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Ordinary Differential Equations

Anti symmetric solutions of non-linear laminar flow between parallel permeable disks

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Abstract

The equations describing similarity solutions for flow between infinite parallel permeable disks with equal rates of suction or injection at the walls is derived using the stream function. This leads to a fourth order non-linear Ordinary Differential Equation. This equation is shown to admit anti-symmetric solutions using the moving plane method. ***To cite this article: Adimurthi, A. Karthik, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Solution antisymétrique d'un écoulement laminaire non linéaires entre deux disques parallèles perméables. On étudie des écoulements similaires entre deux disques parallèles infinis, perméables, dans le cas où les taux d'aspiration ou d'injection sont égaux ; on déduit les équations de mouvement en utilisant la fonction de courant. La méthode de plan mobile permet de démontrer l'antisymétrie des solutions d'un problème aux limites pour une équation différentielle d'ordre quatre. L'antisymétrie mise en évidence est conforme aux résultats numériques connus. ***Pour citer cet article : Adimurthi, A. Karthik, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

The study of non-linear laminar flow between parallel permeable disks has been extensively investigated both experimentally and theoretically. These equations form the prototype model for transpiration cooling, boundary layer control, thrust bearing etc. This problem was first studied by Burman [1] using regular perturbation methods valid for small Reynolds number. Later, Sellars [5], Yuan [7], Skalak and Wang [6] and Robinson [4] studied the problem theoretically for large Reynolds number.

To derive the required equation, we consider a steady, two-dimensional, axi-symmetric, non-linear flow of an incompressible fluid between parallel, permeable disks. By introducing a stream function, $\psi(r,\lambda) = \frac{U_0 r^2}{2} u(\lambda)$, we get

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the differential equation $u^{(4)}(\lambda) - Ru^{(3)}(\lambda)u(\lambda) = 0$, where $R = \frac{U_0 h}{\nu}$ is the Reynolds number and $u^{(i)}$ denotes the ith derivative of u. The corresponding boundary conditions on u are: $u(\pm 1) = \pm A$, $u'(\pm 1) = 0$.

In [2], the above problem was studied both theoretically and computationally. The existence of a solution for all R was shown using Leray Schauder degree theory. The Results of numerical simulations of the non-linear equation showed the solution was anti-symmetric in nature which is proved in the following section.

2. Symmetry of the solution

Let $I = (-1, 1)$ and $R > 0$, $A > 0$ and consider the boundary value problem:

$$u^{(4)} = Ruu^{(3)} \quad \text{in } I, \qquad u(\pm 1) = \pm A, \quad u'(\pm 1) = 0. \tag{1}$$

For a solution of (1), define $x_0, \zeta_0 \in I$ by: $u'(x_0) = \max_I u'$, $u^{(3)}(\zeta_0) = \max_I u^{(3)}$. Then we have the following:

Theorem 2.1. *Let u be a solution of* (1)*, then $u(x) = -u(-x)$, for all $x \in I$.*

We adopt the method of moving plane [3] to prove the theorem. The underlying equation is of fourth order and the solutions we are looking for are anti-symmetric; hence we adopt the simultaneous moving plane method for u' and $u^{(3)}$ instead of u. We need some preliminary lemmas before going to the proof.

Lemma 2.2. *Let u be a solution of* (1)*, then there exists a constant $c < 0$ such that*

(i) $u^{(3)} = c\exp(R\int_{-1}^{x} u(t)\,dt)$;
(ii) u', $u^{(3)}$ *are strictly concave functions with $u' > 0$, $u^{(3)} < 0$ in I, and u is a strictly increasing function*;
(iii) x_0, ζ_0 *are unique points with $u''(x_0) = u(\zeta_0) = 0$.*

Proof. Integrating (1) to obtain $c \in \mathbb{R}$ such that (i) holds. If $c = 0$, then u' is linear and hence from (1), $u' = 0$. Therefore u is constant which contradicts (1). Suppose $c > 0$, then u' is strictly convex with $u'(\pm 1) = 0$. Hence $u' < 0$ in I and therefore u is a decreasing function which is a contradiction since $u(-1) = -A < A = u(1)$. Hence $c < 0$ and therefore u' is strictly concave with $u' > 0$ in I. Thus it follows from (3) that $u^{(5)} < 0$ and hence $u^{(3)}$ is strictly concave.

