Delta_x G(0, y) > 0$ for all $y \in B$ and thus arguing exactly as in Theorem 2.2(ii), we have $-\Delta u_1(x) > 0$ in $B(R_1)$ where $R_1 > 0$ is sufficiently small. Choose $R_1 < R$ such that the condition (2.16) is satisfied. Then from Lemma A.2 we have

$$\Delta^2\varphi_s - q(x)\frac{\varphi_s}{|x|^4(\ln \frac{R}{|x|})^2}$$
$$= \frac{\varphi_s}{|x|^4(\ln \frac{R}{|x|})^2(\ln(\ln \frac{R}{|x|}))^2}\left[(1 - q)\left(\ln\left(\ln \frac{R}{|x|}\right)\right)^2 - s(s + 2) + o(1)\right].$$

Hence

$$\Delta^2\varphi_s - q(x)\frac{\varphi_s}{|x|^4(\ln \frac{R}{|x|})^2} \leqslant 0 \quad \text{in } B(R_1).$$

Similarly as in Theorem 2.2(ii) we obtain that $u_1 \geqslant m\varphi_s$ in $B(R_1)$. □

Proof of Theorem 2.3(iii) *(Existence of $W_0^{2,p}(\Omega)$ solution)*. Let $0 < \nu < 1$ and u_ν satisfy $L_{\nu q} u_\nu = \lambda u_\nu$ with $\int_B u_\nu^2 = 1$ for some $\lambda = \lambda_\nu(q)$. The existence of u_ν follows from Theorem 2.1 and by coercivity of the operator $\Delta^2 - \nu \frac{q}{|x|^4(\ln \frac{R}{|x|})^2}$ in $H_0^2(\Omega)$. We will show that $u_\nu \to u_1$ in $W_0^{2,p}(\Omega)$ for all $p < 2$.

First we will prove the following estimates on $u_\nu, \nabla u_\nu, \nabla^2 u_\nu$. Fix $0 < \delta < 1$. Then there exists $R_1 > 0$ such that for $\nu \in (\frac{1}{2}, 1)$

$$\begin{cases} |u_\nu| \leqslant M_1 \left(\ln \frac{R}{|x|}\right)^{\nu/2} \left(\ln\left(\ln \frac{R}{|x|}\right)\right)^\delta & \text{in } B(R_1), \\ |\nabla u_\nu| \leqslant M_2 |x|^{-1} \left(\ln \frac{R}{|x|}\right)^{\nu/2} \left(\ln\left(\ln \frac{R}{|x|}\right)\right)^\delta & \text{in } B(R_1), \\ |\nabla^2 u_\nu| \leqslant M_3 |x|^{-2} \left(\ln \frac{R}{|x|}\right)^{\nu/2} \left(\ln\left(\ln \frac{R}{|x|}\right)\right)^\delta & \text{in } B(R_1), \end{cases} \tag{5.9}$$

where M_1, M_2, M_3 are constants independent of ν.

Let

$$\xi_\nu(x) = \left(\ln \frac{R}{|x|}\right)^{\nu/2} \left(\ln\left(\ln \frac{R}{|x|}\right)\right)^\delta.$$

Then from Lemma A.2 we have

$$\Delta^2 \xi_\nu - \nu q \frac{\xi_\nu}{|x|^4(\ln \frac{R}{|x|})^2} - \lambda_\nu(q) \xi_\nu = \frac{\xi_\nu}{|x|^4(\ln \frac{R}{|x|})^2} \left[\nu(2 - \nu - q) + 4 \frac{\delta(1-\delta)}{(\ln(\ln \frac{R}{|x|}))^2} + o(1) \right].$$

Hence there exists $R_1 > 0$ such that

$$\Delta^2 \xi_\nu - \nu q \frac{\xi_\nu}{|x|^4(\ln \frac{R}{|x|})^2} - \lambda_\nu(q) \xi_\nu \geqslant 0 \quad \text{in } B(R_1).$$

Proceeding exactly as in Theorem 2.2(iii), we have the above result. □

Proof of Theorem 2.4. Let u_ν be a non-negative solution to (2.19) corresponding to $\lambda = \lambda_D(\nu)$. Set $\phi_\nu^1 = |x|^{-(n-4)\beta(\nu)}$. By choice of $\beta(\nu)$ and (2.18) it follows that ϕ_ν^1 satisfies the equation

$$\Delta^2 \phi_\nu^1 - \nu \frac{n^2(n-4)^2}{16} \frac{\phi_\nu^1}{|x|^4} = 0.$$

Let $0 < R < 1$. Since $-\Delta \phi_\nu^1 > 0$ on $\partial B(R)$ we can choose $C_1 > 0$ such that

$$u_\nu \geqslant C_1 \phi_\nu^1, \qquad -\Delta u_\nu \geqslant -C_1 \Delta \phi_\nu^1 \quad \text{on } \partial B(R).$$

Define $w_\nu = u_\nu - C_1 \phi_\nu^1$. Then w_ν satisfies

$$\Delta^2 w_\nu - \nu \frac{n^2(n-4)^2}{16} \frac{w_\nu}{|x|^4} = \lambda(\nu) u_\nu.$$

So we have

$$\begin{cases} \Delta^2 w_\nu - \nu\frac{n^2(n-4)^2}{16}\frac{w_\nu}{|x|^4} \geqslant 0 & \text{in } B(R), \\ w_\nu \geqslant 0, -\Delta w_\nu \geqslant 0 & \text{on } \partial B(R). \end{cases}$$

Hence by the maximum principle (Section 4) we have that $u_\nu \geqslant C_1\phi_\nu^1$ in $B(R)$. Hence

$$C_1 \leqslant \liminf_{x\to 0}\left[\left(\ln\frac{1}{|x|}\right)^{(n-4)\beta(\nu)} u_\nu(x)\right].$$

Now to prove the other inequality. Let us introduce the function

$$\phi_\nu^2(x) = |x|^{\alpha(x)},$$

where $\alpha(x) = -(n-4)\beta(\nu) - |x|$. Let

$$\psi_1 = r^{-(n-4)\beta(\nu)} = r^{-a}, \qquad \psi_2 = r^{-r}.$$

So we have

$$\psi_1' = -ar^{-(a+1)}, \qquad \psi_1'' = a(a+1)r^{-(a+2)},$$
$$\Delta\psi_1 = \psi_1'' + \frac{n-1}{r}\psi_1' = \big[a(a+1) - (n-1)a\big]r^{-(a+2)},$$
$$\psi_2\Delta^2\psi_1 = \nu\frac{n^2(n-4)^2}{16}\frac{\varphi_\nu^2(x)}{|x|^4}.$$

Hence we have

$$\Delta\psi_1 = a(2+a-n)r^{-(a+2)},$$
$$\Delta\psi_1' = -a\Delta\big(r^{-(a+1)}\big) = -a(a+1)(3+a-n)r^{-(a+3)},$$
$$\psi_2' = -(1+\ln r)e^{-r\ln r},$$
$$\psi_2'' = r^{-r}\left\{(1+\ln r)^2 - \frac{1}{r}\right\}, \qquad (\Delta\psi_i)' = \Delta\psi_i' - \frac{n-1}{r^2}\psi_i',$$
$$2\Delta\psi_1\Delta\psi_2 + 4\psi_1'\Delta\psi_2' + 4\psi_2'\Delta\psi_1' + 4\psi_1''\psi_2'' - \frac{2(n-1)}{r^2}\psi_1'\psi_2' = O\left(r^{-(a+3+r)}\left(\ln\frac{1}{r}\right)^3\right).$$

