Entanglement of two distant Bose-Einstein condensates by detection of Bragg-scattered photons

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We show that it is possible to entangle two distant Bose-Einstein condensates by detection of Hanbury Brown-Twiss type correlations in photons Bragg-scattered by the condensates. The generated entanglement is either in quasiparticle number or quadrature phase variables. We present selective results which exhibit complementary squeezing behavior in relative quasi-particle number and phase operators. Our proposed scheme can be generalized for multiple condensates and also for spinor condensates with Bragg scattering of polarized light with the latter capable of producing hyper entanglement.

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I. INTRODUCTION

Quantum entanglement means inseparability of joint wave function of two or more distant objects into a product of wave functions of individual objects - even in the absence of any mutual interaction or communication between them. This epitomizes the underlying nonlocal character of quantum world. One of the consequences of this nonlocal realism is that a single local measurement can not reveal the complete state of an entangled system, since the process of measurement itself forces the wave function to "collapse" into one its measured (eigenstate) state in a probabilistic sense. Thus, measurement process can post-selectively play a role in creation and manipulation of entanglement, and this is the essence of what is called "projective measurement".

Efficient generation of entanglement in many-particle systems and its robust transmission and transfer to other systems is important for quantum information processes. Based on atom-photon interaction and the exchange of photons between the qubits, entanglement in distant atomic states [1–3] and also between photons [4, 5] have been experimentally demonstrated. There is another way of entangling two remote systems without requiring any direct interaction between them: This is based on projective measurement. In a recent experiment, Moehring *et al.* [6] have created entanglement between two distant trapped ions by coincident detection of two photons spontaneously emitted by the two ions. An earlier experiment has shown interference of light emitted by two atoms [7–9] making use of projective measurements. Thiel *et al.* [10] have proposed a scheme of entangling several remote atomic qubits and thereby creating Dicke state [11] of many-atoms by projective measurement of photons using multiple photo-detectors. Dicke states are particularly important for their robustness against particle loss [12, 13] and non-local properties of entangled multipartite states [5, 14–16]. There are several other proposals for projecting distant non-interacting particles into entangled states via photo-detection [17–22]. Continuous variables like the quadratures of a field mode (which are analogous to position and momentum) have also been employed [23] in entanglement studies.

Bose-Einstein condensate (BEC) is a macroscopic quantum object where entanglement arises quite naturally due to two-body interaction. Bogoliubov theory [24] of Bose condensation reveals that in the ground state of condensate, two particles with opposite momentum are maximally entangled [25] in momentum variables as in EPR state [26]. This unique feature makes Bose condensates a good source of entanglement in motional degrees of freedom. Furthermore, in a two-component BEC, one can generate entanglement in hyperfine spin degrees of freedom [27–33]. In order to extract the intrinsic entanglement of a BEC for useful purpose of quantum information processes, it is required to excite quasi-particles in momentum modes by stimulated Raman scattering or Bragg scattering [34, 35]. Then a scattered atom becomes entangled with Bragg-scattered photon [36]. In fact, Bragg spectroscopy can be used as a tool for generating entanglement of different kinds in a variety of physical situations. For instance, tripartite entanglement among two momentum modes of BEC and one electromagnetic field mode can be produced [25]. Furthermore, it has been shown that when a common laser beam passes through two spatially separated condensates, photon scattered by the first condensate carries and transfers quantum information to the second one and thereby two condensates become entangled [37]. Similar experimental scheme has been used to produce and subsequently measure phase difference between two spatially separated condensates [38].

Here we propose a new scheme for generation of entanglement in quasi-particle number and phase variables between two remote condensates by projective measurement on Bragg-scattered photons. Our scheme relies on

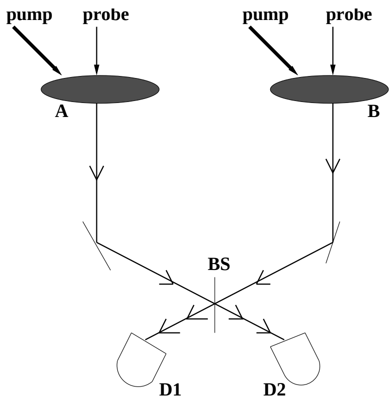

FIG. 1: A scheme for entangling two separate BECs by Bragg scattering and photo-detection. A and B are two BECs which scatter photons from pump beams into probe ones. D_1 and D_2 are two photo-detectors and 'BS' stands for beam splitter.

coincident detection of two Bragg-scattered photons coming from two remote condensates in a Hanbury Brown-Twiss type experimental arrangement as schematically illustrated in Fig.1 A and B are two remote single-component condensates. Bragg scattering of pump photons occur independently at the two condensates, the scattering is stimulated by probe beams. This gives rise to the generation of quasi-particles in both the condensates in particular momentum modes determined by the relative angle between pump and probe light beams. The probe beams are assumed to have the same mode, that is, same frequency and polarization property. The scattered photons in the probe modes are allowed to pass through a beam splitter in order to coalesce them to lose which path information. Then the two photons coming out of the output ports of the beam spitter are detected by two photo-detectors D_1 and D_2 in a coincident way.

