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Role of the fundamental solution in Hardy–Sobolev-type inequalities

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Let $n \geqslant 3$, $\Omega \subset \mathbb{R}^n$ be a domain with $0 \in \Omega$, then, for all $u \in H_0^1(\Omega)$, the Hardy–Sobolev inequality says that

$$\int_\Omega |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \geqslant 0$$

and equality holds if and only if $u = 0$ and $((n-2)/2)^2$ is the best constant which is never achieved. In view of this, there is scope for improving this inequality further. In this paper we have investigated this problem by using the fundamental solutions and have obtained the optimal estimates. Furthermore, we have shown that this technique is used to obtain the Hardy–Sobolev type inequalities on manifolds and also on the Heisenberg group.

1. Introduction

Let $n \geqslant 3$ and $0 \in \Omega \subset \mathbb{R}^n$ be a domain. Then the classical Hardy–Sobolev (HS) inequality [9, 15, 16, 22] states that, for all $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \geqslant 0 \tag{1.1}$$

and $(n-p/p)^p$ is the best constant in (1.1) and it is never achieved. In view of this, is there scope for improving this inequality by replacing the zero term by some nontrivial functional of u in (1.1)? Recently, there has been considerable interest in this question and one of the important improvements was obtained by Brezis and Vasquez [8]. They showed that if Ω is a bounded domain, then there exists a $C > 0$, such that, for all $u \in H_0^1(\Omega)$,

$$\int_\Omega |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \geqslant C \int_\Omega |u|^2. \tag{1.2}$$

Furthermore, if $\lambda(\Omega)$ denotes the best choice of C in (1.2), then $\lambda(\Omega)$ is never achieved. Again we can ask whether there is a scope for further improvement of this inequality. In this direction Brezis and Vasquez raised the following question:

What is the best possible remainder term one can expect of (1.2)?

Recently, this question was answered in [4–6, 11], where the following inequality was proved.

Let Ω be a bounded domain and $1 < p \leqslant n$. Let R be sufficiently large. There then exists a $C > 0$ depending on n, p and R such that

$$\int_\Omega |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \geqslant C \int_\Omega \frac{|u|^p}{|x|^p (\log(R/|x|))^\gamma} \tag{1.3}$$

for every $u \in W_0^{1,p}(\Omega)$ if and only if

$$\gamma \geqslant \begin{cases} 2 & \text{for } 1 < p < n, \\ n & \text{for } p = n. \end{cases}$$

Moreover, for $\gamma = 2$, the right-hand side of (1.3) can be improved by adding an appropriate finite or infinite series. It was shown in [3] that if $p = 2$ and $\gamma = 2$, then $C = \frac{1}{4}$, and if $p = n$ and $\gamma = n$, then $C = ((n-1)/n)^n$ is the best constant for (1.3). These results were extended in [1, 2] to spaces $W^{1,p}(\Omega)$. The perturbed eigenvalue problem corresponding to the Euler–Lagrange equations associated to (1.3) was studied in [2, 3, 17]. Here one can find the condition on the perturbed coefficient in order to guarantee the existence of an eigenvalue of the corresponding operator in $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$ with Neumann boundary condition.

Our interests in this are twofold. For the sake of simplicity, first consider $p = 2$.

(i) What is the analogous HS inequality if we replace $|\nabla u|^2$ by a general bilinear form
$$a(u,u) = \sum_{1 \leqslant i,j \leqslant n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j},$$
coming from a positive definite matrix $((a_{ij}(x)))$?

(ii) What is the analogous HS inequality if we replace $|\nabla u|^2$ by $\sum_{j=1}^{l} |Z_j u|^2$, where the Z_j are smooth vector fields?

The methodology adopted in [4–6, 11] is not suitable for answering the questions above. This is because, in these papers, either symmetrization is used or the function is decomposed into its radial and nonradial components, whereas the method in [2] does not involve either of these methods and is most suitable for tackling the above questions. Basically, the fundamental solution for the Laplacian is used to derive HS-type inequalities.

