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GENERALIZED HARDY–RELLICH INEQUALITIES IN CRITICAL DIMENSION AND ITS APPLICATIONS

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In this paper, we study the Hardy–Rellich inequalities for polyharmonic operators in the critical dimension and an analogue in the p-biharmonic case. We also develop some optimal weighted Hardy–Sobolev inequalities in the general case and discuss the related eigenvalue problem. We also prove $W^{2,q}(\Omega)$ estimates in the biharmonic case.

Keywords: Hardy inequalities; biharmonic operator; $W^{2,q}(\Omega)$ estimates.

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1. Introduction

Inequalities involving integrals of a function and its derivatives appear frequently in various branches of mathematics and represent a useful tool, e.g., in the theory and practice of differential equations, in the theory of approximation etc. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $0 \in \Omega$. Let us recall that the Hardy–Rellich inequality states that for all $u \in H_0^2(\Omega)$

$$\int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \geq 0, \quad n \geq 5 \tag{1.1}$$

where $\frac{n^2(n-4)^2}{16}$ is the best constant in (1.1) and it is never achieved in any domain $\Omega \subset \mathbb{R}^n$. This inequality was first proved by Rellich [15] for $u \in H_0^2(\Omega)$ and it was extended to functions in $H^2(\Omega) \cap H_0^1(\Omega)$ by Dold *et al.* in [9].

The main questions related to this inequality are many folds and are as follows:

(i) extend the inequality (1.1) in all dimensions,
(ii) replace "2" by "p",
(iii) extend this to polyharmonic case.

In this direction, Davis and Hinz [8] generalized (1.1) and showed that for any $p \in (1, \frac{n}{2})$, there holds

$$\int_\Omega |\Delta u|^p - \left(\frac{n(p-1)(n-2p)}{p^2}\right)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} \geq 0, \quad u \in C_0^\infty(\Omega\backslash\{0\}). \tag{1.2}$$

In [12], the inequality (1.2) was proved for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $1 < p < \frac{n}{2}$. Also it was extended these inequalities to the polyharmonic case with weights and is as follows. Let $0 < k < \frac{n}{2}$ be an integer and $u \in W_0^{k,2}(\Omega)$. Then if $k = 2m$

$$\int_\Omega (\Delta^m u)^2 dx \geq \left(\prod_{l=1}^{2m} \frac{(n+4m-4l)^2}{4}\right) \int_\Omega \frac{u^2}{|x|^{4m}} dx. \tag{1.3}$$

If $k = 2m+1$, then

$$\int_\Omega |\nabla \Delta^m u|^2 dx \geq \left(\prod_{l=1}^{2m+1} \frac{(n+4m+2-4l)^2}{4}\right) \int_\Omega \frac{u^2}{|x|^{4m+2}} dx. \tag{1.4}$$

Let $\theta \in \mathbb{R}, p > 1, k \geq 1, n > s > 2(1 + p(k-1))$ and $n > \theta + 2$, define

$$d_{p,\theta} = \frac{(n-2-\theta)[(p-1)(n-2)+\theta]}{p^2} \tag{1.5}$$

$$c_{p,s}^k = \left[\frac{1}{p^{2k}} \prod_{i=0}^{k-1} (n+2pi-s)(n(p-1)-2p(i+1)+s)\right]^p \tag{1.6}$$

then for all $u \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{|\Delta u|^p}{|x|^{\theta+2-2p}} dx \geq d_{p,\theta}^p \int_\Omega \frac{|u|^p}{|x|^{\theta+2}} dx \tag{1.7}$$

$$\int_\Omega \frac{|\nabla(\Delta u)|^p}{|x|^{\theta+2-3p}} dx \geq d_{p,\theta}^p \left(\frac{n-(\theta+2-2p)}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{\theta+2}} dx \tag{1.8}$$

$$\int_\Omega \frac{|(\Delta^k u)|^p}{|x|^{s-2pk}} dx \geq c_{p,s}^k \int_\Omega \frac{|u|^p}{|x|^s} dx. \tag{1.9}$$

Also, he proved that if $n > 2kp$, then

$$\int_\Omega |(\Delta^k u)|^p dx \geq e_{p,k} \int_\Omega \frac{|u|^p}{|x|^{2pk}} dx \tag{1.10}$$

where

$$e_{p,k} = \left[\frac{1}{p^{2k}} \prod_{i=0}^{k-1} (n-2p(i+1))(n(p-1)+2pi)\right]^p.$$

It is to be noted that all the constants appearing in the above inequalities are sharp.

Another important Hardy–Rellich type inequality is when the entire boundary is considered as the singularity. Let $d(x) = d(x, \partial\Omega)$ denotes the distance function to the boundary $\partial\Omega$ of Ω. For $t \in (0, 1]$, define for $i \geq 2$,

$$\begin{aligned} X_1(t) &= (1 - \ln t)^{-1} \\ X_i(t) &= X_1(X_{i-1}(t)). \end{aligned}$$

For a convex domain Ω, it has been shown in [14] that for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\Delta u|^2 \geq \frac{9}{16} \int_\Omega \frac{u^2}{d(x)^4}, \quad u \in C_0^\infty(\Omega). \tag{1.11}$$

This inequality has been improved in [5]. It has been shown that there exist a $D_0 > \sup_{x\in\Omega} d(x)$ such that for all $u \in C_0^\infty(\Omega)$ with $X_j = X_j\left(\frac{d(x)}{D}\right)$, there holds

$$\int_\Omega |\Delta u|^2 \geq \frac{1}{4} \int_\Omega \frac{|\nabla u|^2}{d(x)^2} + \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{|\nabla u|^2}{d(x)^2} X_1^2 X_2^2 \cdots X_i^2, \tag{1.12}$$

$$\int_\Omega |\Delta u|^2 \geq \frac{9}{16} \int_\Omega \frac{|u|^2}{d(x)^4} + \frac{5}{8} \sum_{i=1}^\infty \int_\Omega \frac{|u|^2}{d(x)^4} X_1^2 X_2^2 \cdots X_i^2. \tag{1.13}$$

Now we come to the question (i), that is, what happens to [1] when $n = 4$? Surprisingly, it was shown in [3] that this inequality differs when compared to $n \geq 5$. Basically, the idea of the proof relies on the fundamental solution of Δ^2 which was used earlier in [1] to generalize Hardy–Sobolev inequality on Riemannian manifolds.

Motivated by [3], in this paper we discuss the description of Hardy–Rellich inequalities in the critical dimension. Furthermore, we prove the Vazquez and Zuazua [18] type of inequalities for the biharmonic case.

2. Main Theorems

Before stating the main theorems, we introduce the following definitions and notations. Let $e^0 = 1$, $e^{(1)} = e$, $e^{(k)} = e^{e^{(k-1)}}$ for $k \geq 1$. Let $a > 0$ and define,

$$\begin{aligned} \ln_{(1)} a &= \ln(a), \\ \ln_{(k)} a &= \ln \ln_{(k-1)}(a), \quad \text{for } k \geq 2, \\ \ln^{(k)}(a) &= \prod_{j=1}^k \ln_{(j)}(a), \qquad \text{if } a > e^{(k-1)}, \\ W_r^{2m,2}(B) &= \{u \in W^{2m,2}(B) : u \text{ is radial}\}, \\ W_r^{2m+1,2}(B) &= \{u \in W^{2m+1,2}(B) : u \text{ is radial}\}. \end{aligned}$$

Recall that $W_0^{k,2}(\Omega)$ is equipped with the norm

$$\|u\|_{W_0^{k,2}(\Omega)} = \begin{cases} \int_\Omega (\Delta^m u)^2 dx; & k = 2m, \quad m \in \mathbb{N}, \\ \int_\Omega |\nabla \Delta^m u|^2 dx; & k = 2m+1, \quad m \in \mathbb{N} \cup \{0\}. \end{cases}$$

We consider two situations: firstly, $W^{2,p}(\Omega)$ where $n = 2p$; and secondly, $W^{k,2}(\Omega)$ where $n = 2k$. Let B be the unit ball in $\mathbb{R}^n$.

Then we have the following main results.

Theorem 2.1. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2p$. Then there exist a constant $C > 0$ such that for any k, we can find a $R = R(k) > 0$ and for all $u \in W_0^{2,p}(\Omega)$*

$$\int_\Omega |\Delta u|^p dx - \frac{2^p(p-1)^{2p}}{p^p} \int_\Omega \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} dx \geq C \int_\Omega \left(\sum_{j=2}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right) \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p dx} \quad \text{if } p \geq 2, \tag{2.1}$$

$$\int_\Omega |\Delta u|^p dx - \frac{2^p(p-1)^{2p}}{p^p} \int_\Omega \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} dx \geq 0 \quad \text{if } 1 < p < 2. \tag{2.2}$$

The constant $-\frac{2^p(p-1)^{2p}}{p^p}$ (the coefficient of $\int_\Omega \frac{|u|^p}{|x|^{2p}(\ln \frac{R}{|x|})^p} dx$) is the best constant and is never achieved by any nontrivial function $u \in W_0^{2,p}(\Omega)$.