The paper is organized in the following way. Since Bragg scattering holds the key for generating quasi-particles and atom-photon entanglement, we first give a brief introduction to Bragg scattering in Sec.II in order to reveal the essential physical processes involved in Bragg scattering. We then formulate our theoretical model of projective measurement and its effect in generating entanglement between two remote condensates in Sec.III. The paper is concluded in Sec.IV.

II. BRAGG SCATTERING

In a Bose condensate of weakly interacting atomic gases, zero momentum ($\mathbf{k} = 0$) state is macroscopically occupied. Therefore, atom-atom collision occurs primarily between zero- and non-zero momentum atoms. In stimulated Raman or Bragg scattering, two far-off resonant laser beams with a small frequency difference is impinged on a trapped BEC. There are basically two physical processes in Bragg scattering. In the first process, a photon from the laser beam with higher frequency is scattered into a photon of the other laser mode. This causes transformation of a zero-momentum atom into an atom of momentum $\mathbf{q}$, where $\mathbf{q}$ is the difference in photon momentum of the two beams. In the second process, an atom moving with a momentum $-\mathbf{q}$ is scattered into a zero-momentum state. Because of bosonic stimulation, the scattering of atoms from zero- to $\mathbf{q}-$ momentum state will be dominant process. There also occur the processes which are opposite to above two processes, but these are subdued due to phase mismatch. Thus Bragg scattering generates quasiparticles [34], predominantly in two momentum side-modes $\mathbf{q}$ and $-\mathbf{q}$, Bragg spectroscopy [35] with coherent or classical light produces coherent states of quasiparticles in a BEC. When these quasiparticles are projected into particle domain, they form two-mode squeezed as well as entangled state [25] in particle number variables.

Two remote condensates A and B are subjected to Bragg scattering with pairs of Bragg pulses. The frequencies and the directions of propagation of the laser beams are so chosen such that Bragg resonance (phase matching) conditions of scattering in both the condensates are fulfilled. We assume the pulse with higher frequency has higher intensity and hence can act as a pump. The other laser beam which acts as stimulant for scattering is of much lower intensity and so can be considered as probe beam. We treat pump beams classically. Since scattering at the two condensates occur independently, the Hamiltonian is simply the sum of the Hamiltonian H^A and H^B corresponding to condensates A and B respectively. Let $\hat{a}_q$ and $\hat{b}_q$ represent the annihilation operators for particles with momentum $\mathbf{q}$ in condensate A and B, respectively. Let the corresponding Bogoliubov quasiparticles be denoted by $\hat{\alpha}_q$ and $\hat{\beta}_q$, respectively. In terms of these quasiparticle operators, the effective Hamiltonian as derived in the Appendix can be written as

$$\begin{aligned} H^J_{eff} = & \ \hbar\omega^B_q \ \left(\hat{\chi}^\dagger_{\mathbf{q}}\hat{\chi}_{\mathbf{q}} + \hat{\chi}^\dagger_{-\mathbf{q}}\hat{\chi}_{-\mathbf{q}}\right) - \hbar\delta_j\hat{c}^\dagger_j\hat{c}_j \\ + & \ \left[\hbar\eta\hat{c}^\dagger_j(\hat{\chi}^\dagger_{\mathbf{q}} + \hat{\chi}_{-\mathbf{q}}) + \mathrm{H.}c.\right] \end{aligned} \tag{1}$$

where the superscript J stands for condensate A or B, $\hat{c}_j$ (with $j = a, b$) is photon annihilation operator for the probe beam applied to condensate J. The particle operators $\hat{\pi}_q(\equiv \hat{a}_q, \hat{b}_q)$ are related to the quasi-particle operators $\hat{\chi}_q(\equiv \hat{\alpha}_q, \hat{\beta}_q)$ by Bogoliubov's transformation

$$\hat{\pi}_q = u_q\hat{\chi}_q - v_q\hat{\chi}^\dagger_{-q} \tag{2}$$

with

$$v^2_q = (u^2_q - 1) = \frac{1}{2}\left(\frac{\hbar\omega_q + \mu}{\hbar\omega^B_q} - 1\right) \tag{3}$$

and

$$\hbar\omega^B_q = \left[(\hbar\omega_q + \mu)^2 - \mu^2\right]^{1/2} \tag{4}$$

is the energy of Bogoliubov's quasiparticle. Here $\hbar\omega_q = \hbar^2q^2/(2m)$ is the kinetic energy of a single atom, $\mu =$

$\frac{\hbar^2\xi^{-2}}{2m}$ is the chemical potential with $\xi = (8\pi n_0 a_s)^{-1/2}$ being the healing or coherence length, δ_j is the detuning between the pump and probe frequencies, $\eta = \sqrt{N} f_q \Omega$, where $f_q = u_q - v_q$ and Ω is the two-photon Rabi frequency. We assume the Bragg resonance condition ($\delta \simeq \omega_q$). The hamiltonian can be solved exactly in Heisenberg picture. The Heisenberg equations of motion for a triad of operators $X = \left(\hat{\alpha}_{\mathbf{q}} \;\; \hat{\alpha}^\dagger_{-\mathbf{q}} \;\; \hat{c}^\dagger_j\right)^T$ can be written in a matrix form $\dot{X} = i\omega^B_q \mathbf{M} X$, where $\mathbf{M}$ is a 3×3 matrix