Then

$$\Delta^2\varphi_\nu^2(x) - \nu\frac{n^2(n-4)^2}{16}\frac{\varphi_\nu^2(x)}{|x|^4} - \lambda(\nu)\varphi_\nu^2 = \psi_1\Delta^2\psi_2 + O\left(r^{-(a+3+r)}\left(\ln\frac{1}{r}\right)^3\right)$$

and

$$\psi_1\Delta^2\psi_2 = Cr^{-(a+3+r)}\left(\ln\frac{1}{r}\right)^4,$$

where $C > 0$. Hence we can choose $R > 0$ such that

$$\Delta^2\phi_\nu^2 - \nu\frac{n^2(n-4)^2}{16}\frac{\phi_\nu^2}{|x|^4} - \lambda(\nu)\phi_\nu^2 \geqslant 0 \quad \text{in } B(R).$$

Since $-\Delta\phi_\nu^2 > 0$ on $\partial B(R)$ we can choose $C_2 > 0$ so that

$$\begin{cases} u_\nu \leqslant C_2\phi_\nu^2 & \text{on } \partial B(R), \\ -\Delta u_\nu \leqslant -C_2\Delta\phi_\nu^2 & \text{on } \partial B(R), \end{cases}$$

and proceeding exactly as in Theorem 2.2(i), we have

$$u_\nu \leqslant C_2\phi_\nu^2 \quad \text{in } B(R),$$
$$\limsup_{x\to 0}\left[\left(\ln\frac{1}{|x|}\right)^{(n-4)\beta(\nu)} u_\nu(x)\right] \leqslant C_2. \qquad \square$$

Proof of Theorem 2.5. Let u_ν be a non-negative solution to (2.20) corresponding to $\lambda = \lambda_D(\nu)$.

Set $\phi_\nu^1 = (\ln\frac{R}{|x|})^{\frac{1-\sqrt{1-\nu}}{2}}$. It follows that ϕ_ν^1 satisfies the equation

$$\Delta^2\phi_\nu^1 - \nu\frac{\phi_\nu^1}{|x|^4(\ln\frac{R}{|x|})^2} = -A\frac{\phi_\nu^1}{|x|^4(\ln\frac{R}{|x|})^4},$$

where A is a positive constant. Let $0 < R_1 < 1$ and choose $C_1 > 0$ such that

$$\begin{cases} u_\nu \geqslant C_1\phi_\nu^1 & \text{on } \partial B(R_1), \\ -\Delta u_\nu \geqslant -C_1\Delta\phi_\nu^1 & \text{on } \partial B(R_1). \end{cases}$$

Define $w_\nu = u_\nu - C_1\phi_\nu^1$. Then w_ν satisfies

$$\Delta^2 w_\nu - \nu\frac{w_\nu}{|x|^4(\ln\frac{R}{|x|})^2} \geqslant 0.$$

So we have

$$\begin{cases} \Delta^2 w_\nu - \nu\frac{w_\nu}{|x|^4(\ln\frac{R}{|x|})^2} \geqslant 0 & \text{in } B(R_1), \\ w_\nu \geqslant 0,\ -\Delta w_\nu \geqslant 0 & \text{on } \partial B(R_1). \end{cases}$$

Hence again by the maximum principle (Section 4) we have $u_\nu \geqslant C_1\phi_\nu^1$ in $B(R_1)$. Then

$$C_1 \leqslant \liminf_{x\to 0}\left[\left(\ln\frac{R}{|x|}\right)^{\frac{-1+\sqrt{1-\nu}}{2}} u_\nu(x)\right].$$

In order to prove the other inequality we define

$$\phi_\nu^2(x) = \left(\ln \frac{R}{|x|}\right)^{\beta(x)},$$

where $\beta(x) = \frac{1-\sqrt{1-\nu}}{2} + |x|$. Let

$$\psi_1(x) = \left(\ln \frac{R}{|x|}\right)^{\alpha} \quad \text{and} \quad \psi_2(x) = \left(\ln \frac{R}{|x|}\right)^{|x|},$$

where $\alpha = \frac{1-\sqrt{1-\nu}}{2}$. Then we have

$$\Delta^2\phi_\nu^2 = \psi_2\Delta^2\psi_1 + \psi_1\Delta^2\psi_2 + 2\Delta\psi_1\Delta\psi_2 + 4\{\psi_1'\Delta\psi_2' + \psi_2'\Delta\psi_1'\} + 4\psi_1''\psi_2'' - 6\frac{\psi_1'\psi_2'}{r^2}.$$

Note that the major term is

$$\psi_1\Delta^2\psi_2 = C\frac{1}{r^3}\phi_\nu^2\left(\ln\left(\ln\frac{R}{r}\right)\right)^4,$$

where $C > 0$ and hence we have

$$\Delta^2\phi_\nu^2 - \nu\frac{\phi_\nu^2}{|x|^4(\ln\frac{R}{|x|})^2} - \lambda(\nu)\phi_\nu^2 = \frac{C}{r^3}\phi_\nu^2\left(\ln\left(\ln\frac{R}{r}\right)\right)^4 + O\left(\frac{1}{r^3}\phi_\nu^2\left(\ln\left(\ln\frac{R}{r}\right)\right)^3\right).$$

Hence we can choose $R_1 > 0$ small enough such that

$$\Delta^2\phi_\nu^2 - \nu\frac{\phi_\nu^2}{|x|^4(\ln\frac{R}{|x|})^2} - \lambda(\nu)\phi_\nu^2 \geqslant 0 \quad \text{in } B(R_1).$$

Since $-\Delta\phi_\nu^2 > 0$ on $\partial B(R_1)$ we can choose $C_2 > 0$ so that

$$\begin{cases} u_\nu \leqslant C_2\phi_\nu^2 & \text{on } \partial B(R_1), \\ -\Delta u_\nu \leqslant -C_2\Delta\phi_\nu^2 & \text{on } \partial B(R_1), \end{cases}$$

and proceeding exactly as in Theorem 2.3(i) we have

$$u_\nu \leqslant C_2\phi_\nu^2 \quad \text{in } B(R_1), \qquad \limsup_{x\to 0}\left[\left(\ln\frac{R}{|x|}\right)^{\frac{-1+\sqrt{1-\nu}}{2}} u_\nu(x)\right] \leqslant C_2.$$

Then the claim follows. □

Proof of Theorem 2.6. Let $n \geqslant 5$. Let u_1 and u_2 be two non-negative solutions of $L_q u = \lambda u$ for $\lambda = \lambda(q)$.

$$\begin{cases} \Delta^2 u_1 - \frac{n^2(n-4)^2}{16} q(x) \frac{u_1}{|x|^4} = \lambda(q) u_1 & \text{in } B, \\ \Delta^2 u_2 - \frac{n^2(n-4)^2}{16} q(x) \frac{u_2}{|x|^4} = \lambda(q) u_2 & \text{in } B, \\ u_1 = u_2 = 0, \ \frac{\partial u_1}{\partial \gamma} = \frac{\partial u_2}{\partial \gamma} = 0 & \text{on } \partial B. \end{cases}$$

Then by Theorem 5.1 $u_1 > 0$ and $u_2 > 0$ in B. Then by regularity result $u \in C^2(\overline{B} \setminus 0)$ and by Hopf's lemma [13] for fourth order equations, we have $\frac{\partial^2 u_1}{\partial \gamma^2} > 0$ and $\frac{\partial^2 u_2}{\partial \gamma^2} > 0$ on ∂B.

