PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 4, Pages 1013–1019
S 0002-9939(03)07301-5
Article electronically published on November 10, 2003

ASYMPTOTIC ESTIMATES FOR A TWO-DIMENSIONAL PROBLEM WITH POLYNOMIAL NONLINEARITY

ADIMURTHI AND MASSIMO GROSSI

(Communicated by David S. Tartakoff)

ABSTRACT. In this paper we give asymptotic estimates of the least energy solution u_p of the functional

$$J(u) = \int_\Omega |\nabla u|^2 \quad \text{constrained on the manifold} \quad \int_\Omega |u|^{p+1} = 1$$

as p goes to infinity. Here Ω is a smooth bounded domain of $\mathbb{R}^2$. Among other results we give a positive answer to a question raised by Chen, Ni, and Zhou (2000) by showing that $\lim_{p\to\infty} ||u_p||_\infty = \sqrt{e}$.

1. INTRODUCTION

In this paper we consider the following elliptic minimization problem. Let us define a C^2-functional on $H_0^1(\Omega)$:

$$J(u) = \int_\Omega |\nabla u|^2 \quad \text{constrained on the manifold} \quad \int_\Omega |u|^{p+1} = 1 \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and p is a real number greater than 1. Then we define

$$S_p = \inf_{u \in H_0^1(\Omega)} J(u). \tag{1.2}$$

By standard results it is easy to see that S_p is achieved at a function $u_p \in H_0^1(\Omega)$ that satisfies

$$\begin{cases} -\Delta u_p = S_p u_p^p & \text{in } \Omega, \\ u_p > 0 & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

By Lemma 2.1 we get $S_p = O\left(\frac{1}{p}\right)$ for p large. Setting $v_p = S_p^{\frac{1}{p-1}} u_p$ we are in the framework of [8], [9] and [6] where some asymptotic results about this problem were obtained.

Received by the editors September 7, 2002.
2000 *Mathematics Subject Classification.* Primary 35J20, 35B40.
Supported by M.U.R.S.T., project "Variational methods and nonlinear differential equations".

In particular, it was proved in [8] and [9] that the minimizer u_p looks like a sharp "spike". More precisely it was shown that, for a suitable sequence $p_n \to \infty$,

$$||u_{p_n}||_\infty \le C, \tag{1.4}$$

and in the sense of distribution,

$$\frac{u_{p_n}^{p_n}}{||u_{p_n}||_{p_n}^{p_n}} \to \delta_{x_0}. \tag{1.5}$$

Moreover, the point x_{p_n} where the minimizer u_{p_n} achieves its maximum converges to a critical point of the Robin function.

In this paper we obtain estimates of a different nature, greatly improving some partial results obtained in [5], where uniqueness and qualitative properties of the least energy solution were proved.

Here we use the two-dimensional blow-up technique introduced in [1]. The blow-up function is obtained by linearizing the nonlinear term plog u_p around the point of maximum of u_p. More precisely let us define the function $z_p(x) : \Omega_p = \frac{\Omega - x_p}{\varepsilon_p} \mapsto \mathbb{R}$,

$$z_p(x) = \frac{p}{u_p(x_p)}\big(u_p(\varepsilon_p x + x_p) - u_p(x_p)\big) \tag{1.6}$$

where x_p is the point where u_p achieves its maximum and $\varepsilon_p^2 = \frac{1}{pS_p u_p(x_p)^{p-1}}$. Then we obtain the following.

Theorem 1.1. *For any sequence z_{p_n} with $p_n \to \infty$, there exists a subsequence of z_{p_n}, still denoted by z_{p_n}, such that $z_{p_n} \to z$ in $C^2_{loc}(\mathbb{R}^2)$, where $z(x) = \log \frac{1}{\left(1+\frac{|x|^2}{8}\right)^2}$.*

The main result of the paper is a consequence of the above theorem.

Theorem 1.2. *Let u_p be a solution to (1.3). Then*

$$\lim_{p\to\infty} ||u_p||_\infty = \sqrt{e}. \tag{1.7}$$

Note that the estimate $\limsup_{p\to\infty} ||u_p||_\infty \le \sqrt{e}$ was proved in [9]. Here we give a positive answer to a question raised in [4], where some numerical computations suggested the validity of (1.7).