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & -\tilde{\eta} \\ 0 & 1 & \tilde{\eta} \\ \tilde{\eta}^* & \tilde{\eta}^* & -\tilde{\delta} \end{pmatrix} \tag{5}$$

where $\tilde{x} = \tilde{x}/\omega^B_q$. Let $\mathbf{D}$ be the diagonalizing matrix of $\mathbf{M}$. The solutions can be explicitly written as

$$X(t) = \mathbf{D}\mathbf{E}(t)\mathbf{D}^{-1}X(0) \tag{6}$$

where $\mathbf{E}$ is a diagonal matrix : $\mathbf{E} = \mathbf{diag}.[\exp(i\lambda_1\tau), \exp(i\lambda_2\tau), \exp(i\lambda_3\tau)]$ with $\tau = \omega^B_q t$ and λ_is being the eigenvalues of $\mathbf{M}$ matrix.

III. ENTANGLEMENT PRODUCED BY PHOTO-DETECTION

Our proposed scheme is shown in Fig.1. Quasiparticles are generated in the condensates A and B due to stimulated light scattering in a pump-probe type Bragg-spectroscopic method. Let $\hat{c}_a$ and $\hat{c}_b$ denote annihilation operators for the two probe light beams scattered by condensates A and B, respectively. Using Eq. (6), the scattered light at the output of the two condensates can be represented by

$$\hat{c}_a(t) = a_q(t)\hat{\alpha}^\dagger_q + a_{-q}(t)\hat{\alpha}_{-q} + a_c(t)\hat{c}_a(0) \tag{7}$$

$$\hat{c}_b(t) = b_q(t)\hat{\beta}^\dagger_q + b_{-q}(t)\hat{\beta}_{-q} + b_c(t)\hat{c}_b(0) \tag{8}$$

where $a_{\pm q}$, $b_{\pm q}$, a_c and b_c are time-dependent coefficients determined by the Eq. (6). The scattered light output coming from the two condensates are passed through a beam-splitter and finally collected at the two detectors at D1 and D2 as shown in Fig.1. Let the reflectivity and transmissivity at left side of the beam-splitter are r and t, respectively, while those at right side are r' and t'. Then the photon annihilation operators at D1 and D2 can be expressed as

$$\hat{C}_{D_1} = t'\hat{c}_b + r\hat{c}_a \tag{9}$$

$$\hat{C}_{D_2} = t\hat{c}_a + r'\hat{c}_b \tag{10}$$

Let the initial state of the total system i.e. two condensates plus the two probe fields, be represented by

$$\mid \Psi_0\rangle = \mid 0,0\rangle_{AB} \mid \alpha,\beta\rangle_{fields} \tag{11}$$

where $\mid 0,0\rangle_{AB}$ indicates a product state with both the condensates in ground states of quasiparticles, where first '0' corresponds to condensate A and the second '0' to condensate B. We assume that both the probe fields are in coherent states $\mid \alpha,\beta\rangle_{fields}$ where the field amplitudes α and β correspond to the probes incident at A and B, respectively. Measurement of two-photon correlation via coincident detection of scattered probe lights at the two detectors will project the two-condensate density operator into

$$\rho_{AB} = \mathcal{N}\mathrm{Tr}_{\mathrm{fields}}\hat{C}_{D_2}\hat{C}_{D_1}\rho_0\hat{C}^\dagger_{D_1}\hat{C}^\dagger_{D_2} \tag{12}$$

where $\mathrm{Tr}_{\mathrm{fields}}$ implies tracing over the field states, $\rho_0 = \mid \Psi_0\rangle\langle\Psi_0 \mid$ and $\mathcal{N}$ denotes a normalization factor. Now, substituting 9 and 10 into 12 and using the relations 7 and 8, we obtain $\rho_{AB} = \mid \Phi\rangle\langle\Phi \mid$ where

$$\begin{aligned} \mid \Phi\rangle &= \sqrt{\mathcal{N}}\langle\alpha,\beta \mid \hat{C}_{D_2}\hat{C}_{D_1} \mid \Psi_0\rangle \\ &= \sqrt{\mathcal{N}}\left(r't' \mid 0_A, S_B(1_q,2_q)\rangle + rt \mid S_A(1_q,2_q), 0_B\rangle\right) \\ &+ \sqrt{\mathcal{N}}(r'r + t't) \mid \Sigma_A(1_q), \Sigma_B(1_q)\rangle. \end{aligned} \tag{13}$$

The states $\mid S_j(1_q,2_q)\rangle$ denotes a superposition state of ground, one and two q-phonon excited states of condensate $j(\equiv A,B)$. Similarly, $\mid \Sigma_j(0,1_q),\rangle$ is another superposition state of ground and one phonon excited states of condensate 'j'. Explicitly, these superposition states can be expressed as

$$\begin{aligned} \mid S_A\rangle &= \sqrt{2}a_q^2 \mid 2_q\rangle_A + 2a_q a_c\alpha \mid 1_q\rangle_A \\ &+ a_c^2\alpha^2 \mid 0\rangle_A \end{aligned} \tag{14}$$

$$\mid \Sigma_A\rangle = a_q \mid 1_q\rangle_A + a_c\alpha \mid 0\rangle_A \tag{15}$$