We will proceed by contradiction. Define

$$m := \min_{x \in \partial B} \frac{\partial^2 u_1 / \partial \gamma^2 (x)}{\partial^2 u_2 / \partial \gamma^2 (x)}.$$

This implies there exists $x_0 \in \partial B$ such that $\frac{\partial^2 u_1}{\partial \gamma^2}(x_0) = m \frac{\partial^2 u_2}{\partial \gamma^2}(x_0)$. If possible let $u_1 \neq m u_2$. Define $v := u_1 - m u_2$. Then v satisfies

$$\begin{cases} \Delta^2 v - \frac{n^2(n-4)^2}{16} q(x) \frac{v}{|x|^4} = \lambda(q) v & \text{in } B, \\ v = 0, \ \frac{\partial v}{\partial \gamma} = 0 & \text{on } \partial B. \end{cases}$$

Then v is a minimizer in (2.4). Hence by Theorem 5.1 v does not change sign in B and by Hopf's lemma $\frac{\partial^2 v}{\partial \gamma^2} \neq 0$ on ∂B which is a contradiction.

Proceeding similarly as above we have the result for $n = 4$. □

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Appendix A

Lemma A.1. *Let $n \geqslant 5$ and $\xi_\nu(x) = |x|^{-\frac{n-4}{2}\nu}(\ln \frac{1}{|x|})^\delta$, $\delta \in \mathbb{R}$. Then*

(i) *for $0 < R < 1$, $\xi_\nu \in H^2(B(R))$ if and only if $\nu < 1$ or $\nu = 1$ and $\delta < -\frac{1}{2}$;*
(ii) *for $x \neq 0$,*

$$\begin{aligned} \Delta^2 \xi_\nu = \frac{\xi_\nu}{|x|^4} \Bigg[& \frac{(n-4)^2}{16} \big(4 + \nu(n-4)\big)(2-\nu)\big(2(n-2) - (n-4)\nu\big) \\ & - \frac{n-4}{2} \frac{\{\nu^2(n-4)^2 - 2\nu(n-4)^2 - 4(n-2)\}(1-\nu)\delta}{(\ln \frac{1}{|x|})} \\ & - \frac{\delta(1-\delta)}{2} \frac{\{3\nu^2(n-4)^2 - 6\nu(n-4)^2 + 2(n(n-10)+20)\}}{(\ln \frac{1}{|x|})^2} \\ & - \frac{2\delta(1-\delta)(n-4)(1-\nu)}{(\ln \frac{1}{|x|})^3} - \frac{\delta(1-\delta)(2-\delta)(3-\delta)}{(\ln \frac{1}{|x|})^4} \Bigg]. \end{aligned} \tag{A.1}$$

Proof. (i) $\xi_\nu \in H^2(B(R))$ iff

$$\int_{B(R)} |x|^{-(n-4)\nu-4}\left(\ln\frac{1}{|x|}\right)^{2\delta} < \infty$$

and this happens iff $\nu < 1$ or $\nu = 1$ and $\delta < -\frac{1}{2}$. This proves (i).

(ii) Let $\xi_\nu(x) = |x|^{-\frac{(n-4)}{2}\nu}(\ln\frac{1}{|x|})^\delta$ and $\delta \in \mathbb{R}$. Then we have the following:

$$\xi_\nu = |x|^{-\frac{n-4}{2}\nu}\left(\ln\frac{1}{|x|}\right)^\delta,$$

$$\xi_{\nu,r} = r^{-(\frac{n-4}{2}\nu+1)}\left\{-\frac{n-4}{2}\nu\left(\log\frac{1}{r}\right)^\delta - \delta\left(\ln\frac{1}{r}\right)^{\delta-1}\right\},$$

$$\begin{aligned}\xi_{\nu,rr} = r^{-(\frac{n-4}{2}\nu+2)}\bigg\{&\left(\frac{n-4}{2}\nu\right)\left(\frac{n-4}{2}\nu+1\right)\left(\ln\frac{1}{r}\right)^\delta \\ &+ \delta\big((n-4)\nu+1\big)\left(\ln\frac{1}{r}\right)^{\delta-1} + \delta(-1+\delta)\left(\ln\frac{1}{r}\right)^{\delta-2}\bigg\},\end{aligned}$$

$$\begin{aligned}\Delta\xi_\nu = r^{-(\frac{n-4}{2}\nu+2)}\bigg\{&\left(\frac{n-4}{2}\nu\right)\left(\frac{n-4}{2}\nu-(n-2)\right)\left(\ln\frac{1}{r}\right)^\delta \\ &+ \delta\big((n-4)\nu-(n-2)\big)\left(\ln\frac{1}{r}\right)^{\delta-1} + \delta(\delta-1)\left(\ln\frac{1}{r}\right)^{\delta-2}\bigg\},\end{aligned}$$

$$\begin{aligned}(\Delta\xi_\nu)_r = r^{-(\frac{n-4}{2}\nu+3)}\bigg\{&-\left(\frac{n-4}{2}\nu\right)\left(\frac{n-4}{2}\nu-(n-2)\right)\left(\frac{n-4}{2}\nu+2\right)\left(\ln\frac{1}{r}\right)^\delta \\ &- \delta\left(\ln\frac{1}{r}\right)^{\delta-1}\left\{\frac{3}{4}(n-4)^2\nu^2 - \nu(n-4)^2 - 2(n-2)\right\} \\ &+ \delta(\delta-1)\left(\ln\frac{1}{r}\right)^{\delta-2}\left\{\left(\frac{3}{2}\nu-1\right)(n-4)\right\} + \delta(\delta-1)(\delta-2)\left(\ln\frac{1}{r}\right)^{\delta-3}\bigg\},\end{aligned}$$

$$\begin{aligned}(\Delta\xi_\nu)_{rr} = r^{-(\frac{n-4}{2}\nu+4)}\bigg\{&\left(\frac{n-4}{2}\nu\right)\left(\frac{n-4}{2}\nu-(n-2)\right)\left(\frac{n-4}{2}\nu+2\right)\left(\frac{n-4}{2}\nu+3\right)\left(\ln\frac{1}{r}\right)^\delta \\ &+ \delta\bigg\{\frac{(n-4)^3}{2}\nu^3 - 3\nu^2\left(\frac{(n-4)^2(n-7)}{4}\right) \\ &- 2(n-4)(n-2)\nu - 3\nu(n-4)^2 - (n-2)^2\bigg\}\left(\ln\frac{1}{r}\right)^{\delta-1} \\ &+ \delta(\delta-1)\left(\ln\frac{1}{r}\right)^{\delta-2}\left\{\frac{3}{2}(n-4)^2\nu^2 - \frac{3}{2}\nu(n-7)(n-4) - \frac{1}{2}(3n-16)(n-2)\right\}\end{aligned}$$

$$- \delta(\delta-1)(\delta-2)\big\{2(n-4)\nu-(n-7)\big\}\left(\ln\frac{1}{r}\right)^{\delta-2}$$

$$+ \delta(\delta-1)(\delta-2)(\delta-3)\left(\ln\frac{1}{r}\right)^{\delta-3}\Bigg\},$$

$$\Delta^2\xi_\nu = \frac{\xi_\nu}{|x|^4}\Bigg[\frac{(n-4)^2}{16}\big(4+\nu(n-4)\big)(2-\nu)\big(2(n-2)-(n-4)\nu\big)$$

$$- \frac{n-4}{2}\frac{\{\nu^2(n-4)^2-2\nu(n-4)^2-4(n-2)\}(1-\nu)\delta}{(\ln\frac{1}{|x|})}$$

$$- \frac{\delta(1-\delta)}{2}\frac{\{3\nu^2(n-4)^2-6\nu(n-4)^2+2(n(n-10)+20)\}}{(\ln\frac{1}{|x|})^2}$$

$$- \frac{2\delta(1-\delta)(n-4)(1-\nu)}{(\ln\frac{1}{|x|})^3} - \frac{\delta(1-\delta)(2-\delta)(3-\delta)}{(\ln\frac{1}{|x|})^4}\Bigg].$$