Here we adopt this method to obtain a general HS-type inequality to answer the two questions in the case of the Heisenberg group. The HS-type inequality for the sub-Laplacian of the Heisenberg group was obtained in [18]. Here we extend this to the p-sub-Laplacian. The advantage of the method we are using is that, even for the standard $|\nabla u|^2$, it gives far more information than the method of (1.1) (see §§ 4.2 and 4.4). This inequality combines both the interior and the boundary HS inequalities (see § 4.2). This method is also applicable for deriving HS-type inequalities for polyharmonic operators.

1.1. Motivation

Before stating the main results, we will illustrate the proof of the classical HS inequality using the fundamental solution of the Laplacian. This is the main philosophy we adopt to obtain our main results in the next section.

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, be a domain and let $0 \leqslant E$ be a fundamental solution of $-\Delta$, i.e.

$$\begin{aligned} -\Delta E &= C\delta_0, \\ E &> 0. \end{aligned}$$

Let $u \in C_0^1(\Omega)$ and define $v = E^{-1/2}u$. Since $E(0) = \infty$, we have $v(0) = 0$ and $u = E^{1/2}v$. Hence,

$$\begin{aligned} \nabla u &= \left(\frac{1}{2}\frac{\nabla E}{E} + \frac{\nabla v}{v}\right)u \\ \int_\Omega |\nabla u|^2 &= \frac{1}{4}\int_\Omega \left|\frac{\nabla E}{E}\right|^2 u^2 + \int_\Omega \frac{\nabla E \cdot \nabla v}{Ev}u^2 + \int_\Omega \frac{|\nabla v|^2}{v^2}u^2 \\ &= \frac{1}{4}\int_\Omega \left|\frac{\nabla E}{E}\right|^2 u^2 + \int_\Omega v(\nabla E \cdot \nabla v) + \int_\Omega |\nabla v|^2 E. \end{aligned}$$

The main point of this calculation is the vanishing of the middle term, namely,

$$\int_\Omega v(\nabla E \cdot \nabla v) = \int_\Omega \tfrac{1}{2}\nabla E \cdot \nabla v^2 = \tfrac{1}{2}Cv^2(0),$$

since E is a fundamental solution. In the radial case this was called a 'magical cancellation' by Brezis and Vazquez [8]; this is merely the property of the fundamental solution, and no symmetrization argument is required in this calculation:

$$\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}u^2 = \int_\Omega |\nabla v|^2 E.$$

Now take

$$E = \frac{1}{|x|^{n-2}}.$$

Then

$$\left|\frac{\nabla E}{E}\right|^2 = \frac{(n-2)^2}{|x|^2}$$

and we recover the classical HS inequality,

$$\int_\Omega |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} = \int_\Omega |\nabla v|^2 E \geqslant 0,$$

and

$$\left(\frac{n-2}{2}\right)^2$$

is the best constant and is never achieved.

1.2. Perspectives

We extend the HS inequalities for general second-order elliptic operators in the divergent form. We also extend it to the case of the sub-Laplacian coming from the Heisenberg group. Finally, note how to extend these equalities on general Riemann manifolds.

2. Main results

Let $1 < p \leqslant n$ and $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $A = ((a_{ij}(x)))$ be a symmetric positive definite matrix with $a_{ij} \in C^1(\bar{\Omega})$. For $u, v \in C^1(\bar{\Omega})$, define the gradient norm associated to A by

$$a(u,v) = \sum_{1\leqslant i,j\leqslant n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \tag{2.1}$$

$$|\nabla u|_A^2 = \sum_{1\leqslant i,j\leqslant n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{2.2}$$