Now, we have the reciprocity relations

$$r^*t' + r't^* = 0, \qquad r^*t + r't'^* = 0$$

$$|r'| = |r|, \quad |t'| = |t|, \quad |r|^2 + |t|^2 = 1$$

Let $t = |t|\exp(i\phi)$ and $t' = |t'|\exp(i\phi')$. Since phase changes by $\pi/2$ on reflection, we have $r = i|r|\exp(i\phi)$ and $r' = i|r'|\exp(i\phi')$. Using the reciprocity relations and considering the field amplitudes α and β as real quantities, we obtain

$$\begin{aligned} \mid \Phi\rangle &= \sqrt{\mathcal{N}}|r||t|\left(\exp(2i\phi') \mid 0_A, S_B\rangle + \exp(2i\phi) \mid S_A, 0_B\rangle\right) \\ &+ \sqrt{\mathcal{N}}\exp[i(\phi+\phi')](|r|^2 - |t|^2) \mid \Sigma_A, \Sigma_B\rangle. \end{aligned} \tag{16}$$

For a 50:50 beam splitter, we then have

$$\mid \tilde{\Phi}\rangle = \sqrt{\mathcal{N}}\frac{1}{2}\left(\mid 0_A, S_B\rangle + \exp(2i\Delta\phi) \mid S_A, 0_B\rangle\right) \tag{17}$$

where $\Delta\phi = \phi - \phi'$ and $\mid \tilde{\Phi}\rangle = \exp(-2i\phi') \mid \Phi\rangle$

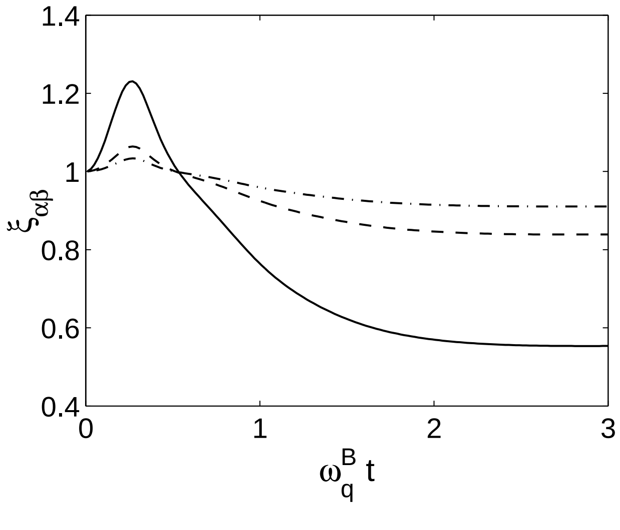

FIG. 2: Entanglement parameter $\xi_{\alpha\beta}$ in quasi-particle number operators as a function of dimensionless interaction time $\omega_q^B t$, for different values of average probe photon numbers $n_p = 10$ (solid), $n_p = 50$ (dashed) and $n_p = 100$ (dashed-dotted). The effective atom-field coupling constants are $\eta_A/\omega_q^B = 7.71$ and $\eta_B = 1.25\eta_A$, the phase-difference $\Delta\phi = 0$.

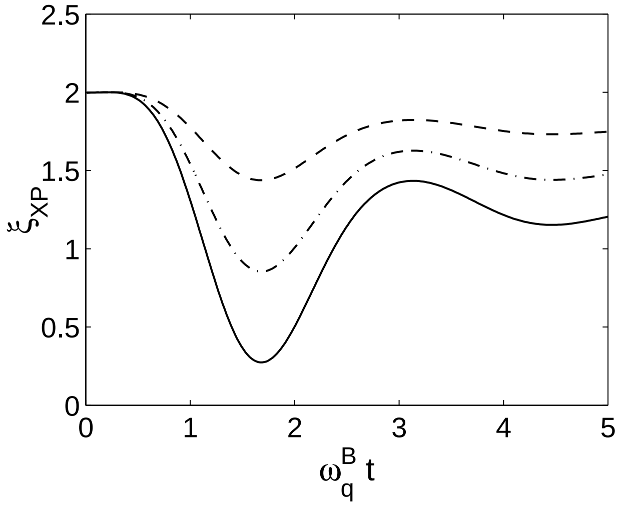

FIG. 3: Entanglement parameter ξ_{XP} in quadrature phase variables as a function of dimensionless interaction time $\omega_q^B t$, for different values of average probe photon numbers $n_p = 150$ (solid), $n_p = 100$ (dashed-dotted) and $n_p = 50$ (dashed). The phase difference is $\Delta\phi = \pi/2$. The effective atom-field coupling constants are $\eta_A/\omega_q^B = 0.75$ and $\eta_B = \eta_A$.