Note that for $\nu = 1$ and replacing δ by $-\delta$, we have

$$\Delta^2\xi_1 = \frac{\xi_1}{|x|^4}\left[\frac{n^2(n-4)^2}{16} - \frac{\delta(\delta+1)(n^2-4n+8)}{2(\ln\frac{1}{|x|})^2} + \frac{\delta(\delta+1)(\delta+2)(\delta+3)}{(\ln\frac{1}{|x|})^4}\right].$$

Lemma A.2. *Let $n = 4$ and $\phi(x) = (\ln\frac{R}{|x|})^a(\ln(\ln\frac{R}{|x|}))^b$, $a, b \in \mathbb{R}$, $R > 0$. Then,*

(i) *for $0 < R_1 < e^{-1}R$, $u \in H^2(B(R_1))$ if and only if $a < 1$ or $a = 1$ and $b < -\frac{1}{2}$;*
(ii) *for $0 < |x| < e^{-1}R$,*

$$\Delta^2\phi - \frac{q(x)\phi(x)}{|x|^4(\log\frac{R}{|x|})^2} = \frac{\phi(x)}{|x|^4(\ln\frac{R}{|x|})^2(\ln(\ln\frac{R}{|x|}))}\Bigg[-\big(4a(a-1)+q\big)\left(\ln\left(\ln\frac{R}{|x|}\right)\right)$$

$$+ 4b(b-a) + \frac{4b(b-1)}{(\ln(\ln\frac{R}{|x|}))} + o(1)\Bigg]. \tag{A.2}$$

Proof. (i) The first part is easy to verify.

(ii) We have

$$\phi(r) = \left(\ln\frac{R}{r}\right)^a\left(\ln\left(\ln\frac{R}{r}\right)\right)^b,$$

$$\phi_r(r) = -\frac{a}{r}\left(\ln\frac{R}{r}\right)^{a-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^b - \frac{b}{r}\left(\ln\frac{R}{r}\right)^{a-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-1},$$

$$\phi_{rr}(r) = -\frac{a}{r^2}\left(\ln\frac{R}{r}\right)^{a-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^b - \frac{b}{r^2}\left(\ln\frac{R}{r}\right)^{a-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-1}$$

$$+ o\left(\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^b\right).$$

Hence $\Delta\phi = \phi_{rr} + \frac{3}{r}\phi_r(r)$ we have,

$$\Delta\phi = -\frac{2a}{r^2}\left(\ln\frac{R}{r}\right)^{a-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b} - \frac{2b}{r^2}\left(\ln\frac{R}{r}\right)^{a-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-1} + o\left(\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b}\right).$$

Let

$$\psi(r) = \frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta} = \frac{\psi_1}{r^2},$$

$$\begin{aligned}
\Delta\psi &= \Delta\left(\frac{1}{r^2}\right)\psi_1 + 2\nabla\left(\frac{1}{r^2}\right)\nabla\psi_1 + \left(\frac{1}{r^2}\right)\Delta\psi_1 \\
&= \psi_1(0) - \frac{4}{r^3}\psi_1 + \frac{1}{r^2}\Delta\psi_1 \\
&= -\frac{4}{r^3}\left[-\frac{\alpha}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta} - \frac{\beta}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta-1}\right] \\
&\quad + \frac{1}{r^2}\left[-\frac{2\alpha}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta} - \frac{2b}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta-1}\right. \\
&\quad \left. + o\left(\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta}\right)\right].
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta\psi &= \frac{4}{r^4}\left(\ln\frac{R}{r}\right)^{\alpha-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta}\left[\alpha + \frac{\beta}{(\ln(\ln\frac{R}{r}))}\right] \\
&\quad - \frac{2}{r^4}\left(\ln\frac{R}{r}\right)^{\alpha-1}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta}\left[\alpha + \frac{\beta}{(\ln(\ln\frac{R}{r}))}\right] + o\left(\frac{1}{r^2}\left(\ln\frac{R}{r}\right)^{\alpha-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{\beta}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta^2\phi &= -2a\Delta\left(\frac{(\ln\frac{R}{r})^{a-1}(\ln(\ln\frac{R}{r}))^{b}}{r^2}\right) - 2b\Delta\left(\frac{(\ln\frac{R}{r})^{a-1}(\ln(\ln\frac{R}{r}))^{b-1}}{r^2}\right) \\
&= -\frac{2a}{r^4}\left[4\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b}\left(a-1+\frac{b}{(\ln(\ln\frac{R}{r}))}\right)\right. \\
&\quad \left. - 2\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b}\left(a-1+\frac{b}{(\ln(\ln\frac{R}{r}))}\right)\right] \\
&\quad - \frac{2b}{r^4}\left[4\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-1}\left(a-1+\frac{b}{(\ln(\ln\frac{R}{r}))}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-1}\left(a-1+\frac{b}{(\ln(\ln\frac{R}{r}))}\right)\Big]\\
&= \frac{(\ln\frac{R}{r})^{a-2}(\ln(\ln\frac{R}{r}))^{b}}{r^4}\big[-8a(a-1)+4a(a-1)\big]\\
&\quad + \frac{(\ln\frac{R}{r})^{a-2}(\ln(\ln\frac{R}{r}))^{b-1}}{r^4}\big[-8ab+4ab-8b(a-1)+4b(a-1)\big]\\
&\quad + \frac{(\ln\frac{R}{r})^{a-2}(\ln(\ln\frac{R}{r}))^{b-2}}{r^4}\big[-8b(b-1)+4b(b-1)\big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2\phi &= -\frac{4a(a-1)}{r^4}\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b} + \frac{4b(1-2a)}{r^4}\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-1}\\
&\quad + \frac{4b(b-1)}{r^4}\left(\ln\frac{R}{r}\right)^{a-2}\left(\ln\left(\ln\frac{R}{r}\right)\right)^{b-2} + o\left(\frac{(\ln\frac{R}{r})^{a-3}(\ln(\ln\frac{R}{r}))^{b}}{r^4}\right)\\
&= -\frac{4a(a-1)}{r^4}\frac{\phi}{(\ln\frac{R}{r})^2} + \frac{4b(1-2a)}{r^4}\frac{\phi}{(\ln\frac{R}{r})^2(\ln(\ln\frac{R}{r}))}\\
&\quad + \frac{4b(b-1)}{r^4}\frac{\phi}{(\ln\frac{R}{r})^2(\ln(\ln\frac{R}{r}))^2} + \frac{E\phi}{(\log\frac{R}{r})^2 r^4}, \quad \text{where } E\to 0 \text{ as } r\to 0.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\Delta^2\phi - \frac{q(x)\phi}{r^4(\log\frac{R}{r})^2} &= \frac{\phi}{r^4(\log\frac{R}{r})^2(\ln(\ln\frac{R}{r}))}\Bigg[-\big(4a(a-1)+q(x)\big)\left(\ln\left(\ln\frac{R}{r}\right)\right)\\
&\qquad +4b(b-a) + \frac{4b(b-1)}{(\ln(\ln\frac{R}{r}))} + o(1)\Bigg]
\end{aligned}
$$

which finishes the proof. □

Thus when $a=\frac{1}{2}$ and $b=-\frac{s}{2}$ we have

$$
\begin{aligned}
&\Delta^2\phi - \frac{q\phi}{|x|^4(\log\frac{R}{|x|})^2}\\
&\quad = \frac{\phi}{|x|^4(\log\frac{R}{|x|})^2(\ln(\ln\frac{R}{|x|}))^2}\left[\big(1-q(x)\big)\left(\ln\left(\ln\frac{R}{|x|}\right)\right)^2 - s(s+2) + o(1)\right].
\end{aligned}
$$