Theorem 2.2. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{R}^n$, $1 \leq q < 2$. Then there exist $R_0 > 0$ and $C_q > 0$ such that $\forall\, R \geq R_0$ and $\forall\, u \in H_0^2(\Omega)$,*

$$\int_\Omega |\Delta u|^2 - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \geq C_q \|u\|^2_{W_0^{2,q}(\Omega)} \quad \text{if } n \geq 5 \tag{2.3}$$

and

$$\int_\Omega |\Delta u|^2 - \int_\Omega \frac{u^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} \geq C_q \|u\|^2_{W_0^{2,q}(\Omega)} \quad \text{if } n = 4. \tag{2.4}$$

Note that Theorem 2.2 is an extension of the Vazquez and Zuazua's inequality [18] in the case of a biharmonic operator and -1 (the coefficient of $\int_\Omega \frac{u^2}{|x|^4(\ln \frac{R}{|x|})^2}$) is the best constant in the inequality (2.4) which follows from [3]. In fact, Theorem 2.2 implies that $H_0^2(\Omega) \hookrightarrow W_0^{2,q}(\Omega)$ is continuous.

Theorem 2.3. (a) *Let B be a unit ball centered at the origin in $\mathbb{R}^n$, $n = 4m$. Then for $m \geq 2$, there exist a constant $C > 0$ such that for any k, we can find a $R = R(k) > 0$ and for all $u \in H_{0,r}^{2m}(B)$*

$$\int_B |\Delta^m u|^2 dx - \frac{n^2}{16}\left[\frac{1}{2^{2m-2}}\prod_{i=0}^{m-2}(4i+2)(8m-4i-6)\right]^2 \int_B \frac{|u|^2}{|x|^{4m}\left(\ln\frac{R}{|x|}\right)^2}dx$$
$$\geq C\int_B \left(\sum_{j=2}^{k}\frac{1}{\left(\ln^{(j)}\frac{R}{|x|}\right)^2}\right)\frac{|u|^2}{|x|^{4m}}dx. \tag{2.5}$$

The constant $-\frac{n^2}{16}\left[\frac{1}{2^{2m-2}}\prod_{i=0}^{m-2}(4i+2)(8m-4i-6)\right]^2$ (the coefficient of $\int_B \frac{|u|^2}{|x|^{4m}(\ln\frac{R}{|x|})^2}dx$) is the best constant and is never achieved by any nontrivial function $u \in H_{0,r}^{2m}(B)$.

(b) *Let B be a unit ball centered at the origin in $\mathbb{R}^n$, $n = 4m+2$. Then there exist a constant $C > 0$ such that for any k, we can find a $R = R(k) > 0$ and for all $u \in H_{0,r}^{2m+1}(B)$*

$$\int_B |\nabla\Delta^m u|^2 dx - \frac{n^2}{16}\left[\frac{1}{2^{2m}}\prod_{i=0}^{m-1}(4m-4i-2)(4m+4i+2)\right]^2$$
$$\times \int_B \frac{|u|^2}{|x|^{4m+2}\left(\ln\frac{R}{|x|}\right)^2}dx$$
$$\geq C\int_B \left(\sum_{j=2}^{k}\frac{1}{\left(\ln^{(j)}\frac{R}{|x|}\right)^2}\right)\frac{|u|^2}{|x|^{4m+2}}dx. \tag{2.6}$$

The constant $-\frac{n^2}{16}\left[\frac{1}{2^{2m}}\prod_{i=0}^{m-1}(4m-4i-2)(4m+4i+2)\right]^2$ (the coefficient of $\int_B \frac{|u|^2}{|x|^{4m+2}(\ln\frac{R}{|x|})^2}dx$) is the best constant and is never achieved by any nontrivial function $u \in H_{0,r}^{2m+1}(B)$.

Next, we study the eigenvalue problems associated with the perturbed Hardy–Rellich operator for the case $n = 4$, which is highly singular and non-compact.

Let $R > 0, \lambda > 0, 0 < \mu < 1, X = H_0^2(\Omega)$ or $H^2(\Omega) \cap H_0^1(\Omega)$ and define

$$\mathcal{F} = \left\{ f \in L^\infty_{\text{loc}}(\Omega \backslash \{0\}) : \lim_{|x|\to\infty} |x|^4 \left(\ln \frac{R}{|x|} \right)^2 f(x) = 0 \right\}$$

$$F = \left\{ f \in L^\infty_{\text{loc}}(\Omega \backslash \{0\}) : \lim_{|x|\to\infty} |x|^4 \left(\ln \frac{R}{|x|} \right)^2 \left(\ln \left(\ln \frac{R}{|x|} \right) \right)^2 f(x) < \infty \right\}.$$

For $f \in \mathcal{F} \cup F$, we look for a weak solution next of the following eigenvalue problems and study the asymptotic behavior of the first eigenvalue as $\mu \to 1$.

$$\begin{cases} L_\mu u = \lambda f(x) u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

$$\begin{cases} L_\mu u = \lambda f(x) u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.8}$$

where

$$L_\mu u = \Delta^2 u - \mu \frac{u}{|x|^4 \left(\ln \dfrac{R}{|x|} \right)^2}. \tag{2.9}$$

Theorem 2.4. *The problem* (2.7) *and* (2.8) *admits a nontrivial weak solution* $u \in X$, *corresponding to the first eigenvalue* $\lambda^1_\mu(f) = \lambda > 0$. *Moreover, as* $\mu \to 1$, $\lambda^1_\mu(f) \to \lambda(f) \geq 0$ *for all* $f \in \mathcal{F}$ *and the limit* $\lambda(f) > 0$ *if* $f \in F$. *Moreover, if* $\Omega = B$, *in problem* (2.7), *then the first eigenfunction is positive and the first eigenvalue is simple. If* $u \in H^2(\Omega) \cap H_0^1(\Omega)$, *the first eigenfunction is positive and the first eigenvalue is positive.*

3. Preliminary Lemmas

In this section, we briefly discuss the Hardy–Sobolev inequalities with weights, which will be required to prove the main theorems. First we introduce the following notations. Let $k \geq 1$ be an integer and $R > e^{(k-1)} \sup_{\partial\Omega} |x|$. Let

$$E(x) = \begin{cases} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3 \\ \ln\left(\dfrac{R}{|x|} \right) & \text{if } n = 2, \end{cases}$$

$$E_1(x) = \begin{cases} \dfrac{1}{|x|^{n-4}} & \text{if } n \geq 5 \\ \ln\left(\dfrac{R}{|x|} \right) & \text{if } n = 4. \end{cases}$$

Let $\alpha \in \mathbb{R}$ and $m \geq 0$ be a measurable function. Let $2^* = \frac{2n}{n-2}$ if $n \geq 3$, $2^\sharp = \frac{2n}{n-4}$ if $n \geq 5$ and define

$$L^p(\Omega, m) = \left\{ u \text{ measurable}; \int_\Omega |u|^p m dx < \infty \right\}$$

$$\mathcal{D}^{1,2}_\alpha(\Omega) = \{u \in L^{2^*}(\Omega); \nabla u \in L^2(\Omega,\, E^{1-2\alpha})\}$$

$$\mathcal{D}^{2,2}_\alpha(\Omega) = \{u \in L^{2^*}(\Omega); \Delta u \in L^2(\Omega,\, E_1^{1-2\alpha})\}$$

$$\mathcal{D}^{1,2}_{\alpha,0}(\Omega) = \{u \in \mathcal{D}^{1,2}_\alpha(\Omega) : u = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{D}^{2,2}_{\alpha,0}(\Omega) = \left\{ u \in \mathcal{D}^{2,2}_\alpha(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

Let us first recall the basic result which will be needed in the proof of the main theorems. Let $p \in (1, \infty)$ and $M > 1$. Then there exist constants $\alpha_1, \alpha_2 > 0$ such that for all $a, b \in \mathbb{R}^n$, $|a| = 1$ we have

$$|a + b|^p - 1 - p\langle a, b\rangle \geq \alpha_1 |b|^2 + \alpha_2 |b|^p \quad \text{if } p \geq 2 \tag{3.1}$$

$$|a + b|^p - 1 - p\langle a, b\rangle \geq \begin{cases} \alpha_1 |b|^2 & \text{if } |b| \leq M \quad p \in (1, 2] \\ \alpha_2 |b|^p & \text{if } |b| \geq M \quad p \in (1, 2). \end{cases} \tag{3.2}$$

If χ denotes the characteristic function, then the above formula can be written as

$$|a + b|^p - 1 - p\langle a, b\rangle \geq \alpha_1 |b|^2 \chi_{\{|b| \leq M\}} + \alpha_2 |b|^p \chi_{\{|b| \geq M\}}.$$

Furthermore if $p = 2$, then $\alpha_1 = \alpha_2 = \frac{1}{2}$ is the best choice in (3.1), (3.2). The following lemma will be used to obtain the remainder terms in Theorems 2.1 and 2.3.