A. Entanglement in quasi-particle picture

Here we consider entanglement in quasiparticle number operators: $\hat{n}_\alpha = \hat{\alpha}_q^\dagger \hat{\alpha}_q$ and $\hat{n}_\beta = \hat{\beta}_q^\dagger \hat{\beta}_q$ with $\hat{n}_\alpha \mid 0\rangle_A = 0, \hat{n}_\beta \mid 0\rangle_B = 0, \hat{n}_\alpha \mid n_q\rangle_A = n \mid n_q\rangle_A$ and $\hat{n}_\beta \mid n_q\rangle_B = n \mid n_q\rangle_B$. How to quantify entanglement is an open question. Although there are various criteria for characterizing entangled states, there seems to be no universal measure for entanglement. The usual characterization is in terms of two-mode squeezing in position and momentum variables or similar canonically conjugate variables of two sub-systems. Consider squeezed state of a single-mode electromagnetic field. The squeezing occurs in two canonically conjugate quadrature phase variables (which are analogous to position and momentum variables) of field amplitudes, the quantum fluctuation in one phase variable is decreased below the standard quantum limit (Poissonian level) at the expense of enhanced fluctuation in the other variable and thus fulfilling Hesienberg uncertainty. In such a situation, the photon statistics of the field becomes sub-Poissonian which is a distinctive feature of nonclassical physics. When squeezing occurs between two variables which are some linear combination of variables of two sub-systems such as $(\hat{X}_1 - \hat{X}_2)/\sqrt{2}$ and $-i(\hat{P}_1 + \hat{P}_2)/\sqrt{2}$, then the joint state of the two sub-systems may be inseparable. Now, if the sum of fluctuations in these two variables is below twice the standard quantum limit, then the joint state of two sub-systems is definitely entangled. This implies that squeezing or sub-Poissonian nature of quantum states may be utilized for characterizing an entangled state. Now, when entanglement occurs in occupation number variables, then squeezing is expected in relative number operators of two subsystems. Since number and phase variables are two canonically conjugate variables satisfying number-phase uncertainty, the entanglement in number variables necessarily negates entanglement in phase variables and vice-versa.

Following Gasenzer *et al.* [36], we define an entanglement parameter in number variables in terms of fluctuation in relative number operators

$$\xi_{\alpha\beta} = \frac{\langle(\hat{n}_\alpha - \hat{n}_\beta)^2\rangle - \langle\hat{n}_\alpha - \hat{n}_\beta\rangle^2}{\langle\hat{n}_\alpha\rangle + \langle\hat{n}_\beta\rangle} \tag{18}$$

The two condensates is said to be entangled in quasi-particle number variables when the condition $\xi_{\alpha\beta} < 1$ is fulfilled. This happens only when $\langle\hat{n}_\alpha\hat{n}_\beta\rangle \neq \langle\hat{n}_\alpha\rangle\langle\hat{n}_\beta\rangle$ implying the non-separability of the joint state of the two condensates when the state is expressed in terms of quasi-particle Fock state basis. Using the state (17), we obtain

$$\xi_{\alpha\beta} = \frac{\langle\hat{n}_\alpha^2\rangle + \langle\hat{n}_\beta^2\rangle - (\langle\hat{n}_\alpha\rangle - \langle\hat{n}_\beta\rangle)^2}{\langle\hat{n}_\alpha\rangle + \langle\hat{n}_\beta\rangle} \tag{19}$$

This shows that there is possibility of entanglement parameter becoming less than unity only when $\langle\hat{n}_\alpha\rangle \neq \langle\hat{n}_\beta\rangle$ which means that the two condensates should either have different densities or different effective field-matter couplings ($\eta_A \neq \eta_B$).

Figure 2 shows the variation of entanglement parameter $\xi_{\alpha\beta}$ as a function of interaction time scaled by inverse Bogoliubov frequency ω_q^B for different values of average probe photon numbers. The probe is in coherent state. At short interaction time there is no entanglement. For longer interaction times, entanglement occurs and saturates in large time limit. Lower the probe photon number

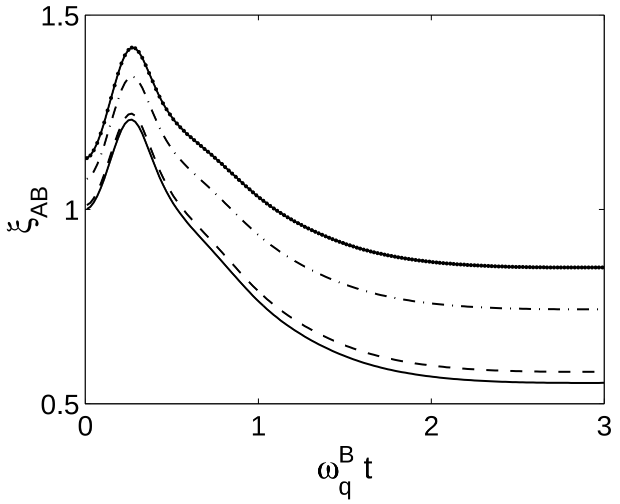

FIG. 4: Entanglement parameter ξ_{AB} in particle number variables as a function of dimensionless interaction time $\omega_q^B t$, for different values momentum transfer $q = 0.4\xi^{-1}$ (solid-dotted), $q = 0.8\xi^{-1}$ (dashed-dotted), $q = 2\xi^{-1}$ (dashed) and $q = 10\xi^{-1}$ (solid). The average number of photons in incident probe beam is $n_p = 10$, and all other parameters are same as in Fig.1.

better is the entanglement. In the large photon number limit, the relative number fluctuation tends to that of coherent state meaning the disappearance of entanglement.