Let

$$L_p(u) = -\sum_{1\leqslant i,j\leqslant n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) |\nabla u|_A^{p-2} \frac{\partial u}{\partial x_i} \right) \tag{2.3}$$

and let E_p be a fundamental solution of L_p [7, 12, 13] given by

$$\left.\begin{aligned} L_p E_p &= \delta_0 \quad \text{in } \Omega, \\ E_p &= 0 \quad \text{in } \partial\Omega. \end{aligned}\right\} \tag{2.4}$$

Then, by the maximum principle and regularity results of [10, 20, 21], it follows that there exists a σ, $0 < \sigma < 1$,

$$E_p \in C^{1,\sigma}_{\text{loc}}(\bar{\Omega} \setminus (0)), \qquad E_p > 0 \quad \text{in } \Omega \setminus (0), \qquad E_p(0) = \infty. \tag{2.5}$$

As in [5, 6, 11], for $0 \leqslant s \leqslant 1$, define

$$h_1(s) = \left(1 + \log\frac{1}{s}\right)^{-1}, \tag{2.6}$$

$$h_k(s) = h_1(h_{k-1}(s)), \tag{2.7}$$

$$\eta_k(s) = h_1(s) \dots h_k(s). \tag{2.8}$$

Let $0 \in \Omega_1 \subset\subset \Omega$ and $R > 0$ be such that

$$\Sigma_R = \{x : E_p(x) = R\} \tag{2.9}$$

is a Lipschitz manifold of dimension $n-1$. Define

$$\rho(x) = \max\left\{E_p(x), \frac{R^2}{E_p(x)}\right\}, \tag{2.10}$$

$$m = \min_{\Omega_1} E_p. \tag{2.11}$$

We then have the following theorem.

THEOREM 2.1. *Let $1 < p \leqslant n$ be fixed and define $E = E_p$. Let $0 \in \Omega_1 \subset\subset \Omega$, Σ_R, ρ and m be as defined above. There then exists a constant $C = C(p,n) > 0$ such that, for any $k \in Z$, $k \geqslant 0$ and for all $u \in W_0^{1,p}(\Omega_1)$, we have the generalized HS inequality*

$$\int_\Omega |\nabla u|_A^p - \left(\frac{p-1}{p}\right)^p \int_\Omega \left|\frac{\nabla E}{E}\right|_A^p |u|^p \geqslant 0. \tag{2.12}$$

For the remainder-term estimate we have that, if $2 \leqslant p \leqslant n$, then

$$\int_\Omega |\nabla u|_A^p - \left(\frac{p-1}{p}\right)^p \int_\Omega \left|\frac{\nabla E}{E}\right|_A^p |u|^p \geqslant C \sum_{i=1}^k \int_\Omega \eta_i \left(\frac{R}{\rho}\right)^2 \left|\frac{\nabla E}{E}\right|_A^p |u|^p - \frac{k}{R^{p-1}} \int_{\Sigma_R} |\nabla E|^{p-1} |u|^p. \tag{2.13}$$

If $1 < p < n$, then, for all $u \in W_0^{1,p}(\Omega)$, we have

$$\int_{\Omega_1} |\nabla u|_A^p - \left(\frac{p-1}{p}\right)^p \int_{\Omega_1} \left|\frac{\nabla E}{E}\right|_A^p |u|^p \geqslant C \sum_{i=1}^k \int_{\Omega_1} \eta_i \left(\frac{m}{E}\right)^2 \left|\frac{\nabla E}{E}\right|_A^p |u|^p. \tag{2.14}$$

Next we generalize the HS inequality to the sub-Laplacian operator defined on the Heisenberg group $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Here the sub-Laplacian is a hypoelliptic operator and the corresponding gradient norm is given by the sum of the squares of left-invariant vector fields (for details see [19]). In order to state the main result, we now recall some definitions, notation and properties related to the Heisenberg group:

$$\mathbb{H}^n = \{\psi = (x,y,t) \mid x, y \in \mathbb{R}^n,\ t \in \mathbb{R}\}, \tag{2.15}$$

$$z = x + \mathrm{i}y, \quad |z|^2 = |x|^2 + |y|^2,$$

$$d(\psi) = (|z|^4 + t^2)^{1/4}. \tag{2.16}$$

Let $\psi_1 = (x_1, y_1, t_1)$ and $\psi_2 = (x_2, y_2, t_2)$. Then the group law is defined as

$$\psi_1 \dot{\psi}_2 = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + \langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