2. Proof of Theorem 1.1

In this section we recall some results about the solution u_p, and then we give the proof of Theorem 1.1.

We start by recalling the following estimates on S_p, due to Ren and Wei ([8]).

Lemma 2.1. *Let S_p be defined as in (1.2). Then*

$$\lim_{p\to\infty} pS_p = 8\pi e. \tag{2.1}$$

Proof. Setting $v_p = (S_p)^{\frac{1}{p-1}} u_p$ we have that v_p also achieves S_p and satisfies

$$\begin{cases} -\Delta v_p = v_p^p & \text{in } \Omega, \\ v_p > 0 & \text{in } \Omega, \\ v_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

From Corollary 2.3 of [8] we get $\lim_{p\to\infty} p\int_\Omega v_p^{p+1} = 8\pi e$. Hence, recalling that $\int_\Omega u_p^{p+1} = 1$ we derive that $\lim_{p\to\infty} p(S_p)^{\frac{p+1}{p-1}} = 8\pi e$, which implies (2.1). $\square$

Let us denote by x_p the point where $u_p(x_p) = ||u_p||_\infty$. By Lemma 4.1 of [8] we know that x_p is far away from the boundary of Ω. The next lemma provides additional information on the rate of $u_p(x_p)$.

Lemma 2.2. *We have that*

$$\lim_{p\to\infty} u_p(x_p)^{p-1} = +\infty. \tag{2.3}$$

Proof. Let us denote by $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Then we have

$$1 = \int_\Omega |u_p|^{p+1} \le u_p(x_p)^{p-1} \int_\Omega |u_p|^2 \le u_p(x_p)^{p-1} \lambda_1(\Omega)^{-1} \int_\Omega |\nabla u_p|^2. \tag{2.4}$$

Recalling that $\int_\Omega |\nabla u_p|^2 = S_p$ and Lemma 2.1, we deduce that $\int_\Omega |\nabla u_p|^2 \to 0$ as p goes to infinity. By (2.4), we obtain the claim. □

Proof of Theorem 1.1. For any sequence $p_n \to +\infty$, let us set $z_n : \Omega_n = \frac{\Omega - x_{p_n}}{\varepsilon_n} \mapsto \mathbb{R}$,

$$z_n(x) = \frac{p_n}{u_{p_n}(x_{p_n})}\big(u_{p_n}(\varepsilon_n x + x_{p_n}) - u_{p_n}(x_{p_n})\big) \tag{2.5}$$

where $\varepsilon_n^2 = \frac{1}{p_n S_{p_n} u_{p_n}(x_{p_n})^{p_n-1}}$. From Lemma 2.1 and Lemma 2.2, we get $\varepsilon_n \to 0$ as $n \to \infty$ and "$\Omega_n \to \mathbb{R}^2$" as $n \to \infty$. Now let us write down the equation satisfied by z_n,

$$\begin{cases} -\Delta z_n = \left(1 + \frac{z_n}{p_n}\right)^{p_n} & \text{in } \Omega_n, \\ 0 < 1 + \frac{z_n}{p_n} \le 1 & \text{in } \Omega_n, \\ z_n = -p_n & \text{on } \partial\Omega_n. \end{cases} \tag{2.6}$$

We want to pass to the limit in (2.6). To do this we use some ideas in [2]. Let $B(0,R)$ be the ball centered at the origin with radius R, and let w_n be the solution of

$$\begin{cases} -\Delta w_n = \left(1 + \frac{z_n}{p_n}\right)^{p_n} & \text{in } B(0,R), \\ w_n = 0 & \text{on } \partial B(0,R). \end{cases} \tag{2.7}$$

By the maximum principle and the standard regularity theory, we have that $0 \le w_n \le C$ with C independent of n. For $x \in B(0,R)$ set $\psi_n(x) = z_n(x) - w_n(x)$. Hence ψ_n is a sequence of harmonic functions which are uniformly bounded above. Hence by Harnack's inequality [7] we have the alternative:
either
i) a subsequence of ψ_n is bounded in $L^\infty_{loc}(B(0,R))$,
or
ii) ψ_n converges uniformly to $-\infty$ on compact subsets of $(B(0,R)$.