A.1. $H^2(\Omega)$ Hardy–Rellich inequality

Let Ω be a smooth domain of $\mathbb{R}^4$ and consider the problem

$$
\begin{cases}
\Delta^2 w = 0 & \text{in } \Omega,\\
w = f & \text{on } \partial\Omega,\\
\frac{\partial w}{\partial\nu} = g & \text{on } \partial\Omega.
\end{cases}
\tag{A.3}
$$

Then, by elliptic regularity, there exist unique Green functions $K_1(x,y)$, $K_2(x,y)$ smooth for $x \in \Omega$, $y \in \partial\Omega$ and some constants $C_1 > 0$, $C_2 > 0$ such that $\forall x \in \Omega$, $y \in \partial\Omega$

$$\left|K_1(x,y)\right| \leqslant \frac{C_1}{|x-y|}, \qquad \left|K_2(x,y)\right| \leqslant \frac{C_2}{|x-y|^2}. \tag{A.4}$$

Moreover,

$$w(x) = \int_{\partial\Omega} K_1(x,y) f(y)\, d\sigma(y) + \int_{\partial\Omega} K_2(x,y) g(y)\, d\sigma(y). \tag{A.5}$$

Then we have the following lemma.

Lemma A.3. *Consider the above problem. Then, $\forall \alpha \geqslant 0$ there exists a constant $C > 0$ such that*

$$\int_\Omega w^2 \leqslant C\left\{\int_{\partial\Omega} f^2 + \int_{\partial\Omega} g^2\right\}, \tag{A.6}$$

$$\int_\Omega \frac{|\ln(\ln\frac{R}{|x|})|^\alpha w^2}{|x|^4(\ln\frac{R}{|x|})^2}\,dx \leqslant C\left\{\int_{\partial\Omega} f^2 + \int_{\partial\Omega} g^2\right\}. \tag{A.7}$$

Proof. (A.6) follows easily.

For (A.7) we proceed in the following manner. We have

$$\int_\Omega \frac{|\ln(\ln\frac{R}{|x|})|^\alpha w^2}{|x|^4(\ln\frac{R}{|x|})^2}\,dx \leqslant \int_{\partial\Omega} f^2(y)\left(\int_\Omega \frac{|\ln(\ln\frac{R}{|x|})|^\alpha |K_1(x,y)|}{|x|^4(\ln\frac{R}{|x|})^2}\,dx\right) d\sigma(y)$$
$$+ \int_{\partial\Omega} g^2(y)\left(\int_\Omega \frac{|\ln(\ln\frac{R}{|x|})|^\alpha |K_2(x,y)|}{|x|^4(\ln\frac{R}{|x|})^2}\,dx\right) d\sigma(y).$$

Let $\overline{B(0,R_1)} \subset \Omega$. Then

$$\int_\Omega \frac{|\ln(\ln\frac{R}{|x|})|^\alpha w^2}{|x|^4(\ln\frac{R}{|x|})^2}\,dx \leqslant C\int_{\partial\Omega} f^2(y)\left(\int_{B(0,R_1)} \cdots + \int_{\Omega\setminus B(0,R_1)} \cdots\right) d\sigma(y)$$
$$+ C\int_{\partial\Omega} g^2(y)\left(\int_{B(0,R_1)} \cdots + \int_{\Omega\setminus B(0,R_1)} \cdots\right) d\sigma(y),$$

$$\int_\Omega \frac{|\ln(\ln\frac{R}{|x|})|^\alpha w^2}{|x|^4(\ln\frac{R}{|x|})^2}\,dx \leqslant C\int_{\partial\Omega} f^2(y)\left(\int_0^{R_1} \frac{|\ln(\ln\frac{R}{r})|^\alpha}{r(\ln\frac{R}{r})^2} + \int_{\Omega\setminus B(0,R_1)} \frac{1}{|x-y|}\right) d\sigma(y)$$
$$+ C\int_{\partial\Omega} g^2(y)\left(\int_0^{R_1} \frac{|\ln(\ln\frac{R}{r})|^\alpha}{r(\ln\frac{R}{r})^2} + \int_{\Omega\setminus B(0,R_1)} \frac{1}{|x-y|^2}\right) d\sigma(y).$$

Hence we have

$$\int_{\Omega} \frac{|\ln(\ln\frac{R}{|x|})|^{\alpha} w^2}{|x|^4(\ln\frac{R}{|x|})^2}\,dx \leqslant C\left\{\int_{\partial\Omega} f^2 + \int_{\partial\Omega} g^2\right\}.$$

The proof of lemma is finished. □

In order to state the next result we introduce certain notations. Let $e^{(0)} = 1, e^{(1)} = e, e^{(k)} = e^{e^{(k-1)}}$ for $k \geqslant 1$. Let $b > 0$ and define

$$\ln^1(b) = \ln b, \qquad \ln^k(b) = \ln\big(\ln^{k-1}(b)\big).$$

Then we have the following.

Theorem A.1. *Let Ω be a bounded domain with smooth boundary and $0 \in \Omega$. Let $R > e^{(k+1)\sup_{x\in\partial\Omega}|x|}$, then there exist constants $\lambda_1 > 0$, $\lambda_2 > 0$ such that for all $u \in H^2(\Omega)$*

$$\int_{\Omega} (\Delta u)^2\,dx - \frac{n^2(n-4)^2}{16}\int_{\Omega}\frac{u^2}{|x|^4}\,dx - \frac{n(n-4)}{8}\int_{\Omega}\frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2}$$
$$- \frac{n(n-4)}{8}\sum_{i=2}^{\infty}\int_{\Omega}\frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2}X_2^2X_3^2\cdots X_i^2$$
$$\geqslant -\lambda_1\int_{\partial\Omega} u^2 - \lambda_2\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2, \quad \text{for } n \geqslant 5; \tag{A.8}$$

$$\int_{\Omega} (\Delta u)^2\,dx - \int_{\Omega}\frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2}\,dx - \sum_{l=1}^{k}\int_{\Omega}\frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2(\ln^l\frac{R}{|x|})^2}$$
$$\geqslant -\lambda_1\int_{\partial\Omega} u^2 - \lambda_2\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2, \quad n = 4. \tag{A.9}$$

Proof of (A.8). In order to prove (A.8) we borrow ideas from [2]. Let $E := |x|^{-(n-4)}$, $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ and $u = E^{1/2}v_0$. Then $v_0(0) = 0$ and

$$|\nabla u|^2 = \frac{(n-4)^2}{4}\frac{u^2}{|x|^2} + |\nabla v_0|^2 E - (n-4)\left\langle \frac{x}{|x|^2}, \nabla v_0\right\rangle E v_0,$$
$$\frac{|\nabla u|^2}{|x|^2} = \frac{(n-4)^2}{4}\frac{u^2}{|x|^4} + \frac{|\nabla v_0|^2}{|x|^2}E - \frac{(n-4)}{2}\left\langle \frac{x}{|x|^n}, \nabla v_0^2\right\rangle,$$
$$\int_{\Omega}\frac{|\nabla u|^2}{|x|^2}\,dx = \frac{(n-4)^2}{4}\int_{\Omega}\frac{u^2}{|x|^4}\,dx + \int_{\Omega}\frac{|\nabla v_0|^2}{|x|^2}E\,dx - \frac{(n-4)}{2}\int_{\partial\Omega}\frac{\langle x,\nu\rangle}{|x|^n}v_0^2,$$

$$\int_{\Omega} \frac{|\nabla u|^2}{|x|^2}\,dx = \frac{(n-4)^2}{4}\int_{\Omega}\frac{u^2}{|x|^4}\,dx + \int_{\Omega}\frac{|\nabla v_0|^2}{|x|^2}E\,dx - \frac{(n-4)}{2}\int_{\partial\Omega}\frac{\langle x,\nu\rangle}{|x|^4}u^2. \tag{A.10}$$