Lemma 3.1. *Let $w_1 \in C^1(\overline{\Omega})$ and for $k \geq 2$, define the sequence $\{w_i\}_i$ by*

$$w_2 = \left(\ln \frac{R}{|x|}\right)^{-\frac{1}{2}} w_1, \ldots, w_k = \left(\ln_{(k-1)} \frac{R}{|x|}\right)^{-\frac{1}{2}} w_{k-1}, \ldots$$

then

$$\int_\Omega \frac{|\nabla w_1|^2}{|x|^{n-2}} = \frac{1}{4} \int_\Omega \left(\sum_{j=1}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right) \frac{w_1^2}{|x|^n} - \frac{1}{2} \int_{\partial\Omega} \left(\sum_{j=1}^{k} \frac{\langle x, \nu\rangle}{\left(\ln^{(j)} \frac{R}{|x|}\right)} \right) \frac{w_1^2}{|x|^n}$$

$$+ \int_\Omega \frac{|\nabla w_{k+1}|^2}{|x|^{n-2}} \left(\ln^{(k)} \frac{R}{|x|}\right) \tag{3.3}$$

and

$$\int_\Omega \frac{|\nabla w_2|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) = \frac{1}{4}\int_\Omega \frac{w_2^2}{|x|^n}\left(\sum_{j=2}^{k}\frac{1}{\left(\ln^{(j)}\frac{R}{|x|}\right)^2}\right)\left(\ln\frac{R}{|x|}\right)$$
$$-\frac{1}{2}\int_{\partial\Omega}\frac{w_2^2}{|x|^n}\left(\sum_{j=2}^{k}\frac{\langle x,\nu\rangle}{\ln^{(j)}\frac{R}{|x|}}\right)\left(\ln\frac{R}{|x|}\right)$$
$$+\int_\Omega \frac{|\nabla w_{k+1}|^2}{|x|^{n-2}}\left(\ln^{(k)}\frac{R}{|x|}\right). \tag{3.4}$$

Proof. We prove (3.4). The proof of (3.3) follows similarly. From the identity $w_2 = (\ln_2 \frac{R}{|x|})^{\frac{1}{2}} w_3$ and taking the logarithmic derivative, we have,

$$\frac{\nabla w_2}{w_2} = -\frac{1}{2}\frac{x}{|x|^2\left(\ln\frac{R}{|x|}\right)\left(\ln\left(\ln\frac{R}{|x|}\right)\right)} + \frac{\nabla w_3}{w_3}.$$

Hence we have

$$\frac{|\nabla w_2|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) = \frac{1}{4}\frac{w_2^2}{|x|^n\left(\ln\frac{R}{|x|}\right)^2\left(\ln\left(\ln\frac{R}{|x|}\right)\right)^2}\left(\ln\frac{R}{|x|}\right)$$
$$+\frac{|\nabla w_3|^2}{|w_3|^2}\frac{1}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) - \frac{1}{2}\left\langle\frac{x}{|x|^n},\nabla w_3^2\right\rangle. \tag{3.5}$$

Let $|S^{n-1}|$ denotes the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$ and δ_0 is the Dirac distribution at the origin. Then, by integrating (3.5) and using the fact that $\mathrm{div}(\frac{x}{|x|^n}) = |S^{n-1}|\delta_0$, we have

$$\int_\Omega \frac{|\nabla w_2|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) = \frac{1}{4}\int_\Omega \frac{w_2^2}{|x|^n\left(\ln^{(2)}\frac{R}{|x|}\right)^2}\left(\ln\frac{R}{|x|}\right)$$
$$+\int_\Omega \frac{|\nabla w_3|^2}{|w_3|^2}\frac{w_2^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right)dx - \frac{1}{2}\int_{\partial\Omega}\frac{\langle x,\nu\rangle}{|x|^n}w_3^2$$
$$\int_\Omega \frac{|\nabla w_2|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) = \frac{1}{4}\int_\Omega \frac{w_2^2}{|x|^n\left(\ln^{(2)}\frac{R}{|x|}\right)^2}\left(\ln\frac{R}{|x|}\right)$$
$$+\int_\Omega \frac{|\nabla w_3|^2}{|x|^{n-2}}\left(\ln^2\frac{R}{|x|}\right)dx - \frac{1}{2}\int_{\partial\Omega}\frac{\langle x,\nu\rangle}{|x|^n}\frac{w_2^2}{\left(\ln^{(2)}\frac{R}{|x|}\right)}\left(\ln\frac{R}{|x|}\right).$$

Observing the fact that $\nabla(\ln_k \frac{R}{|x|}) = -\big(\frac{1}{\ln^{(k-1)} \frac{R}{|x|}}\big)\frac{x}{|x|^2}$ and $w_k = \big(\ln_{(k)} \frac{R}{|x|}\big)^{\frac{1}{2}} w_{k+1}$ we have,

$$\int_\Omega \frac{|\nabla w_k|^2}{|x|^{n-2}}\left(\ln^{(k-1)}\frac{R}{|x|}\right) = \int_\Omega \frac{w_k^2}{|x|^n}\frac{1}{\left(\ln^{(k)}\frac{R}{|x|}\right)^2}\left(\ln\frac{R}{|x|}\right)$$
$$+\int_\Omega \frac{|\nabla w_{k+1}|^2}{|x|^{n-2}}\left(\ln^{(k)}\frac{R}{|x|}\right) - \frac{1}{2}\int_\Omega \left\langle \nabla w_{k+1}^2, \frac{x}{|x|^n}\right\rangle. \tag{3.6}$$

Hence by induction the inequality (3.4) follows. □

Lemma 3.2. *Let Ω be a bounded domain with smooth boundary and $0 \in \Omega$. If $R > e^{(k-1)}\sup_{\partial\Omega}|x|$, then there exist a constant $\lambda = \lambda(\Omega, R) < 0$ such that $\forall\, u \in \mathcal{D}_\alpha^{1,2}(\Omega)$ and $n \geq 3$ we have*

$$\int_\Omega E^{1-2\alpha}|\nabla u|^2 dx - \alpha^2(n-2)^2\int_\Omega \frac{u^2}{|x|^2}E^{1-2\alpha}dx$$
$$\geq \frac{1}{4}\int_\Omega \left(\sum_{i=1}^{k}\frac{1}{\left(\ln^{(j)}\frac{R}{|x|}\right)^2}\right)\frac{u^2}{|x|^2}E^{1-2\alpha} + \lambda\int_{\partial\Omega} u^2 \tag{3.7}$$

and if $n = 2$

$$\int_\Omega E^{1-2\alpha}|\nabla u|^2 dx - \alpha^2\int_\Omega \frac{u^2}{|x|^2\left(\ln\frac{R}{|x|}\right)^2}E^{1-2\alpha}dx$$
$$\geq \frac{1}{4}\int_\Omega \left(\sum_{i=2}^{k}\frac{1}{\left(\ln^{(j)}\frac{R}{|x|}\right)^2}\right)\frac{u^2}{|x|^2}E^{1-2\alpha} + \lambda\int_{\partial\Omega} u^2. \tag{3.8}$$

The constant $-\alpha^2(n-2)^2$ (the coefficient of $\int_\Omega \frac{u^2}{|x|^2}E^{1-2\alpha}dx$) is the best constant and is never achieved by any nontrivial function $u \in \mathcal{D}_{\alpha,0}^{1,2}(\Omega)$ in the case $n \geq 3$. Moreover, if $n = 2$, then $-\alpha^2$ (the coefficient of $\int_\Omega \frac{u^2}{|x|^2(\ln\frac{R}{|x|})^2}E^{1-2\alpha}dx$) is the best constant and is never achieved by any nontrivial function $u \in \mathcal{D}_{\alpha,0}^{1,2}(\Omega)$.

Proof. Let $n \geq 3$. Let $u = E^\alpha v_1$. Then $v_1(0) = 0$ and

$$\frac{\nabla u}{u} = \alpha\frac{\nabla E}{E} + \frac{\nabla v_1}{v_1}.$$

This implies that

$$\frac{|\nabla u|^2}{u^2} = \alpha^2 \frac{|\nabla E|^2}{E^2} + \frac{|\nabla v_1|^2}{v_1^2} + 2\alpha \left\langle \frac{\nabla E}{E}, \frac{\nabla v_1}{v_1} \right\rangle.$$

Hence

$$|\nabla u|^2 = \alpha^2 (n-2)^2 \frac{u^2}{|x|^2} + |\nabla v_1|^2 E^{2\alpha} + \alpha \langle \nabla E, \nabla v_1^2 \rangle E^{2\alpha - 1}.$$

Thus

$$|\nabla u|^2 E^{2\beta} = \alpha^2 (n-2)^2 \frac{u^2}{|x|^2} E^{2\beta} + |\nabla v_1|^2 E^{2(\alpha+\beta)} + \alpha \langle \nabla E, \nabla v_1^2 \rangle E^{2(\alpha+\beta)-1}. \tag{3.9}$$

Let $\alpha + \beta = \frac{1}{2}$. Since $v_1(0) = 0$ and E is a fundamental solution, integrating by parts, (3.9) yields

$$\begin{aligned} \int_\Omega |\nabla u|^2 E^{2\beta} dx = \alpha^2 (n-2)^2 \int_\Omega \frac{u^2}{|x|^2} E^{2\beta} dx + \int_\Omega \frac{|\nabla v_1|^2}{|x|^{n-2}} dx \\ - \alpha (n-2) \int_{\partial\Omega} \frac{\langle x, \nu \rangle}{|x|^{n-2}} v_1^2. \end{aligned} \tag{3.10}$$

Substituting $v_1 = w_1$ in (3.3) and estimating the boundary integral to obtain the required inequality. For the optimality of the constant consider the family of functions $u_\delta(x) = E^{\alpha-\delta}\eta(x)$ where $\eta \in C_0^\infty(\Omega)$ and $\eta = 1$ in a neighborhood to zero.