Since $\psi_n(0) = z_n(0) - w_n(0) = -w_n(0) \ge -C$, case ii) cannot occur. Hence, up to a subsequence, which we denote again by ψ_n, we have that ψ_n is bounded in $L^\infty(B(0,R))$ for any $R > 0$ and the same holds for z_n. From (2.6), and the standard regularity theory, we derive that z_n is bounded in $C^2_{loc}(\mathbb{R}^2)$, and then it converges to a function $z \in C^2(\mathbb{R}^2)$. Passing to the limit in (2.6), we get that z satisfies

$$-\Delta z = e^z \quad \text{in } \mathbb{R}^2. \tag{2.8}$$

Let us prove that $\int_{\mathbb{R}^2} e^z < +\infty$. To do this we observe that, since $z_n \to z$ in $C^2_{loc}(\mathbb{R}^2)$, then

$$p_n\Big(\log\big(1+\frac{z_n}{p_n}\big)-\frac{z_n}{p_n}\Big)\to 0 \quad \text{pointwise in } \mathbb{R}^2. \tag{2.9}$$

Hence

$$z_n+p_n\Big(\log\big(1+\frac{z_n}{p_n}\big)-\frac{z_n}{p_n}\Big)\to z \quad \text{pointwise in } \mathbb{R}^2. \tag{2.10}$$

By Fatou's Lemma, we deduce

$$\begin{aligned}\int_{\mathbb{R}^2} e^z &\le \lim_{n\to\infty}\int_{\Omega_n} e^{z_n+p_n\Big(\log\big(1+\frac{z_n}{p_n}\big)-\frac{z_n}{p_n}\Big)} = \lim_{n\to\infty}\int_{\Omega_n}\Big(1+\frac{z_n}{p_n}\Big)^{p_n}\\ &= \lim_{n\to\infty}\frac{1}{\varepsilon_n^2 u_{p_n}^{p_n}(x_{p_n})}\int_\Omega u_{p_n}^{p_n} \le \lim_{n\to\infty}\frac{p_nS_{p_n}}{u_{p_n}(x_{p_n})}|\Omega|^{\frac{1}{p_n+1}}\le C\end{aligned} \tag{2.11}$$

since $u_{p_n}(x_{p_n})\ge C$ in Ω for n large (see [8], p. 755).

By a result of Chen and Li ([3]), the solutions of (2.8) satisfying $\int_{\mathbb{R}^2} e^z < +\infty$ are given by

$$z(x)=\log\frac{\mu}{\big(1+\frac{\mu}{8}|x-x_0|^2\big)^2} \quad \text{for } \mu>0 \text{ and } x_0\in\mathbb{R}^2. \tag{2.12}$$

Since $z(x)\le z(0)=0$ for any $x\in\mathbb{R}^2$, we derive that $\mu=1$ and $x_0=0$ in (2.12), and this gives the claim of Theorem 1.1. □

3. Proof of Theorem 1.2

The next estimate plays a role in the proof of Theorem 1.2. This estimate was proved in [9] but we stress that it follows easily by Theorem 1.1.

Lemma 3.1. *We have that*

$$\limsup_{n\to\infty}||u_{p_n}||_\infty \le \sqrt{e}. \tag{3.1}$$

Proof. It follows directly by Theorem 1.1.