Now,

$$\begin{aligned}\int_{\Omega}\frac{|\nabla u|^2}{|x|^2}\,dx &= \int_{\Omega}\frac{\nabla u\cdot\nabla u}{|x|^2}\,dx\\ &= -\int_{\Omega}\nabla\left(\frac{\nabla u}{|x|^2}\right)u\,dx + \int_{\partial\Omega}\frac{u}{|x|^2}\langle\nabla u,\nu\rangle\\ &= -\int_{\Omega}\frac{u\Delta u}{|x|^2}\,dx + 2\int_{\Omega}u\nabla u\cdot\frac{x}{|x|^4} + \int_{\partial\Omega}\frac{u}{|x|^2}\frac{\partial u}{\partial\nu}\\ &= -\int_{\Omega}\frac{u\Delta u}{|x|^2}\,dx - (n-4)\int_{\Omega}\frac{u^2}{|x|^4} + \int_{\partial\Omega}\frac{u^2}{|x|^4}\langle x,\nu\rangle + \int_{\partial\Omega}\frac{u}{|x|^2}\frac{\partial u}{\partial\nu}. \end{aligned}\tag{A.11}$$

Putting the value of (A.10) in (A.11) we have

$$\begin{aligned}-\int_{\Omega}\frac{u\Delta u}{|x|^2}\,dx &= \frac{n(n-4)}{4}\int_{\Omega}\frac{u^2}{|x|^4} - \frac{n-2}{2}\int_{\partial\Omega}\frac{u^2}{|x|^4}\langle x,\nu\rangle\,dx\\ &\quad - \int_{\partial\Omega}\frac{u}{|x|^2}\frac{\partial u}{\partial\nu} + \int_{\Omega}\frac{|\nabla v_0|^2}{|x|^2}E\,dx. \end{aligned}\tag{A.12}$$

Using the above substitution and Schwartz inequality, we obtain

$$\begin{aligned}\int_{\Omega}|\Delta u|^2\,dx &\geqslant \frac{n^2(n-4)^2}{16}\int_{\Omega}\frac{u^2}{|x|^4} - \frac{(n-4)(n-2)}{2}\int_{\partial\Omega}\frac{u^2}{|x|^4}\langle x,\nu\rangle\,dx\\ &\quad - \frac{n-4}{2}\int_{\partial\Omega}\frac{u}{|x|^2}\frac{\partial u}{\partial\nu} + \frac{n(n-4)}{2}\int_{\Omega}\frac{|\nabla v_0|^2}{|x|^2}E\,dx. \end{aligned}$$

Let $v_1 = {X_1}^{1/2}v_0$. Then we have

$$\int_{\Omega}\frac{|\nabla v_0|^2}{|x|^2}E\,dx = \frac{1}{4}\int_{\Omega}\frac{|u|^2}{|x|^4}X_1^2 - \int_{\partial\Omega}\frac{u^2}{|x|^4}\langle x,\nu\rangle X_1 + \int_{\Omega}\frac{|\nabla v_1|^2}{|x|^2}EX_1.$$

Similarly for $i\geqslant 2$ define $v_i(x) = {X_i}^{1/2}v_{i-1}$. Then

$$\int_{\Omega}\frac{|\nabla v_1|^2}{|x|^2}EX_1\,dx = \frac{1}{4}\int_{\Omega}\frac{|u|^2}{|x|^4}X_1^2X_2^2 - \int_{\partial\Omega}\frac{u^2}{|x|^4}\langle x,\nu\rangle X_1X_2 + \int_{\Omega}\frac{|\nabla v_2|^2}{|x|^2}EX_1X_2.$$

Let

$$\lambda_1 = \inf_{x\in\partial\Omega} \frac{|\langle x, \nu\rangle|}{|x|^4}\left\{\frac{(n-4)(n-2)}{2} + \sum_{i=1}^{\infty} X_1\cdots X_i\right\} + \frac{(n-4)^2}{4}.$$

Similarly as in Theorem 2.1,

$$\int_\Omega |\Delta u|^2\,dx \geqslant \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4} + \frac{n(n-4)}{8}\int_\Omega \frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2} - \lambda_1\int_{\partial\Omega} u^2$$
$$- \lambda_2\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2 + \frac{n(n-4)}{8}\sum_{i=2}^{\infty}\frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2}X_2^2X_3^2\cdots X_i^2. \qquad \square$$

Proof of (A.9). In order to prove Theorem A.1, we require the following lemma and the proof of the theorem will follow as a consequence. □

Let $H = \{u \in H^2(\Omega)\colon\ \Delta^2 u = 0\}$. Then $H^2(\Omega) = H_0^2(\Omega) \oplus H$. For $v \in H_0^2(\Omega)$, $u \in H$ we have $\int_\Omega \Delta u \Delta v = 0$.

Lemma A.4. *Let $u \in H^2(\Omega)$, $v \in H_0^2(\Omega)$, $w \in H$ such that $u = v + w$, where w satisfies*

$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega,\\ w = u & \text{on } \partial\Omega,\\ \frac{\partial w}{\partial\nu} = \frac{\partial u}{\partial\nu} & \text{on } \partial\Omega.\end{cases}$$

Then there exists $C > 0$ such that

$$\int_\Omega (\Delta u)^2 = \int_\Omega (\Delta v)^2 + \int_\Omega (\Delta w)^2, \tag{A.13}$$

$$\int_\Omega w^2 \leqslant C\left\{\int_{\partial\Omega} u^2 + \int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\right\}. \tag{A.14}$$

Let $k \in \mathbb{N}$, $R > e^{(k+1)}\sup_\Omega |x|$, then there exist $C_1 > 0$, $C_2 > 0$ such that $\forall u \in H^2(\Omega)$

$$\int_\Omega (\Delta u)^2 - \int_\Omega (\Delta w)^2 + C_1\int_{\partial\Omega} u^2 + C_2\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2 - \int_\Omega \frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2}$$
$$\geqslant \sum_{l=1}^{k}\int_\Omega \frac{u^2}{|x|^4(\ln\frac{R}{|x|})^2(\ln^l\frac{R}{|x|})^2}. \tag{A.15}$$

Proof. The first part follows trivially.