Next we consider entanglement in quadrature operators $\hat{X}_{\chi=\alpha,\beta} = (1/2)(\hat{\chi}+\hat{\chi}^\dagger)$ and $\hat{P}_{\chi=\alpha,\beta} = (1/2i)(\hat{\chi}-\hat{\chi}^\dagger)$. The entanglement parameter in quadratures is defined as

$$\xi_{XP} = \frac{1}{2}\left[\langle\Delta(\hat{X}_\alpha + \hat{X}_\beta)^2\rangle + \langle\Delta(\hat{P}_\alpha - \hat{P}_\beta)^2\rangle\right]. \quad (20)$$

The condition for the occurrence of entanglement is $\xi_{XP} < 1$. Now, we have

$$\xi_{XP} = 1 + \langle n_\alpha\rangle + \langle n_\beta\rangle + \langle\alpha_q\beta_q\rangle + \langle\alpha_q^\dagger\beta_q^\dagger\rangle - (\langle\alpha_q^\dagger\rangle\langle\alpha_q\rangle + \langle\beta_q^\dagger\rangle\langle\beta_q\rangle + \langle\alpha_q\rangle\langle\beta_q\rangle + \langle\alpha_q^\dagger\rangle\langle\beta_q^\dagger\rangle). (21)$$

For the state (17), the terms $\langle\alpha_q\beta_q\rangle = \langle\alpha_q^\dagger\beta_q^\dagger\rangle = 0$ while $\langle\alpha_q\rangle = \exp[2i\Delta\phi]\langle 0_A \mid \hat{\alpha}_q \mid S_A\rangle\langle S_B \mid 0_B\rangle = \exp[2i\Delta\phi]2a_q a_c \alpha b_c^2\beta^2$.

The variation of ξ_{XP} as a function of interaction time for different probe photon numbers is plotted in Fig. 3. In contrast to the behavior of $\xi_{\alpha\beta}$ in Fig.2, we notice that the entanglement in phase variable is better for larger number of probe photons in the intermediate interaction time regime. Furthermore, entanglement disappears in the long time limit which is also in sharp contrast to the case of number variables.

B. Entanglement in particle picture

We have so far considered entanglement in quasi-particle picture. Now, we turn our attention in particle picture described by the operators $\hat{a}_q$ and $\hat{b}_q$.

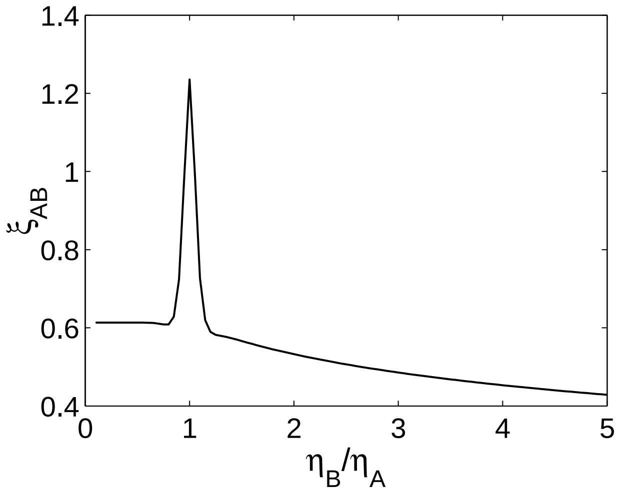

FIG. 5: Entanglement parameter ξ_{AB} as a function of ratio of the two coupling constants η_B/η_A for a fixed interaction time $\omega_q^B t = 3.0$, $n_p = 10$ and $q = 2\xi^{-1}$. The other parameters are same as in Fig.1.

To see whether this state is entangled in number operators or not, we calculate the the two-mode squeezing parameter

$$\xi_{AB} = \langle[\Delta(\hat{n}_A - \hat{n}_B)]^2\rangle/(\langle\hat{n}_A\rangle + \langle\hat{n}_B\rangle), \quad (22)$$

where $\langle(\Delta\hat{n})^2\rangle = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$ and $\hat{n}_{A,B}$ are the particle number operators in momentum mode q: $\hat{n}_A = \hat{a}_{\mathbf{q}}^\dagger\hat{a}_{\mathbf{q}}$ and $\hat{n}_B = \hat{b}_{\mathbf{q}}^\dagger\hat{b}_{\mathbf{q}}$.

After some algebra, and assuming the Bogoliubov's amplitudes (u, v) be the same for both the condensates, we obtain

$$\xi_{AB} = \zeta\left[u^2\xi_{\alpha\beta} + v^2 + \frac{2v^2}{\langle\hat{n}_\alpha\rangle + \langle\hat{n}_\beta\rangle}\right] \quad (23)$$

where

$$\zeta = 1 + \frac{2v^2}{u^2(\langle\hat{n}_\alpha\rangle + \langle\hat{n}_\beta\rangle)} \quad (24)$$

Figure 4 exhibits the behavior of entanglement parameter as a function of interaction time in particle number variables for different fixed momentum transfers. We infer that the entanglement is better for larger momentum transfers. This implies that entanglement is more significant in quadratic dispersion regime (free particle-like) than to linear dispersion (phonon) regime.