Now at ζ_0, $0 = u^{(4)}(\zeta_0) = u(\zeta_0)u^{(3)}(\zeta_0)$ implies that $u(\zeta_0) = 0$. Since u is strictly increasing and u'' is strictly decreasing and hence x_0 and ζ_0 are unique. This proves the lemma. $\square$

Let $J \subset I$ be an interval and consider the equation:

$$w^{(4)} = Rww^{(3)} \quad \text{in } J. \tag{2}$$

Let u, v be solution of (2) and $h = u - v$. Then differentiating (2), we obtain

$$w^{(5)} = Rw'w^{(3)} + \frac{(w^{(4)})^2}{w^{(3)}}, \quad \text{and} \quad h^{(5)} - \frac{u^{(4)} + v^{(4)}}{u^{(3)}}h^{(4)} - \left(Rv' - \frac{(v^{(4)})^2}{u^{(3)}v^{(3)}}\right)h^{(3)} = Rh'u^{(3)}. \tag{3}$$

Let $\lambda \in I$, $2\lambda - x \in I$; let us define:

$$I_-(\lambda) = (-1, \lambda), \quad I_+(\lambda) = (\lambda, 1), \quad u_\lambda(x) = -u(2\lambda - x), \quad w_\lambda(x) = u'(x) - u'_\lambda(x), \tag{4}$$

$$A_\lambda(x) = -\frac{u^{(4)}(x) + u_\lambda^{(4)}(x)}{u^{(3)}(x)}, \qquad B_\lambda(x) = -\left(Ru'_\lambda(x) - \frac{(u_\lambda^{(4)}(x))^2}{u^{(3)}(x)u_\lambda^{(3)}(x)}\right). \tag{5}$$

Observe if u is a solution of (2), then u_λ is also a solution of (2) and thus by taking $v = u_\lambda$ in (3), we obtain,

$$w_\lambda^{(4)} + A_\lambda w_\lambda^{(3)} + B_\lambda w_\lambda^{(2)} = Rw_\lambda u^{(3)}. \tag{6}$$

Lemma 2.3. *Let $\lambda, \mu \in I$, then w_λ and $w_\mu^{(2)}$ satisfies*:

(i) $w'_\lambda(\lambda) = 0$ *if and only if* $\lambda = x_0$;
(ii) $w_\mu^{(3)}(\mu) = 0$ *if and only if* $\mu = \zeta_0$;

(iii) *Let* $\lambda < 0$ *and suppose* $w'_\lambda(\lambda) \neq 0$, $w_\lambda < 0$ *in* $[-1, \lambda)$ *for* $\lambda < 0$ (*or in* $(\lambda, 1]$ *for* $\lambda > 0$)*, then there exist a* $\tilde{\lambda} > \lambda$ ($\tilde{\lambda} < \lambda$) *such that* $w_\eta < 0$ *in* $I_-(\eta)$, $\forall \eta \in (-1, \tilde{\lambda}]$ (*respectively in* $I_+(\eta)$ $\forall \eta \in [\tilde{\lambda}, 1)$);

(iv) *Suppose* $w_\mu^{(3)}(\mu) \neq 0$*, then* (iii) *holds if we replace* w_λ *by* $w_\mu^{(2)}$ *in* (iii).

Proof. Since $w'_\lambda(\lambda) = 2u''(\lambda)$ and $w_\mu^{(3)}(\mu) = 2u^{(4)}(\mu) = 2Ru(\mu)u^{(3)}(\mu)$, hence (i) and (ii) follow from Lemma 2.2.

Suppose (iii) does not hold, then there exist sequences $\lambda_n > \lambda$, $x_n < y_n < \lambda_n$ in $[-1, \lambda_n)$ such that, as $n \to \infty$, $\lambda_n \to \lambda$, $x_n \to x$, $y_n \to y$ and $w_{\lambda_n}(x_n) = w'_{\lambda_n}(y_n) = w_{\lambda_n}(\lambda_n) = 0$. Hence as $n \to \infty$, $w_\lambda(x) = w'_\lambda(y) = 0$, $x \leqslant y \leqslant \lambda$. Since $w_\lambda(x) < 0$ in $[-1, \lambda)$, hence $x = y = \lambda$ and $w'_\lambda(\lambda) = 0$ which contradicts the hypothesis. This proves the lemma. Similarly (iv) follows. □