For the second inequality, let $n = 2$ and $u = E^\alpha v_1$ we have similarly as above

$$|\nabla u|^2 = \alpha^2 \frac{u^2}{|x|^2 \left(\ln \frac{R}{|x|}\right)^2} + |\nabla v_1|^2 E^{2\alpha} + \alpha \langle \nabla E, \nabla v_1^2 \rangle E^{2\alpha-1}$$

and we get

$$\begin{aligned} \int_\Omega |\nabla u|^2 E^{2\beta} dx = \alpha^2 \int_\Omega \frac{u^2}{|x|^2 \left(\ln \frac{R}{|x|}\right)^2} E^{2\beta} dx + \int_\Omega |\nabla v_1|^2 \left(\ln \frac{R}{|x|}\right) dx \\ - \alpha \int_{\partial\Omega} \frac{\langle x, \nu \rangle}{|x|^2} v_1^2. \end{aligned} \tag{3.11}$$

Using an identity (3.4), we have the required inequality. From this, it follows that the best constants are never achieved in $\mathcal{D}^{1,2}_{\alpha,0}$. For the optimality of the constant, consider the family of functions $u_\delta(x) = E^{\alpha-\delta}\eta$ where $\eta \in C_0^\infty(\Omega)$ and $\eta = 1$ in a neighborhood to zero. □

Lemma 3.3. (a) *Let $n = 4m$ and $B \subset \mathbb{R}^n$ be the unit ball centered at zero then for all $u \in H^{2m}_{0,r}(B)(W^{2m,2}_{0,r}(B))$*

$$\int_B |\Delta^m u|^2 dx \geq \frac{n^2}{4} c^{m-1}_{2,4m-2} \int_B \frac{|\nabla u|^2}{|x|^{4m-2}} dx \tag{3.12}$$

where

$$c_{2,4m-2}^{m-1} = \left[\frac{1}{2^{2m-2}} \prod_{i=0}^{m-2} (4i+2)(8m-4i-6) \right]^2$$

and equality holds iff $u \equiv 0$.

(b) *Let* $n = 4m+2$ *and* $B \subset \mathbb{R}^n$ *be the unit ball centered at zero then for all* $u \in H_{0,r}^{2m+1}(B)(W_{0,r}^{2m+1,2}(B))$

$$\int_B |\nabla \Delta^m u|^2 dx \geq \frac{n^2}{4} e_{2,4m} \int_B \frac{|\nabla u|^2}{|x|^{4m}} dx \tag{3.13}$$

where

$$e_{2,4m} = \left[\frac{1}{2^{2m}} \prod_{i=0}^{m-1} (4m-4i-2)(4m+4i+2) \right]^2$$

and equality holds iff $u \equiv 0$.

(c) *Let* $1 < p < n$ *and* $B \subset \mathbb{R}^n$ *be the unit ball centered at zero then for all* $u \in W_{0,r}^{2,p}(B)$ *and* $u \in W_r^{2,p}(B) \cap W_{0,r}^{1,p}(B)$,

$$\int_B |\Delta u|^p dx \geq \frac{n^p (p-1)^p}{p^p} \int_B \frac{|\nabla u|^p}{|x|^p} dx \tag{3.14}$$

and equality holds iff $u \equiv 0$. *Hence in particular* (3.14) *holds for the case* $p = \frac{n}{2}$.

Proof. (a) From [3, Lemma 3.1], we have $\int_B |\Delta w|^2 \geq \frac{n^2}{4} \int_B \frac{|\nabla w|^2}{|x|^2}$. Hence we have

$$\int_B |\Delta^m u|^2 dx = \int_B |\Delta(\Delta^{m-1} u)|^2 dx \geq \frac{n^2}{4} \int_B \frac{|\nabla \Delta^{m-1} u|^2}{|x|^2} dx$$
$$= \frac{n^2}{4} \int_B \frac{|\Delta^{m-1} \nabla u|^2}{|x|^2} dx.$$

Thus we have from (1.9)

$$\int_B |\Delta^m u|^2 dx \geq \frac{n^2}{4} c_{2,4m-2}^{m-1} \int_B \frac{|\nabla u|^2}{|x|^{4m-2}} dx$$

where

$$c_{2,4m-2}^{m-1} = \left[\frac{1}{2^{2m-2}} \prod_{i=0}^{m-2} (4i+2)(8m-4i-6) \right]^2.$$

(b) When $n = 4m+2$, applying the above inequality for each component of ∇u, using (1.10) and summing to obtain

$$\int_B |\nabla \Delta^m u|^2 = \int_B |\Delta^m (\nabla u)|^2 \geq e_{2,4m} \int_B \frac{|\nabla^2 u|^2}{|x|^{4m}} dx$$

where

$$e_{2,4m} = \left[\frac{1}{2^{2m}} \prod_{i=0}^{m-1} (4m - 4i - 2)(4m + 4i + 2) \right]^2 .$$

Let $2\alpha(n-2) = (n-4m)$ and applying the weighted Hardy–Sobolev inequality (3.7) to $v = \nabla u \in H^{2m}_{0,r}(B)$, then the above inequality yields

$$\int_B |\nabla(\Delta^m u)|^2 dx \geq \frac{n^2(n-4m)^2}{16} e_{2,4m} \int_B \frac{|\nabla u|^2}{|x|^{4m}} dx = \frac{n^2}{4} e_{2,4m} \int_B \frac{|\nabla u|^2}{|x|^{4m}} dx.$$

(c) For the case $p \in (1, n)$. Let $u \in W^{2,p}_r(B) \cap W^{1,p}_{0,r}(B)$. Since $C^2_r(\overline{B}) \cap C^1_{0,r}(B)$ is dense in $u \in W^{2,p}_r(B) \cap W^{1,p}_{0,r}(B)$ it is enough to prove the inequality for the case $u \in C^2_r(\overline{B}) \cap C^1_{0,r}(B)$. Let $\Delta u \leq 0$. Then by Hopf's lemma $u_r < 0$. Let $v = r^{\frac{n-p}{p}} u_r$. Then $v(0) = 0$. Then

$$\begin{aligned} u_{rr} &= -\frac{(n-p)}{p} \frac{u_r}{r} + \frac{u_r v_r}{v} \\ \Delta u &= \frac{n(p-1)}{p} \frac{u_r}{r} + r^{-\frac{n-p}{p}} v_r \\ \Delta u &= \frac{n(p-1)}{p} \frac{u_r}{r} \left\{ 1 + \frac{p}{n(p-1)} \frac{r^{-\frac{n-p}{p}} r v_r}{u_r} \right\}. \end{aligned}$$

Using the fact that $(1+x)^p \geq 1 + px$ for $x \geq -1$, we have,

$$|\Delta u|^p \geq \frac{n^p (p-1)^p}{p^p} \frac{|u_r|^p}{r^p} \left\{ 1 + p \frac{p}{n(p-1)} \frac{v_r}{u_r} r r^{-\frac{n-p}{p}} \right\}.$$

Hence we have,

$$|\Delta u|^p \geq \frac{n^p (p-1)^p}{p^p} \frac{u_r^p}{r^p} + p \left(\frac{p}{n(p-1)} \right)^{p-1} v|v|^{p-2} v_r r^{1-n}.$$

Since $p \int_0^1 v|v|^{p-2} v_r dr = \int_0^1 (|v|^p)_r dr = |v(1)|^p - |v(0)|^p = |v(1)|^p$, hence

$$\int_B |\Delta u|^p dx \geq \frac{n^p (p-1)^p}{p^p} \int_B \frac{|\nabla u|^p}{|x|^p} dx.$$

This proves the lemma. □

We also prove an weighted Hardy–Rellich inequality, in order to stress the fact how the fundamental solution plays a key role in deriving this inequality. It should be noted that Lemma 3.4 is not required in the course of proof of the main theorems.

Lemma 3.4. *Let $n \geq 5$ and Ω be a bounded domain with smooth boundary and $0 \in \Omega$. If $R > e^{(k-1)}\sup_{\partial\Omega}|x|$, then there exist constants $\lambda_1 = \lambda_1(\Omega, R) < 0$ and $\lambda_2 = \lambda_2(\Omega, R) < 0$ such that $\forall\, u \in \mathcal{D}^{2;2}_{\alpha}(\Omega)$, we have*

$$\int_\Omega |\Delta u|^2 E_1^{1-2\alpha} dx \geq \left(\alpha^2(n-4)^2 + \frac{1}{2}\theta(n-\theta-2)\right)^2 \int_\Omega \frac{u^2}{|x|^4} E_1^{1-2\alpha} dx$$
$$+ \frac{1}{4}\big(2\alpha^2(n-4)^2 + \theta(n-\theta-2)\big) \int_\Omega \left(\sum_{i=1}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right)$$
$$\times \frac{u^2}{|x|^4} E_1^{1-2\alpha} + \lambda_1 \int_{\partial\Omega} u^2 + \lambda_2 \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \tag{3.15}$$

where $E_1 = \frac{1}{|x|^{n-4}}$ and $\theta = (n-4)(1-2\alpha) + 2$. The constant $(\alpha^2(n-4)^2 + \frac{1}{2}\theta(n-\theta-2))^2$ (the coefficient of $\int_\Omega \frac{u^2}{|x|^4} E_1^{1-2\alpha} dx$) is the best constant and is never achieved by any nontrivial function $u \in \mathcal{D}^{2,2}_{\alpha,0}(\Omega)$.