Since quadrature phase variables are superpositions of particle annihilation and creation operators, in the particle picture the operators

$$\hat{X}_a = \frac{1}{\sqrt{2}}\left[\hat{a}_q + \hat{a}_q^\dagger\right] = u_q\hat{X}_\alpha(q) - v_q\hat{X}_\alpha(-q) \quad (25)$$

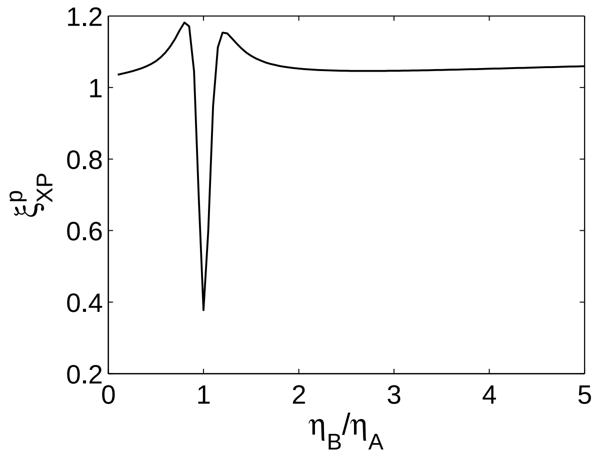

FIG. 6: Entanglement parameter ξ^p_{XP} as a function of ratio of the two coupling constants η_B/η_A for $\omega_q^B t = 1.5$, $n_p = 150$ and $q = 8.33\xi^{-1}$. The other parameters are same as in Fig.2.

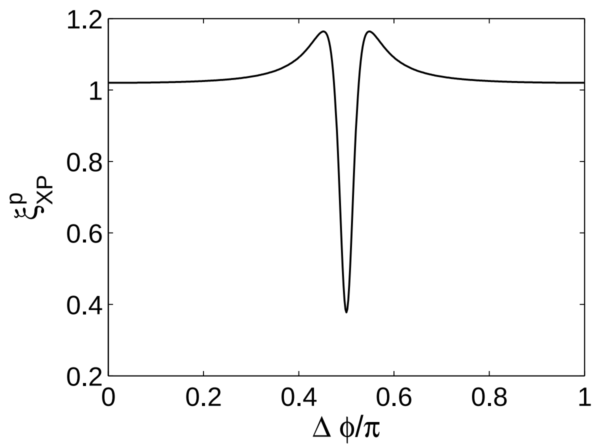

FIG. 7: Entanglement parameter ξ^p_{XP} as a function of phase difference $\Delta\phi$ for $\eta_A = \eta_B$. The other parameters are same as in Fig.2.

where $\hat{X}_\alpha(\pm q) = 1/\sqrt{2}(\alpha_{\pm q} + \alpha^\dagger_{\pm q})$ are again superposition of quasi-particle $\hat{X}$ variables corresponding to two momentum-modes $\pm q$. Since $-q$ mode in both the condensates is in quasi-particle vacuum, the two condensates are not entangled in this mode in either quadrature phase or number variables. It then follows that the entanglement parameter $\xi^{\rm p}_{XP}$ in particle picture is related to that (ξ_{XP}) in quasi-particle picture by a scale factor $|u_q|^2$, apart from an additional term $|v_q|^2$. For $q \to \infty$, the entangled parameters in the two pictures will be identical since $|u_q| \to 1$ and $|v_q| \to 0$ in this limit.

Figures 5 shows the behavior of entanglement in particle number variable as a function of the ratio of the two effective atom-filed coupling constants, while Fig.6 exhibits the same for quadrature phase variables. In Fig.7, we display the variation of entanglement in quadrature phase variable as a function of phase-difference between the two optical paths. This shows that the entanglement occurs only when phase difference $\Delta\phi$ is equal or nearly equal to $\pi/2$. In contrast, entanglement in number variable does not depend on phase difference.

Comparing Fig.2 with Fig.3, and also Fig.5 with Fig.6, we notice a few things about entanglement in number vis-a-vis phase variables. First, entanglement in quasi-particle or particle number variables arises only when $\eta_A \neq \eta_B$. In contrast, entanglement in phase variables peaks at $\eta_A = \eta_B$ and vanishes even if η_A differs slightly from η_B. Second, as we increase the intensity of stimulating probe laser, entanglement in number variables decreases. But exactly opposite effect is observed in case of entanglement in phase variables. From this comparison, we may infer that entanglements in number and phase variables have some complementary effects. We further notice that the entanglement parameter in both variables become saturated when light-matter interaction time is large. Furthermore, Fig. 4 shows that entanglement in particle number variables is better for large q rather than small q. In quasi-particle picture, entanglement in quasi-particle number operators can occur in any value of q, because it only depends on quasi-particle number fluctuations, and not on q. However, in the large momentum transfer regime, the entanglement in both the pictures would be same. In the small momentum transfer regime, we then have more significant entanglement in phonon number variables as compared to that in particle number variables.