Setting $u_{p_n}=u_n$ and $L=\limsup_{n\to\infty}||u_n||_\infty$, by using Fatou's Lemma, we obtain

$$\begin{aligned}1&=\int_\Omega u_n^{p+1}=u_n(x_n)^{p_n+1}\varepsilon_n^2\int_{\Omega_n}\big(1+\frac{z_n}{p_n}\big)^{p_n+1}\\ &=\frac{u_n(x_n)^2}{p_nS_{p_n}}\int_{\Omega_n}\big(1+\frac{z_n}{p_n}\big)^{p_n+1}\ge\frac{L^2}{8\pi e}\int_{\mathbb{R}^2}e^z.\end{aligned} \tag{3.2}$$

Recalling that $\int_{\mathbb{R}^2} e^z = 8\pi$, we deduce the claim. □

Let us consider the linearized operator associated to (1.3), i.e., $L_p : H^1_0(\Omega)\to H^{-1}(\Omega)$,

$$L_p=-\Delta-pS_pu_p^{p-1}(x)I, \quad x\in\Omega, \tag{3.3}$$

and let us denote by $\lambda_1(L_p)$, $\lambda_2(L_p)$ the first and the second eigenvalue of L_p. Now let us recall a property of $\lambda_2(L_p)$.

Lemma 3.2. *We have that*

$$\lambda_2(L_p)\ge 0. \tag{3.4}$$

Proof. The proof is standard since u_p is a minimizer of J on the manifold $\int\limits_{\Omega} |u|^{p+1} = 1$. ◻

We consider, for $D \subset \Omega_p$, $L_{p,D} : H_0^1(D) \to H^{-1}(D)$,

$$L_{p,D} = -\Delta - \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} I, \quad x \in D, \tag{3.5}$$

and let us denote by $\lambda_1(L_{p,D})$, $\lambda_2(L_{p,D})$ the first and the second eigenvalue of $L_{p,D}$.

Lemma 3.3. *We have that*

$$\lambda_2(L_{p,\Omega_p}) \geq 0. \tag{3.6}$$

Proof. Using the scaling $x \to \varepsilon_p x + x_p$ we get $\lambda_2(L_{p,\Omega_p}) = \varepsilon_p^2 \lambda_2(L_p)$ and (3.6) follows by Lemma 3.2. ◻

Lemma 3.4. *Let us denote by $B_1 = B(0,1)$. Let $p_n \to \infty$ such that $z_{p_n} \to z$ in $C^1_{loc}(\mathbb{R}^2)$. Then for large p_n, we have*

$$\lambda_1(L_{p_n,B_1}) < 0. \tag{3.7}$$

Proof.

$$w_p = x \cdot \nabla z_p + \frac{2}{p-1} z_p + \frac{2p}{p-1}. \tag{3.8}$$

By direct computation we get that w_p satisfies

$$-\Delta w_p = \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} w_p. \tag{3.9}$$

Moreover, $w_{p_n}(0) \to 2$ and for $|x| = 1$, $w_{p_n}(x) \to -\frac{4}{9}$ as $p_n \to \infty$.

Hence, if we denote by $A_p = \{x \in B_1 : w_p > 0\}$ and

$$\tilde{w}_p = \begin{cases} w_p & \text{if } x \in A_p, \\ 0 & \text{if } x \in B_1 \setminus \bar{A}_p, \end{cases} \tag{3.10}$$

we derive that for p_n large, $\tilde{w}_{p_n} \in H_0^1(B_1)$. From (3.9) we get

$$\int_{B_1} |\nabla \tilde{w}_{p_n}|^2 - \int_{B_1} \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n} x + x_{p_n})}{u_{p_n}^{p_n-1}(x_{p_n})} \tilde{w}_{p_n}^2 = 0, \tag{3.11}$$

and this implies that $\lambda_1(L_{p_n,B_1}) < 0$. ◻

Lemma 3.5. *Let p_n be a sequence as in Lemma 3.4. Then for p_n large we have*

$$\lambda_1(L_{p_n,\Omega_{p_n} \setminus B_1}) > 0. \tag{3.12}$$

Proof. By contradiction let us suppose that $\lambda_1(L_{p_n,\Omega_{p_n}\setminus B_1}) \leq 0$. Then from Lemma 3.4, for large p_n, $\lambda_1(L_{p_n,B_1}) < 0$ and hence $\lambda_2(L_{p_n,\Omega_{p_n}}) < 0$. This gives a contradiction with Lemma 3.3. ◻

Remark 3.6. Lemma 3.4 implies that the operator $L_{p_n}, \Omega_{p_n \setminus B_1}$ satisfies the maximum principle in $\Omega_{p_n} \setminus B_1$.