Proof. Let $u = E_1^{\alpha} v$. Then $v(0) = 0$ and we have for $n \geq 5$

$$\frac{|\nabla u|^2}{|x|^2} E_1^{2\beta} = \alpha^2(n-4)^2 \frac{|u|^2}{|x|^4} E_1^{2\beta} + \frac{|\nabla v|^2}{|x|^2} E_1^{2(\alpha+\beta)}$$
$$- \alpha(n-4) \left\langle \frac{x}{|x|^n}, \nabla v^2 \right\rangle E_1^{2(\alpha+\beta)-1}.$$

Choosing $\alpha + \beta = \frac{1}{2}$ and integrating we have

$$\int_\Omega \frac{|\nabla u|^2}{|x|^2} E_1^{2\beta} = \alpha^2(n-4)^2 \int_\Omega \frac{|u|^2}{|x|^4} E_1^{2\beta} + \int_\Omega \frac{|\nabla v|^2}{|x|^{n-2}} - \alpha(n-4) \int_{\partial\Omega} \frac{\langle x, \nu\rangle}{|x|^n} v^2. \tag{3.16}$$

Choosing $\theta = 2(n-4)\beta + 2$, (3.16) reduces to

$$\int_\Omega \frac{|\nabla u|^2}{|x|^\theta} = \alpha^2(n-4)^2 \int_\Omega \frac{|u|^2}{|x|^{\theta+2}} + \int_\Omega \frac{|\nabla v|^2}{|x|^{n-2}} - \alpha(n-4) \int_{\partial\Omega} \frac{\langle x, \nu\rangle}{|x|^n} v^2. \tag{3.17}$$

Now integrating by parts

$$\int_\Omega \frac{|\nabla u|^2}{|x|^\theta} = -\int_\Omega \frac{u\Delta u}{|x|^\theta} - \frac{\theta}{2}(n-\theta-2) \int_\Omega \frac{|u|^2}{|x|^{\theta+2}} + \int_{\partial\Omega} \frac{u}{|x|^\theta} \frac{\partial u}{\partial \nu} + \frac{\theta}{2} \int_{\partial\Omega} \frac{u^2 \langle x \cdot \nu\rangle}{|x|^{\theta+1}}. \tag{3.18}$$

Substituting the value of (3.17) in (3.18) we have

$$-\int_\Omega \frac{u\Delta u}{|x|^\theta} = \left\{\alpha^2(n-4)^2 + \frac{1}{2}\theta(n-\theta-2)\right\}\int_\Omega \frac{u^2}{|x|^{\theta+2}} + \int_\Omega \frac{|\nabla v|^2}{|x|^{n-2}}$$
$$-\int_{\partial\Omega} \frac{u}{|x|^\theta}\frac{\partial u}{\partial\nu} - \alpha(n-4)\int_{\partial\Omega}\frac{\langle x,\nu\rangle}{|x|^n}v^2 - \frac{\theta}{2}\int_{\partial\Omega}\frac{u^2\langle x,\nu\rangle}{|x|^{\theta+1}}.$$

Applying Cauchy–Schwarz's inequality we have

$$\frac{1}{2\varepsilon}\int_\Omega \frac{|\Delta u|^2}{|x|^{\theta-2}} + \frac{\varepsilon}{2}\int_\Omega \frac{|u|^2}{|x|^{\theta+2}} \geq \left\{\alpha^2(n-4)^2 + \frac{1}{2}\theta(n-\theta-2)\right\}\int_\Omega \frac{u^2}{|x|^{\theta+2}}$$
$$+ \int_\Omega \frac{|\nabla v|^2}{|x|^{n-2}} - \int_{\partial\Omega}\frac{u}{|x|^\theta}\frac{\partial u}{\partial\nu}$$
$$- \alpha(n-4)\int_{\partial\Omega}\frac{\langle x,\nu\rangle}{|x|^n}v^2 - \frac{\theta}{2}\int_{\partial\Omega}\frac{u^2\langle x,\nu\rangle}{|x|^{\theta+1}}. \quad (3.19)$$

Choosing $\varepsilon = \alpha^2(n-4)^2 + \frac{\theta}{2}(n-\theta-2)$ we have

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{\theta-2}} \geq \left(\alpha^2(n-4)^2 + \frac{\theta}{2}(n-\theta-2)\right)^2 \int_\Omega \frac{u^2}{|x|^{\theta+2}}$$
$$+ \left(2\alpha^2(n-4)^2 + \theta(n-\theta-2)\right)\int_\Omega \frac{|\nabla v|^2}{|x|^{n-2}}$$
$$+ \lambda_1\int_{\partial\Omega} u^2 + \lambda_2\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2. \quad (3.20)$$

Using (3.3), we have the required inequality. It follows clearly from this inequality the best constant is never achieved for $u \in \mathcal{D}^{2,2}_{\alpha,0}(\Omega)$. For the optimality of the constant consider the family of functions $u_\delta(x) = E_1^{\alpha-\delta}\eta$ where $\eta \in C_0^\infty(\Omega)$ and $\eta = 1$ in a neighborhood to zero. □

4. Proof of the Main Theorems

Proof of Theorem 2.1. First we prove for $u \in W^{2,p}_{0,r}(B)$. Let $u = \left(\ln\frac{R}{|x|}\right)^{\frac{n-2}{n}} v$. Then $v(0) = 0$. Then

$$|\nabla u|^p = \left(\frac{n-2}{n}\right)^p \frac{|u|^p}{|x|^p\left(\ln\frac{R}{|x|}\right)^p}\left|\frac{x}{|x|} - \frac{n}{n-2}\frac{\nabla v}{v}|x|\left(\ln\frac{R}{|x|}\right)\right|^p.$$

For the case $p \geq 2$, we have from (3.1),

$$|\nabla u|^p \geq \left(\frac{n-2}{n}\right)^p \frac{|u|^p}{|x|^p\left(\ln\frac{R}{|x|}\right)^p}\left\{1 - p\left(\frac{n}{n-2}\right)\left\langle x, \frac{\nabla v}{v}\right\rangle\left(\ln\frac{R}{|x|}\right)\right.$$
$$\left. + \alpha_1\left(\frac{n}{n-2}\right)^2\frac{|x|^2|\nabla v|^2}{v^2}\left(\ln\frac{R}{|x|}\right)^2 + \alpha_2\left(\frac{n}{n-2}\right)^p\frac{|x|^p|\nabla v|^p}{v^p}\left(\ln\frac{R}{|x|}\right)^p\right\}.$$

Thus we have

$$\frac{|\nabla u|^p}{|x|^p} \geq \left(\frac{n-2}{n}\right)^p \frac{|u|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p} - p\left(\frac{n-2}{n}\right)^{p-1}\left\langle \frac{x}{|x|^{2p}}, \nabla v\right\rangle |v|^{p-2}v$$
$$+ \frac{4\alpha_1}{p^2}\left(\frac{n-2}{n}\right)^{p-2}\frac{|\nabla v^{\frac{p}{2}}|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) + \alpha_2\frac{|\nabla v|^p}{|x|^p}\left(\ln\frac{R}{|x|}\right)^{p-1}.$$

Hence we have

$$\frac{|\nabla u|^p}{|x|^p} \geq \left(\frac{n-2}{n}\right)^p \frac{|u|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p} - \left(\frac{n-2}{n}\right)^{p-1}\left\langle \frac{x}{|x|^{n}}, \nabla |v|^p\right\rangle$$
$$+ \frac{4\alpha_1}{p^2}\left(\frac{n-2}{n}\right)^{p-2}\frac{|\nabla v^{\frac{p}{2}}|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right) + \alpha_2\frac{|\nabla v|^p}{|x|^p}\left(\ln\frac{R}{|x|}\right)^{p-1}. \tag{4.1}$$

Since $v(0) = v|_{\partial\Omega} = 0$ and hence integral of the second term vanishes. Therefore integrating (4.1) and choosing $v^{\frac{p}{2}} = w_1$, we have

$$\int_B \frac{|\nabla u|^p}{|x|^p} \geq \left(\frac{n-2}{n}\right)^p \int_B \frac{|u|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p} + \frac{4\alpha_1}{p^2}\left(\frac{n-2}{n}\right)^{p-2}\int_B \frac{|\nabla w_1|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right)$$
$$+ \alpha_2\int_B \frac{|\nabla v|^p}{|x|^p}\left(\ln\frac{R}{|x|}\right)^{p-1} \tag{4.2}$$

which implies that

$$\int_B \frac{|\nabla u|^p}{|x|^p} \geq \left(\frac{n-2}{n}\right)^p \int_B \frac{|u|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p} + C_1\int_B \frac{|\nabla w_1|^2}{|x|^{n-2}}\left(\ln\frac{R}{|x|}\right).$$

Using (3.4) on the second term in the above inequality, we obtain

$$\int_B \frac{|\nabla u|^p}{|x|^p}dx \geq \left(\frac{n-2}{n}\right)^p \int_B \frac{|u|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p}dx$$
$$+ C\int_B \frac{|u|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^{p-2}}\left(\sum_{j=2}^{k}\frac{1}{\left(\ln^{(j)}\frac{R}{|x|}\right)^2}\right)dx. \tag{4.3}$$

Hence combining (4.3) and (3.14) and noting the fact that $(\ln \frac{R}{|x|}) \geq 1$, we have

$$\int_B |\Delta u|^p \geq \frac{2^p(p-1)^{2p}}{p^p} \int_B \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} dx$$

$$+ C \int_B \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} \left(\sum_{j=2}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right) dx.$$

Hence from the inequality it is clear that the $\frac{2^p(p-1)^{2p}}{p^p}$ is not achieved, otherwise the remainder term is zero, which will imply that $u \equiv 0$, a contradiction. Later on, we prove that in fact $\frac{2^p(p-1)^{2p}}{p^p}$ is the best constant. This proves the inequalities (2.1) and (2.2) hold for all $u \in W^{2,p}_{0,r}(B)$. Note that we are only using the fact that $u = 0$ on ∂B and hence the above inequalities are true for the case $u \in W^{2,p}_r(B) \cap W^{1,p}_{0,r}(B)$.