IV. CONCLUSION

In conclusion, we have shown that two remote independent condensates can be made entangled in number and phase variables by projective measurement on two photons Bragg-scattered by the two condensates. The generated entanglement shows complementarity in relative number and phase fluctuations indicating occurrence of relative number-phase squeezing. The entanglement in number variables can be detected by out-coupling the quasi-particles of condensates A and B by pairs of Bragg pulses with large momentum transfer as shown by Ketterle's group [39] and then measuring the relative number fluctuations via absorption imaging. Entanglement in phase variables may be detected by phase-sensitive measurement of Bragg-scattered probe beams. Our proposed scheme can be easily generalized for multiple condensates. Furthermore, it may be interesting to generate entanglement both in spin and motional degrees of freedom using spinor condensates and polarized lights in our scheme. Then one can explore interesting interplay between the two types of entanglement.

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APPENDIX

Here we derive the effective hamiltonian for Bragg scattering in a condensate. The total hamiltonian of a condensate interacting with two single-mode light fields is $H = H_A + H_F + H_{AF}$, where $H_F = \hbar\omega_1 \hat{c}^\dagger_{\mathbf{k}_1}\hat{c}_{\mathbf{k}_1} + \hbar\omega_2 \hat{c}^\dagger_{\mathbf{k}_2}\hat{c}_{\mathbf{k}_2}$ corresponds to the two fields described by the operators $\hat{c}_{\mathbf{k}_1}$ and $\hat{c}_{\mathbf{k}_2}$ with photon momentum $\mathbf{k}_1$ and $\mathbf{k}_2$ and the frequencies ω_1 and ω_2, respectively. We assume $\omega_1 > \omega_2$. The free part of the atomic hamiltonian

$$H_A = \sum_k \hbar\omega_k \hat{\pi}^\dagger_{\mathbf{k}}\hat{\pi}_{\mathbf{k}} + \frac{4\pi\hbar^2 a_s}{2mV} \times \sum_{\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} \hat{\pi}^\dagger_{\mathbf{k}_3}\hat{\pi}^\dagger_{\mathbf{k}_4}\hat{\pi}_{\mathbf{k}_5}\hat{\pi}_{\mathbf{k}_6}\delta_{\mathbf{k}_3+\mathbf{k}_4,\mathbf{k}_5+\mathbf{k}_6} \quad (26)$$

governs the dynamics of a weakly interacting atomic condensate and

$$H_{AF} = \hbar\Omega \hat{c}^\dagger_{\mathbf{k}_2}\hat{c}_{\mathbf{k}_1} \sum_k \left(\hat{\pi}^\dagger_{\mathbf{q}+\mathbf{k}}\hat{\pi}_{\mathbf{k}} + \hat{\pi}_{-\mathbf{q}+\mathbf{k}}\hat{\pi}^\dagger_{\mathbf{k}}\right) + \text{H.c.} \quad (27)$$

describes atom-field interaction. Here $\hat{\pi}_k(\hat{\pi}^\dagger_k)$ is the annhilation(creation) operator of an atom with momentum $\mathbf{k}$ and frequency $\omega_k = \frac{\hbar k^2}{2m}$; $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$, $\Omega = (\vec{E}_1.\vec{d}_{13})(\vec{E}_2.\vec{d}_{32})/(\hbar^2\Delta)$ is the two-photon Rabi frequency, where $E_{1(2)}$ are the field amplitudes, the $\vec{d}_{ij}$ is the electronic transition dipole moment between the states $|i\rangle$ and $|j\rangle$ of an atom and Δ is the detuning of the first laser field (with frequency ω_1) from the transition frequency between the electronic ground ($|1\rangle$) and excited ($|3\rangle$) levels of the atom. For a single-component condensate the electronic ground states $|1\rangle$ and $|2\rangle$ are the same. Here a_s is the s-wave scattering length of the atoms and V is the volume of the condensate.

Using Bogoliubov's prescription $\hat{\pi}_0, \hat{\pi}^\dagger_0 \rightarrow \sqrt{N_0}$, and keeping the number density $n_0 = N_0/V$ fixed in the thermodynamic limit, one can transform the hamiltonian H_A into a quadratic form. Further, applying Bogoliubov's transformation it is possible to diagonalize H_A and rewrite the entire hamiltonian in terms of Bogoliubov's quasi-particle operators $\hat{\chi}_{\mathbf{k}}$. Considering the condensate ground state energy as the zero of the energy scale, and treating the laser light with higher frequency (ω_1) classically, the effective hamiltonian can be written as

$$H_{eff} = \hbar\omega^B_q \left(\hat{\chi}^\dagger_{\mathbf{q}}\hat{\chi}_{\mathbf{q}} + \hat{\chi}^\dagger_{-\mathbf{q}}\hat{\chi}_{-\mathbf{q}}\right) - \hbar\delta \hat{c}^\dagger_{\mathbf{k}_2}\hat{c}_{\mathbf{k}_2} + \left[\hbar\eta \hat{c}^\dagger_{\mathbf{k}_2}(\hat{\chi}^\dagger_{\mathbf{q}} + \hat{\chi}_{-\mathbf{q}}) + \text{H.}c.\right] \quad (28)$$

where $\delta = \omega_1 - \omega_2$ and $\eta = \sqrt{N} f_q \Omega$; where $f_q = u_q - v_q$. In writing the above equation, we have retained only two dominant momentum side-modes of the condensate under the assumption of Bragg resonance ($\delta \simeq \omega_q$).

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