The left-invariant vector fields are given by

$$\left.\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad 1 \leqslant j \leqslant n, \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad 1 \leqslant j \leqslant n, \\ T &= \frac{\partial}{\partial t}. \end{aligned}\right\} \tag{2.17}$$

Let $\Omega \subset \mathbb{H}^n$ be a domain. For $u \in C^1(\Omega)$ define the subgradient $\nabla_{\mathbb{H}}(u)$ by

$$\nabla_{\mathbb{H}}(u) = (X_1(u), \dots, X_n(u), Y_1(u), \dots, Y_n(u)), \tag{2.18}$$

$$|\nabla_{\mathbb{H}}(u)|^2 = \sum_{j=1}^{n} (|X_j(u)|^2 + |Y_j(u)|^2). \tag{2.19}$$

Let $1 < p < \infty$, and define the sub-Laplacian L_p as follows. Let $u \in C^2(\Omega)$. Then

$$L_p u = -\sum_{j=1}^{n} \left[X_j \left(\left| \frac{\nabla_{\mathbb{H}}(u)}{|z|} \right|^{p-2} X_j u \right) + Y_j \left(\left| \frac{\nabla_{\mathbb{H}}(u)}{|z|} \right|^{p-2} Y_j u \right) \right]. \tag{2.20}$$

2.1. Weighted Folland–Stein spaces, $FS_0^{1,p}(\Omega)$

Let $1 \leqslant p < \infty$ and let $\Omega \subset \mathbb{H}^n$ be an open set. For $u \in C_0^\infty(\Omega)$, define the norm:

$$|u|_{1,p}^p = \int_\Omega \frac{|\nabla_{\mathbb{H}}(u)|^p}{|z|^{p-2}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t. \tag{2.21}$$

Define $FS_0^{1,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the norm (2.21).

Before starting on the main result, we recall some properties of L_p, and they will explain why the weight $|z|^{2-p}$ has to be taken in the definition of $FS_0^{1,p}(\Omega)$.

Let $1 < p < \infty$ and $R > 0$. Define

$$f(z,t) = |z|^4 + t^2, \tag{2.22}$$

$$E_p(z,t) = \begin{cases} f^{(n+2-p)/2(p-1)} & \text{if } p \neq n+2, \\ \log\left(\dfrac{R}{f}\right) & \text{if } p = n+2. \end{cases} \tag{2.23}$$

We then have the following theorem.

THEOREM 2.2. *Let $1 < p < n+2$. Then, for all $u \in FS_0^{1,p}$, the HS-type inequality is given by*

$$\int_{\mathbb{H}^n} \frac{|\nabla_{\mathbb{H}}(u)|^p}{|z|^{p-2}} - \left(\frac{2(n+2-p)}{p} \right)^p \int_{\mathbb{H}^n} \frac{|z|^2 |u|^p}{(|z|^4 + t^2)^{p/2}} \geqslant 0, \tag{2.24}$$

and

$$\left(\frac{2(n+2-p)}{p} \right)^p$$

is the best constant and it is never achieved. Furthermore, if $0 \in \Omega \subset \mathbb{H}^n$ is a bounded domain and $f < R$ on $\bar{\Omega}$, then there exists a $C > 0$ for all $u \in FS_0^{1,p}(\Omega)$, such that the following conditions hold.