Also note that for the case $1 < p < 2$, we cannot obtain the remainder term as in (2.1) but by using (3.2), we can show that the constant $\frac{2^p(p-1)^{2p}}{p^p}$ is not achieved. Next we prove this for the non-radial case by using the ideas in [16] (see [11]). Let $|\Omega| = |B|$. First we may restrict ourselves to $\Omega = B$ and a radial function u. Define $f = -\Delta u$.

$$\begin{cases} -\Delta w = |f|^* & \text{in } B \\ w = 0 & \text{on } \partial B \end{cases} \tag{4.4}$$

where f^* denotes the Schwarz symmetrization of f. Then $w \in W^{2,p}_r(B) \cap W^{1,p}_{0,r}(B)$. By [17], we have $w \geq |u|^* \geq 0$. Hence

$$\int_B |\Delta w|^p dx = \int_B (|f|^*)^p dx = \int_\Omega |f|^p dx = \int_\Omega |\Delta u|^p dx,$$

$$\int_B \frac{w^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} dx \geq \int_B \frac{|u|^{*p}}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} dx \geq \int_\Omega \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} dx.$$

Similarly we get

$$\int_B \frac{|w|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} \left(\sum_{j=2}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right) dx$$

$$\geq \int_\Omega \frac{|u|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p} \left(\sum_{j=2}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right) dx.$$

Hence the inequalities (2.1), (2.2) holds for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $u \in W_0^{2,p}(\Omega)$.

Now we prove the sharpness of the previous inequality, i.e. we show the existence of a family of radial functions ψ_δ such that

$$\lim_{\delta \to 0} \frac{\displaystyle\int_\Omega |\Delta \psi_\delta|^p}{\displaystyle\int_\Omega \frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p}} = \frac{2^p(p-1)^{2p}}{p^p}.$$

Let $B(1) \subset \Omega$ and $\varphi \in C_0^\infty(\Omega)$ be radial such that

$$\varphi(x) = \begin{cases} 1 & \text{in } B\left(\frac{1}{2}\right) \\ 0 & \text{on } \Omega \backslash B(1). \end{cases}$$

Define $\psi_\delta(x) = \left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta} \varphi(x)$

$$\Delta \psi_\delta(x) = \left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta} \Delta\varphi + \Delta\left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta} \varphi + 2\left\langle \nabla\left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta}, \nabla\varphi \right\rangle.$$

Then we have

$$\frac{\displaystyle\int_\Omega |\Delta \psi_\delta|^p}{\displaystyle\int_\Omega \frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p}} = \frac{\displaystyle\int_\Omega \left|\Delta\left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta}\right|^p \varphi^p}{\displaystyle\int_\Omega \frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p}} + \frac{O(1)}{\displaystyle\int_\Omega \frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln \frac{R}{|x|}\right)^p}}.$$

Now we have

$$\begin{aligned} \Delta\left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta} &= -\frac{1}{r^2}\left(\frac{p-1}{p} - \delta\right)\left(\frac{1}{p} + \delta\right)\left(\ln \frac{R}{r}\right)^{-\frac{p+1}{p}-\delta} \\ &\quad - \frac{n-2}{r^2}\left(\frac{p-1}{p} - \delta\right)\left(\ln \frac{R}{r}\right)^{-\frac{1}{p}-\delta}. \end{aligned}$$

Putting $n = 2p$, we have

$$\begin{aligned} \Delta\left(\ln \frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta} &= -\frac{1}{r^2}\left(\frac{p-1}{p} - \delta\right)\left(\frac{1}{p} + \delta\right)\left(\ln \frac{R}{r}\right)^{-\frac{p+1}{p}-\delta} \\ &\quad - \frac{2p-2}{r^2}\left(\frac{p-1}{p} - \delta\right)\left(\ln \frac{R}{r}\right)^{-\frac{1}{p}-\delta}. \end{aligned}$$

Hence we have

$$\left|\Delta\left(\ln\frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta}\right|^p = 2^p(p-1)^p\left(\frac{p-1}{p}-\delta\right)^p\left(\ln\frac{R}{r}\right)^{-1-p\delta}$$

$$\times\frac{1}{r^{2p}}\left|1+\frac{1+p\delta}{2p(p-1)}\frac{1}{\left(\ln\frac{R}{r}\right)}\right|^p.$$

Also note that

$$|\psi_\delta|^p = \left(\ln\frac{R}{|x|}\right)^{p-1-p\delta}\varphi^p.$$

Hence

$$\frac{\displaystyle\int_\Omega\left|\Delta\left(\ln\frac{R}{|x|}\right)^{\frac{p-1}{p}-\delta}\right|^p\varphi^p}{\displaystyle\int_\Omega\frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p}} = 2^p(p-1)^p\left(\frac{p-1}{p}-\delta\right)^p + \frac{O(1)}{\displaystyle\int_\Omega\frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p}} \tag{4.5}$$

Taking limit as $\delta\to 0$ in (4.5) and noting that

$$\lim_{\delta\to 0}\int_\Omega\frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p} = \infty,$$

we have

$$\lim_{\delta\to 0}\frac{\displaystyle\int_\Omega|\Delta\psi_\delta|^p}{\displaystyle\int_\Omega\frac{|\psi_\delta|^p}{|x|^{2p}\left(\ln\frac{R}{|x|}\right)^p}} = \frac{2^p(p-1)^{2p}}{p^p}.$$

Hence $\frac{2^p(p-1)^{2p}}{p^p}$ is the best constant in (2.1) and it is never achieved in any bounded domain. □

Proof of Theorem 2.2. As in Theorem 2.1, it is enough to prove it for the radial superharmonic functions when $\Omega = B$ as in (4.4) we have $\|u\|_{W_0^{2,q}(\Omega)} = \|w\|_{W_0^{2,q}(B)}$. Letting $2\alpha(n-2) = 4-n$ in (3.7) for $n\geq 5$, we obtain

$$\frac{n^2}{4}\int_B\frac{|\nabla u|^2}{|x|^2}dx \geq \frac{n^2(n-4)^2}{16}\int_B\frac{u^2}{|x|^4}dx$$

and hence we have

$$\int_B|\Delta u|^2 - \frac{n^2(n-4)^2}{16}\int_B\frac{u^2}{|x|^4} \geq \int_B|\Delta u|^2 - \frac{n^2}{4}\int_B\frac{|\nabla u|^2}{|x|^2}\quad\text{for } n\geq 5.$$

Let $n = 4$. Define $v = \left(\ln \frac{R}{|x|}\right)^{-\frac{1}{2}} u$, then $v(0) = 0$

$$\int_B \frac{|\nabla u|^2}{|x|^2} = \frac{1}{4}\int_B \frac{u^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} - \frac{1}{2}\int_B \left\langle \frac{x}{|x|^4}, \nabla v^2 \right\rangle + \int_B \frac{|\nabla v|^2}{|x|^2}\left(\ln \frac{R}{|x|}\right)$$

$$\geq \frac{1}{4}\int_B \frac{u^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2}$$

since $\operatorname{div}\left(\frac{x}{|x|^4}\right) = C\delta_0$ and $v(0) = 0$. Hence we have

$$\int_B |\Delta u|^2 - \int_B \frac{u^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} \geq \int_B |\Delta u|^2 - 4\int_B \frac{|\nabla u|^2}{|x|^2}.$$

We have from [3, Lemma 3.1], and for $n \geq 4$, for u radial

$$\int_B |\Delta u|^2 - \frac{n^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2} \geq \int_B \frac{|\nabla v|^2}{|x|^{n-2}} dx \tag{4.6}$$

where $u_r = r^{-\frac{n-2}{2}} v$. Let $v = (\ln \frac{R}{|x|})^{\frac{1}{2}} w$. Then from (3.3) we have

$$\int_B |\Delta u|^2 - \frac{n^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2} - \frac{1}{4}\int_B \frac{|\nabla u|^2}{|x|^2\left(\ln \frac{R}{|x|}\right)^2} dx \geq \int_B \frac{|\nabla w|^2}{|x|^{n-2}}\left(\ln \frac{R}{|x|}\right) dx. \tag{4.7}$$

Let $u_r = \left(\ln \frac{R}{|x|}\right)^{\frac{1}{2}} r^{-\frac{n-2}{2}} w$, then

$$|u_{rr}| = O\left(\frac{\left(\ln \frac{R}{|x|}\right)^{\frac{1}{2}}}{|x|^{\frac{n-2}{2}}} w_r + \frac{\left(\ln \frac{R}{|x|}\right)^{\frac{1}{2}}}{|x|^{\frac{n}{2}}} w\right)$$

$$|u_{rr}|^q = O\left(\frac{\left(\ln \frac{R}{|x|}\right)^{\frac{q}{2}}}{|x|^{\frac{(n-2)q}{2}}} |w_r|^q + \frac{\left(\ln \frac{R}{|x|}\right)^{\frac{q}{2}}}{|x|^{\frac{nq}{2}}} |w|^q\right).$$

Therefore,

$$\omega_n \int_0^1 |u_{rr}|^q r^{n-1} dr = O\left(\int_B \frac{\left(\ln \frac{R}{|x|}\right)^{\frac{q}{2}}}{|x|^{\frac{(n-2)q}{2}}} |\nabla w|^q + \int_B \frac{\left(\ln \frac{R}{|x|}\right)^{\frac{q}{2}}}{|x|^{\frac{nq}{2}}} |w|^q\right). \tag{4.8}$$