By strict concavity of u', u' is strictly increasing from $[-1, x_0)$ and strictly decreasing in $(x_0, 1]$. Hence there exist $-1 < \eta' < x_0 < \eta'' < 1$ such that $w_\eta < 0$ in $(-1, \eta]$ for $\eta \leqslant \eta'$ and $w_\eta < 0$ in $(\eta, 1]$ for $\eta'' \leqslant \eta < 1$. Since $u^{(3)}$ is also strictly concave and hence similar result holds. Therefore, define $-1 < \lambda_0 \leqslant \min(x_0, 0)$, $-1 < \mu_0 \leqslant \min(\zeta_0, 0)$, $\max(x_0, 0) \leqslant \lambda_1 \leqslant 1$, $\max(\zeta_0, 0) \leqslant \mu_1 < 1$ by

$$\lambda_0 = \sup\{\lambda;\ w_\eta < 0 \text{ in } I_-(\eta)\ \forall \eta \in (-1, \lambda]\}, \qquad \mu_0 = \sup\{\mu;\ w_\eta^{(2)} < 0 \text{ in } I_-(\eta)\ \forall \eta \in (-1, \mu]\},$$
$$\lambda_1 = \inf\{\lambda;\ w_\eta < 0 \text{ in } I_+(\eta)\ \forall \eta \in [\lambda, 1)\}, \qquad \mu_1 = \inf\{\mu;\ w_\eta^{(2)} < 0 \text{ in } I_+(\eta)\ \forall \eta \in [\mu, 1)\}.$$

Let u be a solution of (1). Since $-u(-x)$ is also a solution of (1) and hence without loss of generality we can assume, $u^{(3)}(-1) \leqslant u^{(3)}(1)$.

Lemma 2.4. $\lambda_0 \leqslant \mu_0$ *and* $\lambda_0 = \min(x_0, 0)$.

Proof. Suppose $\mu_0 < \lambda_0$, then $w_{\mu_0} < 0$, $w_{\mu_0}^{(2)} \leqslant 0$ in $I_-(\mu_0)$. If $2\mu_0 + 1 \leqslant \zeta_0$, then $u^{(3)}(-1) < u^{(3)}(2\mu_0 + 1)$. If $\zeta_0 < 2\mu_0 + 1$, and since $u^{(3)}(-1) \leqslant u^{(3)}(1)$, we have $u^{(3)}(-1) \leqslant u^{(3)}(1) < u^{(3)}(2\mu_0 + 1)$. In either case, we have $w_{\mu_0}^{(2)}(-1) < 0 = w_{\mu_0}^{(2)}(\mu_0)$. Let $h = w_{\mu_0}^{(2)}$, then from (4), $h(\mu_0) = w_{\mu_0}^{(2)}(\mu_0) = 0$ and from (6), h satisfies: $h'' + A_{\mu_0}h' + B_{\mu_0}h \geqslant 0$ with $h \leqslant 0$ in $I_-(\mu_0)$ and $h_{\mu_0}(\mu_0) = 0$ with $h_{\mu_0}(-1) < 0$.

Hence by Strong Maximum principle, $h < 0$ in $I_-(\mu_0)$ and $w_{\mu_0}^{(3)}(\mu_0) = h'(\mu_0) \neq 0$. Since $w_{\mu_0}^{(2)}(-1) \neq 0$ and hence from (iv) of Lemma 2.3, there exist a $\tilde{\mu} > \mu_0$ such that $w_\mu^{(2)} < 0$ in $I_-(\tilde{\mu})$ which contradicts the maximality of μ_0.