(i) *Let $2 \leqslant p < n+2$. Then*

$$\int_\Omega \frac{|\nabla_{\mathbb{H}}(u)|^p}{|z|^{p-2}} - \left(\frac{2(n+2-p)}{p} \right)^p \int_\Omega \frac{|z|^2 |u|^p}{f^{p/2}} \geqslant C \sum_{i=1}^{\infty} \int_\Omega \frac{\eta_i^2(R/f) |z|^2 |u|^p}{f^{p/2}}. \tag{2.25}$$

(ii) *Let $p = n + 2$. Then*

$$\int_\Omega \frac{|\nabla_{\mathbb{H}}(u)|^p}{|z|^{p-2}} - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|z|^2|u|^p}{(\log(R/f))^p f^{p/2}} \geqslant C \sum_{i=1}^{\infty} \int_\Omega \frac{\eta_i^2(R/f)|z|^2|u|^p}{(\log(R/f))^p f^{p/2}}. \tag{2.26}$$

(iii) *Let $1 < p \leqslant 2$, and $k > 0$. Then there exists a $C(k)$ such that*

$$\int_\Omega \frac{|\nabla_{\mathbb{H}}(u)|^p}{|z|^{p-2}} - \left(\frac{2(n+2-p)}{p}\right)^p \int_\Omega \frac{|z|^2|u|^p}{f^{p/2}} \geqslant C \sum_{i=1}^{k} \int_\Omega \frac{\eta_i^2(R/f)|z|^2|u|^p}{f^{p/2}}. \tag{2.27}$$

3. Proof of the theorems

We need to prove some preliminary lemmas first.

LEMMA 3.1. *Let $1 < p \leqslant n$, $0 \in \Omega \subset \mathbb{R}^n$, be a bounded domain with smooth boundary. Let A, E_p, R, h_k, η_k and Σ_R be as defined in (2.5)–(2.9). With abuse of notation, we use $E = E_p$, $|\cdot|_A = |\cdot|$. Let $\omega_1 \in C^1(\bar{\Omega})$ and define $\{\omega_k\}_{k\geqslant 2}$ inductively by*

$$\omega_k(x) = h_k^{-1/2}\left(\frac{R}{\rho(x)}\right)\omega_{k+1}. \tag{3.1}$$

Then

$$\begin{aligned}\int_\Omega \eta_{k-1}^{-1}\left(\frac{R}{\rho(x)}\right)\left|\frac{\nabla E}{E}\right|^{p-2} E^{p-1}|\nabla\omega_k|^2 &= \frac{1}{4}\int_\Omega \eta_k^2\left(\frac{R}{\rho(x)}\right)\left|\frac{\nabla E}{E}\right|^{p} E^{p-1}\omega_1^2 - \int_{\Sigma_R} |\nabla E|^{p-1}\omega_1^2 \\ &\quad - \frac{1}{2}\int_{\partial\Omega} \eta_k\left(\frac{R}{\rho(x)}\right)|\nabla E|^{p-2}\langle\nabla E, \nu\rangle\omega_1^2 \\ &\quad + \int_\Omega \eta_k^{-1}\left(\frac{R}{\rho(x)}\right)\left|\frac{\nabla E}{E}\right|^{p-2} E^{p-1}|\nabla\omega_{k+1}|^2.\end{aligned} \tag{3.2}$$

Proof. From the definition we have the following identities.

$$\frac{h_k'(s)}{h_k(s)} = \frac{\eta_k(s)}{s}, \tag{3.3}$$

$$\frac{\nabla\rho}{\rho} = \begin{cases} \dfrac{\nabla E}{E} & \text{if } E > R, \\ -\dfrac{\nabla E}{E} & \text{if } E < R. \end{cases} \tag{3.4}$$