For the second part define

$$I(u) := \int\limits_{\Omega} (\Delta u)^2 - \int\limits_{\Omega} \frac{u^2}{|x|^4 (\ln \frac{R}{|x|})^2}.$$

Let $0 \leqslant l \leqslant k$ and let C_k denote a generic constant depending on k, Ω, R. Then

$$\begin{aligned}
2\int\limits_{\Omega} \frac{|vw|}{(|x|^2 \ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} &= 2\int\limits_{\Omega} \frac{|v|}{|x|^2 (\ln \frac{R}{|x|})(\ln^{k+1} \frac{R}{|x|})} \frac{|w|(\ln^{k+1} \frac{R}{|x|})}{|x|^2 (\ln \frac{R}{|x|})(\ln^l \frac{R}{|x|})^2} \\
&\leqslant \frac{1}{N}\int\limits_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^{k+1} \frac{R}{|x|})^2} + C_k \int\limits_{\Omega} \frac{|w|^2 (\ln^{k+1} \frac{R}{|x|})^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^4} \\
&\leqslant \frac{1}{N}\int\limits_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^{k+1} \frac{R}{|x|})^2} + C_k \int\limits_{\partial\Omega} \left\{ u^2 + \left(\frac{\partial u}{\partial \nu}\right)^2 \right\} \\
&\leqslant \frac{1}{N}\int\limits_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^{k+1} \frac{R}{|x|})^2} + C_k \int\limits_{\partial\Omega} \left\{ u^2 + \left(\frac{\partial u}{\partial \nu}\right)^2 \right\}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
2\int\limits_{\Omega} \frac{|vw|}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} &\leqslant \frac{1}{N}\int\limits_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^{k+1} \frac{R}{|x|})^2} \\
&\quad + C_k \int\limits_{\partial\Omega} \left\{ u^2 + \left(\frac{\partial u}{\partial \nu}\right)^2 \right\}. \qquad \text{(A.16)}
\end{aligned}$$

Using (A.16), (A.7) we have

$$\begin{aligned}
&I(u) - \sum_{l=1}^{k} \int\limits_{\Omega} \frac{|u|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} \\
&\quad = I(v+w) - \sum_{l=1}^{k} \int\limits_{\Omega} \frac{|u|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} \\
&\quad = I(v) + I(w) - \sum_{l=1}^{k} \int\limits_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} - 2\sum_{l=1}^{k} \int\limits_{\Omega} \frac{vw}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} \\
&\qquad - \sum_{l=1}^{k} \int\limits_{\Omega} \frac{|w|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} - 2\int\limits_{\Omega} \frac{vw}{|x|^4 (\ln \frac{R}{|x|})^2}
\end{aligned}$$

$$\geqslant I(v) - \sum_{l=1}^{k} \int_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} - \frac{(k+1)}{N} \int_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^{k+1} \frac{R}{|x|})^2}$$

$$- C_k \int_{\partial\Omega} \left\{ u^2 + \left(\frac{\partial u}{\partial \nu}\right)^2 \right\} + \int_{\Omega} (\Delta w)^2.$$

Choosing $N = k+1$ we have

$$I(u) - \sum_{l=1}^{k} \int_{\Omega} \frac{|u|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2}$$

$$\geqslant I(v) - \sum_{l=1}^{k+1} \int_{\Omega} \frac{|v|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} + \int_{\Omega} (\Delta w)^2 - C_k \int_{\partial\Omega} \left\{ u^2 + \left(\frac{\partial u}{\partial \nu}\right)^2 \right\}.$$

Finally as $v \in H_0^2(\Omega)$ we have

$$I(u) - \int_{\Omega} (\Delta w)^2 - \sum_{i=1}^{k} \int_{\Omega} \frac{|u|^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln^l \frac{R}{|x|})^2} \geqslant -C_1 \int_{\partial\Omega} u^2 - C_2 \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2.$$

Hence the lemma is proved. □

A.2. $W_0^{1,q}(\Omega)$ estimates

Theorem A.2. *Let $1 \leqslant q < 2$. Then there exist $R_0 > 0$, $C_1 > 0$, $C_2 > 0$ such that $\forall R \geqslant R_0$, $\forall u \in H_0^2(\Omega)$ or $\forall u \in H^2(\Omega) \cap H_0^1(\Omega)$*

$$\int_{\Omega} (\Delta u)^2 \, dx - \frac{n^2(n-4)^2}{16} \int_{\Omega} \frac{u^2}{|x|^4} \, dx - C_1 \int_{\Omega} \frac{u^2}{|x|^4 (\ln \frac{R}{|x|})^2}$$
$$\geqslant C_2 \|u\|^2_{W_0^{1,q}(\Omega)}, \quad n \geqslant 5; \tag{A.17}$$

$$\int_{\Omega} (\Delta u)^2 \, dx - \int_{\Omega} \frac{u^2}{|x|^4 (\ln \frac{R}{|x|})^2} \, dx - \int_{\Omega} \frac{u^2}{|x|^4 (\ln \frac{R}{|x|})^2 (\ln(\ln \frac{R}{|x|}))^2}$$
$$\geqslant C_2 \|u\|^2_{W_0^{1,q}(\Omega)}, \quad n = 4. \tag{A.18}$$

Proof of (A.17). We use the similar ideas as in [3]. Let $n \geqslant 5$. Then we obtain by Theorem A.1,

$$\int_{\Omega} (\Delta u)^2 \, dx - \frac{n^2(n-4)^2}{16} \int_{\Omega} \frac{u^2}{|x|^4} \, dx - C_1 \int_{\Omega} \frac{u^2}{|x|^4 (\ln \frac{R}{|x|})^2} \geqslant C \int_{\Omega} \frac{|\nabla w|^2}{|x|^{n-2}} \left(\ln \frac{R}{|x|}\right),$$

where

$$w = \left(\ln \frac{R}{|x|}\right)^{-1/2} v = \left(\ln \frac{R}{|x|}\right)^{-1/2} |x|^{\frac{n-4}{2}} u.$$

Since

$$u = |x|^{-\frac{n-4}{2}} \left(\ln \frac{R}{|x|}\right)^{1/2} w,$$

we have

$$|\nabla u| = O\left(\frac{(\ln \frac{R}{|x|})^{1/2}}{|x|^{\frac{n-4}{2}}} \nabla w + \frac{(\ln \frac{R}{|x|})^{1/2}}{|x|^{\frac{n-2}{2}}} w\right).$$

Therefore

$$|\nabla u|^q = O\left(\frac{(\ln \frac{R}{|x|})^{q/2}}{|x|^{\frac{n-4}{2}q}} |\nabla w|^q + \frac{(\ln \frac{R}{|x|})^{q/2}}{|x|^{\frac{n-2}{2}q}} |w|^q\right).$$

Let $w \in C_0^\infty(\Omega)$ and $k \geqslant 0$, $\alpha \geqslant 0$. Then

$$\begin{aligned}
\int_\Omega \frac{(\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^q &= \frac{1}{n} \int_\Omega \frac{(\operatorname{div} x)(\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^q \\
&= -\frac{q}{n} \int_\Omega \frac{\langle x, \nabla w\rangle (\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^{q-2} w + \frac{k}{n} \int_\Omega \frac{(\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^q \\
&\quad + \frac{\alpha}{n} \int_\Omega \frac{(\ln \frac{R}{|x|})^\alpha}{|x|^k (\ln \frac{R}{|x|})} |w|^q.
\end{aligned}$$

Let $k < n$ and $R_0 > 0$ such that

$$\frac{\alpha}{n} \sup_{x \in \Omega} \frac{1}{(\ln \frac{R_0}{|x|})} < \frac{1}{2}\left(1 - \frac{k}{n}\right).$$

Then for $R \geqslant R_0$, the above identity gives

$$\frac{1}{2}\left(1 - \frac{k}{n}\right) \int_\Omega \frac{(\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^q \leqslant \frac{q}{n} \int_\Omega \frac{|\nabla w| (\ln \frac{R}{|x|})^\alpha}{|x|^{k-1}} |w|^{q-1},$$

$$\frac{1}{2}\left(1 - \frac{k}{n}\right) \int_\Omega \frac{(\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^q \leqslant \frac{q}{n} \left(\int_\Omega \frac{|w|^q (\ln \frac{R}{|x|})^\alpha}{|x|^k}\right)^{(q-1)/q} \left(\int_\Omega \frac{|\nabla w|^q (\ln \frac{R}{|x|})^\alpha}{|x|^{k-q}}\right)^{1/q}.$$