Suppose $\lambda_0 < \min(x_0, 0)$, then $2\lambda_0 + 1 < 1$ and hence $w_{\lambda_0}(-1) = -u'(2\lambda_0 + 1) < 0$. Since $u^{(3)}(-1) \leqslant u^{(3)}(1)$ and by strict concavity of $u^{(3)}$, we have for any $t \in (-1, 1)$, $u^{(3)}(t) > \min(u^{(3)}(1), u^{(3)}(-1)) = u^{(3)}(-1)$. Therefore we have $w_{\lambda_0}^{(2)}(-1) < 0$. Since $w_{\lambda_0} \leqslant 0$ in $I_-(\lambda_0)$, hence from (6) and the maximum principle implies that $w_{\lambda_0}^{(2)} < 0$ in $I_-(\lambda_0)$. Therefore w_{λ_0} is strictly concave and hence $w_{\lambda_0} < 0$ in $I_-(\lambda_0)$. Since $\lambda_0 < x_0$ and hence $w'_{\lambda_0}(\lambda_0) \neq 0$. Therefore from (iii) of Lemma 2.3 , there exist a $\lambda > \lambda_0$ such that $w_\lambda < 0$ in $I_-(\lambda)$ which contradicts the maximality of λ_0. This proves the lemma. □

Lemma 2.5. *If* $x_0 \geqslant 0$*, then* $u(x) = -u(-x)$.

Proof. From Lemma 2.4, $\lambda_0 = 0$, hence $w_{\lambda_0}^{(2)} \leqslant 0$. Therefore w'_{λ_0} is a decreasing function. Since $x_0 \geqslant 0$, hence for $x \in I_-(\lambda_0)$, $w'_{\lambda_0}(x) \geqslant w'_{\lambda_0}(0) = 2u''(0) \geqslant 2u''(x_0) = 0$. Hence w_{λ_0} is non-decreasing function with $w_{\lambda_0}(0) = w_{\lambda_0}(-1) = 0$. This implies that $w_{\lambda_0} \equiv 0$. Hence $\forall x \in [-1, 0]$, $u(x) + u(-x) = u(-1) + u(1) = 0$. This proves the lemma. □

Lemma 2.6. *If* $x_0 < 0$*, then* $\mu_0 = \min(\zeta_0, 0)$.

Proof. Suppose $\mu_0 < \min(\zeta_0, 0)$, then from (ii) of Lemma 2.3, $w_{\mu_0}^{(3)}(\mu_0) \neq 0$. Since $u^{(3)}(-1) \leqslant u^{(3)}(1)$ and by strict concavity of $u^{(3)}$, we have for any $t \in (-1, 1)$, $u^{(3)}(t) > \min(u^{(3)}(1), u^{(3)}(-1)) = u^{(3)}(-1)$. Therefore we have $w_{\mu_0}^{(2)}(-1) < 0$. Hence from (iv) of Lemma 2.3 and by maximality of μ_0, there exist a $y \in I_-(\mu_0)$ such that $w_{\mu_0}^{(2)}(y) = 0 \geqslant w_{\mu_0}^{(2)}(x)$ for $x \in I_-(\mu_0)$. Hence $w_{\mu_0}^{(3)}(y) = 0$ and this implies that $u(y) = -u(2\mu_0 - y)$. Therefore $\zeta_0 < 2\mu_0 - y$ and from (i) of Lemma 2.1, for $x \in I_-(\mu_0)$, we have $\int_{-1}^{2\mu_0 - x} u(t)\,\mathrm{d}t \leqslant \int_{-1}^{x} u(t)\,\mathrm{d}t$ and $\int_{-1}^{2\mu_0 - y} u(t)\,\mathrm{d}t =$

$\int_{-1}^{y} u(t)\,\mathrm{d}t$. Let $g(x) = \int_{x}^{2\mu_0 - x} u(t)\,\mathrm{d}t$, then from the above equations g satisfies $g(x) \leqslant 0$ in $I_-(\mu_0)$ and $g(y) = 0$. Hence $g'(y) = 0$ and $g''(y) \leqslant 0$. That is

$$u(y) = -u(2\mu_0 - y), \qquad u'(y) \geqslant u'(2\mu_0 - y). \tag{7}$$

Since $w_{\mu_0}^{(2)} \leqslant 0$ in $I_-(\mu_0)$, this implies that w_{μ_0} is a concave function. Since $w_{\mu_0}(-1) < 0$, $w_{\mu_0}(\mu_0) = 0$ and hence there exist a maximal interval $[\eta, \mu_0]$ on which $w_{\mu_0} \geqslant 0$. That is $u'(x) \leqslant u'(2\mu_0 - x)$ for $x \in (\mu_0, 2\mu_0 - \eta)$ and from (7), $y \in [\eta, \mu_0]$. Now $\mu_0 \leqslant \zeta_0 < 2\mu_0 - y$ and hence from (7) and (iii) of Lemma 2.2,