On Σ_R, we have $\rho = R$ and hence

$$\omega_1|_{\Sigma_R} = \omega_2|_{\Sigma_R} = \cdots = \omega_k|_{\Sigma_R}, \tag{3.5}$$

$$\omega_k^2 = \eta_{k-1}\left(\frac{R}{\rho}\right)\omega_1^2. \tag{3.6}$$

From (3.3)–(3.6) we get

$$\begin{aligned}
\frac{\nabla\omega_k}{\omega_k} &= \left(\frac{1}{2}\right)\frac{h_k'(R/\rho)}{h_k(R/\rho)}\frac{R}{\rho^2}\nabla\rho + \frac{\nabla\omega_{k+1}}{\omega_{k+1}} \\
&= \tfrac{1}{2}\eta_k\left(\frac{R}{\rho}\right)\frac{\nabla\rho}{\rho} + \frac{\nabla\omega_{k+1}}{\omega_{k+1}}, \\
|\nabla\omega_k|^2 &= \tfrac{1}{4}\eta_k^2\left(\frac{R}{\rho}\right)\left|\frac{\nabla E}{E}\right|^2\omega_k^2 + \frac{\eta_k(R/\rho)\langle\nabla\rho,\nabla\omega_{k+1}\rangle\omega_k^2}{\rho\omega_{k+1}} \\
&\quad + \left(\frac{\omega_k}{\omega_{k+1}}\right)^2|\nabla\omega_{k+1}|^2, \\
\eta_{k-1}^{-1}\left(\frac{R}{\rho}\right)|\nabla\omega_k|^2 &= \tfrac{1}{4}\eta_k^2\left(\frac{R}{\rho}\right)\left|\frac{\nabla E}{E}\right|^2\omega_1^2 + \frac{1}{2}\left\langle\frac{\nabla\rho}{\rho},\nabla\omega_{k+1}^2\right\rangle \\
&\quad + \eta_k^{-1}\left(\frac{R}{\rho}\right)|\nabla\omega_{k+1}|^2.
\end{aligned}$$

Let ν_0 denote the exterior normal on the boundary of $E > R$. This is given by $\nu_0 = -\nabla E/|\nabla E|$. Then

$$\begin{aligned}
&\int_\Omega\left|\frac{\nabla E}{E}\right|^{p-2}E^{p-1}\left\langle\frac{\nabla\rho}{\rho},\nabla\omega_{k+1}^2\right\rangle \\
&\quad = \int_{E>R}|\nabla E|^{p-2}\langle\nabla E,\nabla\omega_{k+1}^2\rangle - \int_{E<R}|\nabla E|^{p-2}\langle\nabla E,\nabla\omega_{k+1}^2\rangle \\
&\quad = \int_{E>R}(L_pE)\omega_{k+1}^2 - 2\int_{\Sigma_R}|\nabla E|^{p-1}\omega_{k+1}^2 - \int_{E<R}(L_pE)\omega_{k+1}^2 \\
&\qquad - \int_{\partial\Omega}|\nabla E|^{p-2}\langle\nabla E,\nu\rangle\omega_{k+1}^2 \\
&\quad = \omega_{k+1}^2(0) - 2\int_{\Sigma_R}|\nabla E|^{p-1}\omega_1^2 - \int_{\partial\Omega}|\nabla E|^{p-2}\langle\nabla E,\nu\rangle\omega_{k+1}^2.
\end{aligned}$$

Since $h_k(0) = 0$ and $E(0) = \infty$, we have $\omega_{k+1}(0) = 0$. Hence, from the above identity, we have

$$\begin{aligned}
&\int_\Omega\eta_{k-1}^{-1}\left(\frac{R}{\rho}\right)\left|\frac{\nabla E}{E}\right|^{p-2}E^{p-1}|\nabla\omega_k|^2 \\
&\quad = \frac{1}{4}\int_\Omega\eta_k^2\left(\frac{R}{\rho}\right)\left|\frac{\nabla E}{E}\right|^{p}E^{p-1}\omega_1^2 - \int_{\Sigma_R}|\nabla E|^{p-1}\omega_1^2 \\
&\qquad - \frac{1}{2}\int_{\partial\Omega}\eta_k\left(\frac{R}{\rho}\right)|\nabla E|^{p-2}\langle\nabla E,\nu\rangle\omega_1^2 \\
&\qquad + \int_\Omega\eta_k^{-1}\left(\frac{R}{\rho}\right)\left|\frac{\nabla E}{E}\right|^{p-2}E^{p-1}|\nabla\omega_{k+1}|^2. \qquad (3.7)
\end{aligned}$$

This proves the lemma. □

The proof of the following lemma follows exactly in the same manner as that of lemma 3.1. Hence, we state it without proof.