Proof of Theorem 1.2. By Lemma 3.1 we know that (up to a subsequence) $\lim_{n\to\infty} ||u_{p_n}||_\infty \le \sqrt{e}$. By contradiction let us suppose that there exists a subsequence of u_{p_n} (still denoted by u_{p_n}) such that

$$\lim_{n\to\infty} ||u_{p_n}||_\infty < \sqrt{e}. \tag{3.13}$$

Now, we will show that for large p_n, (3.13) implies the following estimate:

$$z_n(x) \le C + \log \frac{1}{\left(1+\frac{|x|^2}{8}\right)^2} \quad \forall x \in \Omega_{p_n} \tag{3.14}$$

where C is a constant independent of n.

By Theorem 1.1, $z_n \to z$ in $C^\circ(\overline{B}_1)$ and hence (3.14) fails for $x \in B_1$. It is enough to prove 3.14 for $x \in \Omega_n \setminus B_1$. To prove this let us observe that the function z satisfies

$$-\Delta z = e^z \ge \left(1+\frac{z}{p}\right)^p \tag{3.15}$$

for any $p > 1$. Furthermore, let us consider $\psi_n = z_n - z$ in Ω_{p_n}. By computing ψ_n on $\partial(\Omega_{p_n} \setminus B_1)$ and by applying the maximum principle, if $x \in \partial\Omega_{p_n}$, we get

$$\begin{aligned}\psi_n(x) = z_n(x) - z(x) = -p_n + 2\log(1+\frac{|x|^2}{8}) \le -p_n + 2\log\frac{1}{\varepsilon_{p_n}^2} + C \\ \le -p_n + 2\log u_{p_n}(x_{p_n})^{p_n-1} + C \le C\end{aligned} \tag{3.16}$$

where we had used $u_{p_n}(x_{p_n}) < \sqrt{e}$.

Now if $x \in \partial B_1$, by Theorem 1.1 we can derive again that $\psi_n(x) \le C$.

Finally, we write down the equation satisfied by ψ_n. Using the convexity of $F(s) = \left(1+\frac{s}{p}\right)^p$ for $p > 1$ we have

$$\begin{aligned}-\Delta\psi_n &= \left(1+\frac{z_n}{p_n}\right)^{p_n} - \left(1+\frac{z}{p_n}\right)^{p_n} \\ &\le \left(1+\frac{z_n}{p_n}\right)^{p_n-1}\psi_n = \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n}x + x_{p_n})}{u_{p_n}^{p_n-1}(x_{p_n})}\psi_n.\end{aligned} \tag{3.17}$$

Since the maximum principle holds in $\Omega_{p_n} \setminus B_1$ for $L_{p_n,\Omega_{p_n}\setminus B_1}$, we now deduce that $\psi_n \le C$ in $\Omega_{p_n} \setminus B_1$ and this gives (3.14).

From (3.14), a contradiction follows easily. Indeed, using Theorem 1.1 and Lebesgue's Theorem we derive

$$\begin{aligned}1 &= \int_\Omega u_n^{p_n+1} = u_n(x_n)^{p_n+1}\varepsilon_n^2 \int_{\Omega_n} \left(1+\frac{z_n}{p_n}\right)^{p_n+1} \\ &= \frac{u_n^2(x_n)}{8\pi e + o(1)}\big(8\pi + o(1)\big),\end{aligned} \tag{3.18}$$

which proves that $\lim_{n\to\infty} ||u_n||_\infty = \sqrt{e}$, a contradiction with (3.13). □

Acknowledgments

This work was done while the second author was visiting the TIFR centre in Bangalore. He would like to thank the School of Mathematics for its support and warm hospitality.

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T.I.F.R. Centre, P.O. Box 1234, Bangalore 560012, India
E-mail address: `aditi@math.tifrbng.res.in`

Università di Roma "La Sapienza", P.le Aldo Moro, 2, 00185 Roma, Italy
E-mail address: `grossi@mat.uniroma1.it`