$$u(2\mu_0 - y) = \int_{\zeta_0}^{2\mu_0 - y} u'(t)\,\mathrm{d}t \leqslant \int_{\zeta_0}^{2\mu_0 - y} u'(2\mu_0 - t)\,\mathrm{d}t = \int_{y}^{2\mu_0 - \zeta_0} u'(t)\,\mathrm{d}t < \int_{y}^{\zeta_0} u'(t)\,\mathrm{d}t = -u(y) = u(2\mu_0 - y),$$

which contradicts (7). This proves the lemma. □

Lemma 2.7. $\mu_0 = \zeta_0$.

Proof. Suppose $\mu_0 < \zeta_0$ then from Lemma 2.6, $\mu_0 = 0$ and $w_{\mu_0}^{(2)}(x) \leqslant 0$ in $I_-(\mu_0)$. Hence w_{μ_0} is a concave function with $w_{\mu_0}(0) = w_{\mu_0}(-1) = 0$. Hence $w_{\mu_0} \geqslant 0$ in $I_-(\mu_0)$. Hence for $x \in (0, 1), u'(x) \leqslant u'(-x)$. Since $0 < \zeta_0$, we have $u(1) = \int_{\zeta_0}^{1} u'(t)\,\mathrm{d}t \leqslant \int_{\zeta_0}^{1} u'(-t)\,\mathrm{d}t = \int_{-1}^{-\zeta_0} u'(t)\,\mathrm{d}t < \int_{-1}^{\zeta_0} u'(t)\,\mathrm{d}t = -u(-1)$, which contradicts the boundary conditions. This proves the lemma. □

Proof of the Main Theorem (Theorem 2.1). Suppose $u(x) \neq -u(-x)$, then from Lemmas 2.5 and 2.6, we have $-1 < x_0 < \mu_0 = \zeta_0 < 0$. Performing the moving plane method from the right, we obtain:

Claim. *$\lambda_1 \leqslant \mu_1$ and $w_{\mu_1}^{(2)}(1) = 0, w_{\mu_1}^{(3)}(1) > 0$.*

Suppose $\mu_1 < \lambda_1$, since $w_{\lambda_1}^{(2)} < 0$ and hence w_{λ_1} is strictly concave with $w_{\lambda_1}(\lambda_1) = 0$, $w_{\lambda_1}(1) = u'(1) - u'(2\lambda_1 - 1) = -u'(2\lambda_1 - 1) < 0$. Hence $w_{\lambda_1} < 0$ in $(\lambda_1, 1]$. Since $x_0 < 0 < \lambda_1$, hence from (i) of Lemma 2.3, $w'_{\lambda_1}(\lambda_1) \neq 0$. Therefore from (iii) of Lemma 2.3 there exist a $\lambda < \lambda_1$such that $w_\lambda < 0$ in $I_+(\lambda)$ which contradicts the maximality of λ_1. Hence $\lambda_1 \leqslant \mu_1$. Therefore, from (6) at $\lambda = \mu_1$ and by the maximum principle, we have $w_{\mu_1}^{(2)} < 0$ in $I_-(\mu_1)$. Since $\zeta_0 < \mu_1$ and hence $w_{\mu_1}^{(3)}(\mu_1) \neq 0$. Therefore from maximality of μ_1 and from (iii) of Lemma 2.3, we have $w_{\mu_1}^{(2)}(1) = 0$ and by maximum principle, $w_{\mu_1}^{(3)}(1) > 0$. This proves the claim.

The claim gives us $u^{(3)}(1) = u^{(3)}(2\mu_1 - 1)$ and $0 < u^{(4)}(1) + u^{(4)}(2\mu_1 - 1) = u^{(3)}(1)(u(1) + u(2\mu_1 - 1))$. Hence $u(1) + u(2\mu_1 - 1) < 0$. From Lemma 2.2, u is an increasing function and hence $|u(x)| \leqslant \max\{|u(1)|, |u(-1)|\} = u(1)$. Hence $u(1) + u(2\mu_1 - 1) \geqslant 0$, which is a contradiction. This proves the theorem. □

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