LEMMA 3.2. *Let $0 \in \Omega_1 \subset\subset \Omega$ and $1 < p \leqslant n$. With the same notation as in the previous lemma, for $\omega_1 \in C^1(\bar{\Omega})$, define the new sequence,*

$$\omega_k(x) = h_k^{-1/2}\left(\frac{m}{E}\right)\omega_{k+1}(x), \tag{3.8}$$

where $m = \inf_{\bar{\Omega}} E$. Then

$$\begin{aligned}\int_{\Omega_1} \eta_{k-1}^{-1}\left(\frac{m}{E}\right)\left|\frac{\nabla E}{E}\right|^{p-2} E^{p-1}|\nabla\omega_k|^2 \\ = \frac{1}{4}\int_{\Omega_1} \eta_k^2\left(\frac{m}{E}\right)\left|\frac{\nabla E}{E}\right|^{p} E^{p-1}\omega_1^2 + \frac{1}{2}\int_{\partial\Omega} \eta_k\left(\frac{m}{E}\right)|\nabla E|^{p-2}\langle\nabla E, \nu\rangle\omega_1^2 \\ + \int_{\Omega_1} \eta_k^{-1}\left(\frac{m}{E}\right)\left|\frac{\nabla E}{E}\right|^{p-2} E^{p-1}|\nabla\omega_{k+1}|^2.\end{aligned} \tag{3.9}$$

We now recall the following elementary inequality (see, for example, [4]): let $1 < p < \infty$ and $x \in \bar{\Omega}$. For $\alpha, \beta \in \mathbb{R}^n$, define

$$\langle\alpha, \beta\rangle_A = \sum_{1\leqslant i,j\leqslant n} a_{ij}(x)\alpha_i\beta_j, \tag{3.10}$$

$$|\alpha|_A^p = (\langle\alpha, \alpha\rangle_A)^{p/2}. \tag{3.11}$$

Then, given $M > 1$, there exist positive constants μ_1 and μ_2, such that, for all $\alpha, \beta \in \mathbb{R}^n$, $x \in \bar{\Omega}$ with $|\alpha|_A = 1$, we have

$$|\alpha + \beta|_A^p - 1 - p\langle\alpha, \beta\rangle_A \geqslant \mu_1|\beta|_A^2 + \mu_2|\beta|_A^p \quad \text{if } 2 \leqslant p < \infty, \tag{3.12}$$

$$|\alpha + \beta|_A^p - 1 - p\langle\alpha, \beta\rangle_A \geqslant \begin{cases} \dfrac{\mu_1}{M^2}|\beta|_A^2 & \text{if } |\beta|_A \leqslant M,\ 1 < p \leqslant 2, \\ \dfrac{\mu_2}{M^p}|\beta|_A^p & \text{if } |\beta|_A \geqslant M,\ 1 < p \leqslant 2. \end{cases} \tag{3.13}$$

Let

$$B(\beta) = \begin{cases} \mu_1|\beta|_A^2 + \mu_2|\beta|_A^p & \text{if } 2 \leqslant p < \infty, \\ \dfrac{\mu_1}{M^2}|\beta|_A^2 & \text{if } |\beta|_A \leqslant M,\ 1 < p \leqslant 2, \\ \dfrac{\mu_2}{M^p}|\beta|_A^p & \text{if } |\beta|_A \geqslant M,\ 1 < p \leqslant 2. \end{cases} \tag{3.14}$$