In order to estimate the right-hand side of (4.8), we need some estimates. Let $w \in C_0^\infty(B)$ and $k \geq 0$, $\alpha \geq 0$. Then

$$\int_B \frac{\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^q = \frac{1}{n}\int_B \frac{(\operatorname{div} x)\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^q$$
$$= -\frac{q}{n}\int_B \frac{\langle x, \nabla w\rangle \left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^{q-2}w + \frac{k}{n}\int_B \frac{\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^q$$
$$+ \frac{\alpha}{n}\int_B \frac{\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k\left(\ln \frac{R}{|x|}\right)}|w|^q.$$

Let $k < n$ and $R_0 > 0$ such that

$$\frac{\alpha}{n}\sup_{x\in B}\frac{1}{\left(\ln \frac{R_0}{|x|}\right)} < \frac{1}{2}\left(1 - \frac{k}{n}\right).$$

Then for $R \geq R_0$, the above identity gives

$$\frac{1}{2}\left(1-\frac{k}{n}\right)\int_B \frac{\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^q \leq \frac{q}{n}\int_B \frac{|\nabla w|\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^{k-1}}|w|^{q-1}$$
$$\frac{1}{2}\left(1-\frac{k}{n}\right)\int_B \frac{\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^q \leq \frac{q}{n}\left(\int_B \frac{|w|^q\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}\right)^{\frac{q-1}{q}}$$
$$\times\left(\int_B \frac{|\nabla w|^q\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^{k-q}}\right)^{\frac{1}{q}}.$$

This implies that there exist a $C = C(k, n, \alpha) > 0$ such that

$$\int_B \frac{\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^k}|w|^q \leq C\left(\int_B \frac{|\nabla w|^q\left(\ln \frac{R}{|x|}\right)^\alpha}{|x|^{k-q}}\right). \tag{4.9}$$

Choose $k = \frac{nq}{2}$ and so $1 \le q < 2$ as $k < n$. Thus from (4.8) and (4.9) we have for $\alpha = \frac{q}{2}$, $R \ge R_0$

$$\omega_n \int_0^1 |u_{rr}|^q r^{n-1} dr \le C \left(\int_B \frac{|\nabla w|^q \left(\ln \frac{R}{|x|}\right)^{\frac{q}{2}}}{|x|^{\frac{(n-2)q}{2}}} \right).$$

Hence applying Hölder's inequality we have

$$\omega_n \int_0^1 |u_{rr}|^q r^{n-1} dr \le C \left(\int_B \frac{|\nabla w|^2 \left(\ln \frac{R}{|x|}\right)}{|x|^{n-2}} dx \right)^{\frac{q}{2}}.$$

This implies

$$\int_B |\Delta u|^q dx \le C \left(\int_B \frac{|\nabla w|^2 \left(\ln \frac{R}{|x|}\right)}{|x|^{n-2}} dx \right)^{\frac{q}{2}}.$$

Combining this with (4.7) completes the proof. □

Remark 4.1. It seems that the inequalities (2.3) and (2.4) can be improved by adding a series of terms on the left-hand side, that is partially visible in (4.7). We will discuss this in a forthcoming paper.

Proof of Theorem 2.3. (a) Let $u = \left(\ln \frac{R}{|x|}\right)^{\frac{1}{2}} v$. Then we have $v(0) = 0$ and for the case $n = 4m$,

$$\int_B \frac{|\nabla u|^2}{|x|^{4m-2}} dx = \frac{1}{4} \int_B \frac{|u|^2}{|x|^{4m} \left(\ln \frac{R}{|x|}\right)^2} dx + \int_B \frac{|\nabla v|^2}{|x|^{4m-2}} \left(\ln \frac{R}{|x|}\right) dx \tag{4.10}$$

and using (3.4) on the second term of the above equality we obtain,

$$\int_B \frac{|\nabla u|^2}{|x|^{4m-2}} dx \ge \frac{1}{4} \int_B \frac{|u|^2}{|x|^{4m} \left(\ln \frac{R}{|x|}\right)^2} dx + C \int_B \left(\sum_{j=2}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2} \right) \frac{|u|^2}{|x|^{4m}} dx \tag{4.11}$$

and applying (3.12), we have the required inequality.

(b) For the case $n = 4m + 2$, we have

$$\int_B \frac{|\nabla u|^2}{|x|^{4m}} dx = \frac{1}{4}\int_B \frac{|u|^2}{|x|^{4m+2}\left(\ln \frac{R}{|x|}\right)^2} dx + \int_B \frac{|\nabla v|^2}{|x|^{4m}}\left(\ln \frac{R}{|x|}\right) dx \tag{4.12}$$

and using (3.4) on the second term of the above equality we obtain

$$\int_B \frac{|\nabla u|^2}{|x|^{4m-2}} dx \geq \frac{1}{4}\int_B \frac{|u|^2}{|x|^{4m}\left(\ln \frac{R}{|x|}\right)^2} dx + C\int_B \left(\sum_{j=2}^{k} \frac{1}{\left(\ln^{(j)} \frac{R}{|x|}\right)^2}\right) \frac{|u|^2}{|x|^{4m+2}} dx \tag{4.13}$$

using (3.13), we have the required inequality.

For the sharpness of the inequalities, consider a family of radial functions

$$\psi_\delta(r) = \begin{cases} \left(\ln \frac{R}{r}\right)^{\frac{c_{4m}}{\prod_{i=1}^{m}(n-2i)} - \delta} \varphi & \text{if } n = 4m, \\ \left(\ln \frac{R}{r}\right)^{\frac{c_{4m+2}}{2m\prod_{i=1}^{m}(n-2i)} - \delta} \varphi & \text{if } n = 4m+2 \end{cases}$$

where $\varphi \in C_0^\infty(B)$ be radial such that

$$\varphi(x) = \begin{cases} 1 & \text{in } B\left(\frac{1}{2}\right) \\ 0 & \text{on } B\backslash B(3/4) \end{cases}$$

$\delta > 0$ and c_{4m}^2, c_{4m+2}^2 denotes the coefficients of

$$\int_B \frac{|u|^2}{|x|^{4m}\left(\ln \frac{R}{|x|}\right)^2} dx$$

and

$$\int_B \frac{|u|^2}{|x|^{4m+2}\left(\ln \frac{R}{|x|}\right)^2} dx$$

in (2.5) and (2.6) respectively. We skip the slightly tedious details. □

Before proving Theorem 2.4, we look into the various difficulties associated with the biharmonic operator.

- Here we deal with the second order Sobolev space $H^2(\Omega)$. Unlike in $H^1(\Omega)$, $H^2(\Omega)$ does not satisfy the property that "$u \in H^2(\Omega)$ implies $|u| \in H^2(\Omega)$". This is a serious block to get *a priori* estimates.
- There is no maximum principle.

Let us recall some known results for biharmonic operator:

Boggio's Principle. Consider the biharmonic equation

$$(F)\begin{cases}\Delta^2 u = f & \text{in } B\\ u = 0 & \text{on } \partial B\\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B\end{cases}$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$ and ν is the outer normal at the boundary of B. Then Boggio's principle [6] states that the Green function associated to the biharmonic problem with zero Dirichlet data in a ball is strictly positive. Hence if $f > 0$ a.e. then $u > 0$ in B. For the weak Boggio's principle see [4].

Note that when we are in the case Ω, a smooth bounded domain

$$(G)\begin{cases}\Delta^2 u = f & \text{in } \Omega\\ u = \Delta u = 0 & \text{on } \partial\Omega\end{cases}$$

there is a natural weak maximum principle.

Lemma 4.2. *If $f \in F$ then there exist $\lambda(f) > 0$ such that*

$$\int_\Omega |\Delta u|^2 dx \geq \int_\Omega \frac{u^2}{|x|^4\left(\ln\dfrac{R}{|x|}\right)^2} dx + \lambda(f)\int_\Omega u^2 f(x) dx \tag{4.14}$$

for all $u \in H_0^2(\Omega), u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. Let $f \in F$, then we have

$$\lim_{\varepsilon\to 0}\ \sup_{x\in B_\varepsilon(0)} |x|^4 f(x)\left(\ln\frac{R}{|x|}\right)^2\left(\ln\ln\frac{R}{|x|}\right)^2 < \infty$$

and hence for sufficiently small $\varepsilon > 0$, there exist a $C > 0$ such that $f \in B_\varepsilon(0)$

$$f(x) < \frac{C}{|x|^4\left(\ln\dfrac{R}{|x|}\right)^2\left(\ln\ln\dfrac{R}{|x|}\right)^2}$$

and otherwise f is bounded. Hence the inequality (4.14) holds. □

Lemma 4.3. *Consider the problem*

$$\begin{cases}\Delta^2 u - \mu\dfrac{u}{|x|^4\left(\ln\dfrac{R}{|x|}\right)^2} = \lambda f(x) u & \text{in } B\\ u \neq 0 & \text{in } B\\ u \in H_0^2(B)\end{cases} \tag{4.15}$$

where B is the unit ball centered at origin. If (4.15) *admits a solution u for some $\lambda = \lambda_\mu^1(f)$, then u does not change sign in B.*

Proof. We prove this lemma for the sake of completeness. A similar version of this lemma can be seen in [3, 10]. Note that proving existence of positive solutions is quite hard in the sense that $u^+, u^- \notin H_0^2(B)$, which played a crucial role in second order equations. Suppose $u \in H_0^2(B)$ solves the above problem with $\lambda = \lambda_\mu^1(f)$ and u changes sign. Define

$$K := \{v \in H_0^2(B) : v \geq 0 \text{ a.e.}\}.$$

Let $a(u,v) = \langle u, v\rangle_{H_0^2(B)} = \int_B \Delta u \Delta v$. $\forall\, u, v \in H_0^2(B)$. Note that K is a closed convex cone. Hence, by [13], there exists a projection $P : H_0^2(B) \to K$ such that for all $u \in H_0^2(B)$, $\forall\, w \in K$

$$a(u - P(u), w - P(u)) \leq 0. \tag{4.16}$$

Since K is a cone we can replace w by tw for $t > 0$ and letting $t \to \infty$ to obtain

$$a(u - P(u), w) \leq \lim_{t\to\infty} \frac{1}{t} a(u - P(u), P(u))$$

which implies that $\Delta^2(u - P(u)) \leq 0$ and by weak Boggio's Principle [4] $u - P(u) \leq 0$.