This implies that there exists $C = C(k, n, \alpha) > 0$ such that

$$\int_\Omega \frac{(\ln \frac{R}{|x|})^\alpha}{|x|^k} |w|^q \leqslant C\left(\int_\Omega \frac{|\nabla w|^q (\ln \frac{R}{|x|})^\alpha}{|x|^{k-q}}\right).$$

Choose $k = \frac{nq}{2}$ where q is such that $k < n$ and $\alpha = \frac{q}{2}$. For $R \geqslant R_0$ we have

$$\int_\Omega |\nabla u|^q = O\left(\int_\Omega \frac{(\ln \frac{R}{|x|})^{q/2}}{|x|^{\frac{n-2}{2}q}} |\nabla w|^q\right).$$

This implies that

$$\int_\Omega |\nabla u|^q \leqslant C\left(\int_\Omega \frac{(\ln \frac{R}{|x|})}{|x|^{n-2}} |\nabla w|^2\right)^{q/2}$$

which follows by Hölder's inequality. Hence we have

$$\int_\Omega (\Delta u)^2\,dx - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}\,dx - C_1\int_\Omega \frac{|u|^2}{|x|^4(\ln \frac{R}{|x|})^2} \geqslant C_2\left(\int_\Omega |\nabla u|^q\right)^{2/q}$$

i.e.,

$$\int_\Omega (\Delta u)^2\,dx - \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}\,dx - C_1\int_\Omega \frac{|u|^2}{|x|^4(\ln \frac{R}{|x|})^2}$$
$$\geqslant C_2\|\nabla u\|^2_{W_0^{1,q}(\Omega)} \quad \forall u \in H_0^2(\Omega). \qquad \square$$

Proof of (A.18). From the Hardy–Rellich inequality for $n = 4$ in Theorem A.1 we have

$$\int_\Omega (\Delta u)^2\,dx - \int_\Omega \frac{u^2}{|x|^4(\ln \frac{R}{|x|})^2}\,dx - \int_\Omega \frac{u^2}{|x|^4(\ln \frac{R}{|x|})^2(\ln(\ln \frac{R}{|x|}))^2}$$
$$\geqslant C_2\int_\Omega \frac{|\nabla w|^2}{|x|^2}\left(\ln \frac{R}{|x|}\right)\left(\ln\left(\ln \frac{R}{|x|}\right)\right)$$
$$\geqslant C_2\int_\Omega |\nabla w|^2\left(\ln \frac{R}{|x|}\right)\left(\ln\left(\ln \frac{R}{|x|}\right)\right),$$

where

$$w = \left(\ln\left(\ln \frac{R}{|x|}\right)\right)^{-1/2} v = \left(\ln\left(\ln \frac{R}{|x|}\right)\right)^{-1/2}\left(\ln \frac{R}{|x|}\right)^{-1/2} u.$$

Hence we have

$$u=\left(\ln\left(\ln\frac{R}{|x|}\right)\right)^{1/2}\left(\ln\frac{R}{|x|}\right)^{1/2}w.$$

Then,

$$|\nabla u|=O\left(\frac{(\ln(\ln\frac{R}{|x|}))^{1/2}}{(\ln\frac{R}{|x|})^{1/2}}w+\nabla w\left(\ln\left(\ln\frac{R}{|x|}\right)\right)^{1/2}\left(\ln\frac{R}{|x|}\right)^{1/2}\right).$$

This implies

$$|\nabla u|^q=O\left(\frac{(\ln(\ln\frac{R}{|x|}))^{q/2}}{(\ln\frac{R}{|x|})^{q/2}}w+\nabla w\left(\ln\left(\ln\frac{R}{|x|}\right)\right)^{q/2}\left(\ln\frac{R}{|x|}\right)^{q/2}\right). \tag{A.19}$$

Let $w\in C_0^\infty(\Omega)$ and $k\geqslant 0$, $\alpha\geqslant 0$ such that

$$\begin{aligned}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}&=\frac{1}{4}\int_\Omega(\operatorname{div}x)\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}\\&=-\frac{q}{4}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^{q-2}w\langle\nabla w,x\rangle}{(\ln\frac{R}{|x|})^k|x|^q}+\frac{q}{4}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}\\&\quad+\frac{\alpha}{4}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^{k+1}(\ln(\ln\frac{R}{|x|}))|x|^q}-\frac{k}{4}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}.\end{aligned}$$

This implies

$$\begin{aligned}\left(1+\frac{k}{4}\right)\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}&=-\frac{q}{4}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^{q-2}w\langle\nabla w,x\rangle}{(\ln\frac{R}{|x|})^k|x|^q}\\&\quad+\frac{\alpha}{4}\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^{k+1}(\ln(\ln\frac{R}{|x|}))|x|^q}.\end{aligned}$$

Choose $R_0>0$ such that

$$\frac{\alpha}{4}\sup_{x\in\Omega}\frac{1}{(\ln\frac{R_0}{|x|})(\ln(\ln\frac{R_0}{|x|}))}<1+\frac{k}{4}.$$

Then by Hölder's inequality we have

$$\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}\leqslant C\left(\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|w|^q}{(\ln\frac{R}{|x|})^k|x|^q}\right)^{(q-1)/q}\left(\int_\Omega\frac{(\ln(\ln\frac{R}{|x|}))^\alpha|\nabla w|^q}{(\ln\frac{R}{|x|})^{k-q}}\right)^{1/q}.$$

Hence we have

$$\int_\Omega \frac{(\ln(\ln\frac{R}{|x|}))^\alpha |w|^q}{(\ln\frac{R}{|x|})^k |x|^q} \leqslant C\left(\int_\Omega \frac{(\ln(\ln\frac{R}{|x|}))^\alpha |\nabla w|^q}{(\ln\frac{R}{|x|})^{k-q}}\right).$$

Choose $k = \frac{q}{2}$, $\alpha = \frac{q}{2}$ then the above inequality becomes

$$\int_\Omega \frac{(\ln(\ln\frac{R}{|x|}))^{q/2} |w|^q}{(\ln\frac{R}{|x|})^{q/2} |x|^q} \leqslant C\left(\int_\Omega |\nabla w|^q \left(\ln\left(\ln\frac{R}{|x|}\right)\right)^{q/2} \left(\ln\frac{R}{|x|}\right)^{q/2}\right).$$

Thus from (A.19) we have

$$\int_\Omega |\nabla u|^q \leqslant C\int_\Omega |\nabla w|^q \left(\ln\left(\ln\frac{R}{|x|}\right)\right)^{q/2} \left(\ln\frac{R}{|x|}\right)^{q/2}.$$

Hence we have

$$\int_\Omega |\nabla u|^q \leqslant C\left(\int_\Omega |\nabla w|^2 \left(\ln\left(\ln\frac{R}{|x|}\right)\right)\left(\ln\frac{R}{|x|}\right)\right)^{q/2},$$

$$\left(\int_\Omega |\nabla u|^q\right)^{2/q} \leqslant C\int_\Omega |\nabla w|^2 \left(\ln\left(\ln\frac{R}{|x|}\right)\right)\left(\ln\frac{R}{|x|}\right)$$

which ends the proof. □

Concluding remark. We have also obtained $W_0^{2,q}(\Omega)$ estimates in [5].

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