Now replacing w by $tP(u)$ for $t > 0$ in (4.17) we have

$$(t-1)a(u - P(u), P(u)) \leq 0$$

and hence $a(u - P(u), P(u)) = 0$.

Hence we can write $u = u_1 + u_2$, $u_1 = P(u) \in K$, $u_2 = u - P(u)$, $u_1 \perp u_2$ and $u_2 \leq 0$. Since u changes sign we have that $u_1 \not\equiv 0$ and $u_2 \not\equiv 0$. Therefore we have,

$$\frac{\displaystyle\int_B |\Delta(u_1 - u_2)|^2 - \mu \int_B \frac{(u_1 - u_2)^2}{|x|^4 \left(\ln \frac{R}{|x|}\right)^2}}{\displaystyle\int_B f(x)(u_1 - u_2)^2} < \frac{\displaystyle\int_B |\Delta(u_1 + u_2)|^2 - \mu \int_B \frac{(u_1 + u_2)^2}{|x|^4 \left(\ln \frac{R}{|x|}\right)^2}}{\displaystyle\int_B f(x)(u_1 + u_2)^2}$$

which contradicts the definition of the first eigenvalue. Then u does not change sign and noting that the Green function is strictly positive we have either $u > 0$ or $u < 0$ in B. □

Similarly as above, we have:

Lemma 4.4. *Consider the problem*

$$\begin{cases} \Delta^2 u - \mu \dfrac{u}{|x|^4 \left(\ln \frac{R}{|x|}\right)^2} = \lambda f(x) u & \text{in } \Omega \\ u \neq 0 & \text{in } \Omega \\ u \in H^2(\Omega) \cap H_0^1(\Omega) \end{cases} \tag{4.17}$$

where $0 \in \Omega$. *If* (4.17) *admits a solution* u *for some* $\lambda = \lambda_\mu^1(f)$, *then* u *does not change sign in* Ω.

Lemma 4.5. *For $f \in \mathcal{F}$, X is compactly embedded in $L^2(\Omega, f)$, where $X = H_0^2(\Omega)$ or $H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. Let $\{u_m\}_{m=1}^{\infty}$ be a bounded sequence in X. Hence along a subsequence $u_m \rightharpoonup u$ (say) in X. Due to the Hardy–Rellich inequality $u_m \rightharpoonup u$ in $L^2\big(\Omega, \frac{1}{|x|^4(\ln\frac{R}{|x|})^2}\big)$ and due to the fact that $X \hookrightarrow L^2(\Omega)$ is compact $u_m \to u$ in $L^2(\Omega)$. Since we have $f \in \mathcal{F}$, for any $\varepsilon > 0$, there exist $\delta > 0$ such that

$$\sup_{B_\delta} |x|^4 \left(\ln \frac{R}{|x|}\right)^2 f(x) \le \varepsilon \tag{4.18}$$

and f is bounded on $\Omega \backslash B_\delta$.

$$\int_\Omega |u_m - u|^2 f(x) = \int_{B_\delta} |u_m - u|^2 f(x) + \int_{\Omega\backslash B_\delta} |u_m - u|^2 f(x).$$

Thus, we have from (4.18)

$$\int_\Omega |u_m - u|^2 f(x) \le \varepsilon \int_{B_\delta} \frac{|u_m - u|^2}{|x|^4\left(\ln\frac{R}{|x|}\right)^2} + C\int_\Omega |u_m - u|^2.$$

By Hardy–Rellich inequality we have

$$\int_\Omega |u_m - u|^2 f(x) \le C\varepsilon \int_\Omega |\Delta u_m - \Delta u|^2 + C\int_\Omega |u_m - u|^2. \tag{4.19}$$

Hence from (4.19) we have $u_m \to u$ in $L^2(\Omega, f)$. □

Proof of Theorem 2.4. We look for critical points of the functional

$$J_\mu(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 dx - \frac{\mu}{2}\int_\Omega \frac{|u|^2}{|x|^4\left(\ln\frac{R}{|x|}\right)^2} dx$$

which is continuous, Gateaux differentiable and coercive on X due to Hardy–Rellich inequality. We minimize this functional on $M = \{u \in X : \int_\Omega |u|^2 f dx = 1\}$. Let $\lambda_\mu^1 = \inf_{u\in M} J_\mu(u)$. Then clearly $\lambda_\mu^1 > 0$. Choosing a minimizing sequence $\{u_m\} \subset M$ with $J_\mu(u_m) \to \lambda_\mu^1$ and the component of $DJ_\mu(u_m)$ restricted to M, tends to 0 strongly in $X^\star$. Since $\mu < 1$, J_μ is coercive which implies u_m is bounded. Hence there exist a subsequence of u_m such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } X \\ u_m \rightharpoonup u & \text{weakly in } L^2\left(\Omega, \dfrac{1}{|x|^4\left(\ln\frac{R}{|x|}\right)^2}\right) \\ u_m \to u & \text{strongly in } L^2(\Omega). \end{cases}$$

Since X is compactly embedded in $L^2(\Omega, f(x))$ and M is weakly closed implies that $u \in M$. Hence

$$\int_\Omega |\Delta u_m|^2 = \int_\Omega |\Delta(u_m - u)|^2 + \int_\Omega |\Delta u|^2 + o(1)$$

$$\int_\Omega \frac{|u_m|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} = \int_\Omega \frac{|u_m - u|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} + \int_\Omega \frac{|u|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} + o(1).$$

Hence we have

$$\lambda_\mu^1 = \int_\Omega |\Delta u_m|^2 - \mu \int_\Omega \frac{|u_m|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} + o(1)$$

$$= \int_\Omega |\Delta(u_m - u)|^2 - \mu \int_\Omega \frac{|u_m - u|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} + \int_\Omega |\Delta u|^2 - \mu \int_\Omega \frac{|u|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} + o(1).$$

Hence we have

$$\lambda_\mu^1 \geq (1 - \mu) \int_\Omega |\Delta(u_m - u)|^2 + \int_\Omega |\Delta u|^2 - \mu \int_\Omega \frac{|u|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} + o(1)$$

$$\lambda_\mu^1 \geq (1 - \mu) \int_\Omega |\Delta(u_m - u)|^2 + \lambda_\mu^1 + o(1).$$

Since $\mu < 1$ we have $u_m \to u$ strongly in X. Hence we have u is a nontrivial solution to the problems (2.7), (2.8) corresponding to $\lambda = \lambda_\mu^1(f)$.

Moreover, if $f \in \mathcal{F}$, then by Lemma 4.2 we have,

$$\lambda_\mu^1(f) \to \lambda(f) = \inf_{u \in X \setminus \{0\}} \frac{\displaystyle\int_\Omega (\Delta u)^2 - \int_\Omega \frac{|u|^2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2}}{\displaystyle\int_\Omega u^2 f(x)} > 0 \quad \text{as } \mu \to 1.$$

In order to prove that $\lambda_\mu^1(f)$ is simple when $\Omega = B$, we proceed in a contrapositive way. Suppose if u_1 and u_2 be two orthogonal eigenfunctions of (2.7) in $H_0^2(B)$ with respect to $\lambda_\mu^1(f)$. Multiplying the equation with u_1 by u_2 and integrating by parts; and noting the fact that $\langle \Delta u_1, \Delta u_2 \rangle_{L^2(B)} = 0$ and $f > 0$ a.e., we have

$$-\mu \int_B \frac{u_1 u_2}{|x|^4\left(\ln \frac{R}{|x|}\right)^2} = \lambda_\mu^1(f) \int_B f u_1 u_2$$

which implies a contradiction, as u_1 and u_2 do not change sign in B by Lemma 4.3. Hence u_1 and u_2 are not linearly independent which implies that they are collinear. □

Remark 4.6. As a result of Theorem 2.2, we can study the eigenvalue problem for the case $n \geq 4$, which is highly singular and non-compact type of the form

$$Lu = \lambda u \quad \text{in } \Omega$$

with zero Dirichlet or Navier boundary conditions; where

$$L = \begin{cases} \Delta^2 u - \dfrac{n^2(n-4)^2}{16} \dfrac{u}{|x|^4} & \text{if } n \geq 5 \\ \Delta^2 u - \dfrac{u}{|x|^4 \left(\ln \dfrac{R}{|x|} \right)^2} & \text{if } n = 4. \end{cases} \tag{4.20}$$

One can easily define the eigenvalues $\{\lambda_k\}$ in the form of Rayleigh quotients and show that $\lambda_k \to \infty$ as $k \to \infty$.

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