Proof of theorem 2.1. Let $1 < p \leqslant n$ and E_p be the fundamental solution of L_p. Let $0 \leqslant u \in C_0^1(\bar{\Omega})$ and define $v = E_p^{-(p-1)/p}u$ Then $v(0) = 0$, $v|_{\partial\Omega} = 0$. For the sake of notational simplification, denote $E = E_p$ and $|\cdot| = |\cdot|_A$. We then have

$$\frac{\nabla u}{u} = \frac{p-1}{p}\frac{\nabla E}{E} + \frac{\nabla v}{v}.$$

and hence from (3.12)–(3.14) we have

$$\begin{aligned}|\nabla u|^p &= u^p\left|\frac{p-1}{p}\frac{\nabla E}{E}+\frac{\nabla v}{v}\right|^p\\ &= \left(\frac{p-1}{p}\right)^p u^p\left|\frac{\nabla E}{E}\right|^p\left|\frac{\nabla E}{|\nabla E|}+\frac{p}{p-1}\frac{E}{|\nabla E|}\frac{\nabla v}{v}\right|^p\\ &\geqslant \left(\frac{p-1}{p}\right)^p u^p\left|\frac{\nabla E}{E}\right|^p\left\{1-\frac{p^2}{p-1}\frac{E}{|\nabla E|^2}\langle\nabla E,\nabla v\rangle+B\left(\frac{p}{p-1}\frac{E}{|\nabla E|}\frac{\nabla v}{v}\right)\right\}.\end{aligned}\tag{3.15}$$

Hence,

$$\int_\Omega|\nabla u|^p-\left(\frac{p-1}{p}\right)^p\int_\Omega\left|\frac{\nabla E}{E}\right|^p u^p\geqslant 0,$$

and equal to zero if and only if $u\equiv 0$. This proves (2.12).

Let $2\leqslant p\leqslant n$. Then, from (3.12), we have, for some constant $\mu_1>0$,

$$\begin{aligned}\int_\Omega u^p\left|\frac{\nabla E}{E}\right|^p B\left(\frac{p}{p-1}\frac{E}{|\nabla E|}\frac{\nabla v}{v}\right) &\geqslant \frac{4\mu_1}{p^2}\int_\Omega u^p\left|\frac{\nabla E}{E}\right|^p\left|\frac{E}{|\nabla E|}\right|^2\left|\frac{\nabla v}{v}\right|^2\\ &= \mu_1\int_\Omega\left|\frac{\nabla E}{E}\right|^{p-2}E^{p-1}\left|\nabla v^{p/2}\right|^2.\end{aligned}$$

Now let $\omega_1=v^{p/2}$ and define

$$\omega_k=h_k^{-1/2}\left(\frac{R}{\rho}\right)\omega_{k+1}.$$

Since $v=u=0$ on $\partial\Omega$, $\omega_1=0$ on $\partial\Omega$. Hence, from lemma 3.1, for any k we obtain

$$\begin{aligned}&\int_\Omega u^p\left|\frac{\nabla E}{E}\right|^p B\left(\frac{p}{p-1}\frac{E}{|\nabla E|}\frac{\nabla v}{v}\right)\\ &\qquad\geqslant \tfrac{1}{4}\mu_1\int_\Omega\sum_{i=1}^k\eta_i^2\left(\frac{R}{\rho}\right)\left|\frac{\nabla E}{E}\right|^p u^p-\frac{k}{R^{p-1}}\int_{\Sigma_R}|\nabla E|^{p-1}u^p.\end{aligned}\tag{3.16}$$

Combining this with (3.15) proves (2.13). Again, with the same method as above (for $2\leqslant p\leqslant n$), (2.14) follows from lemma 3.2.

Let $1<p\leqslant 2$ and $0\leqslant u\in W_0^{1,p}(\Omega)$. Let $M>0$, $v=E^{-(p-1)/p}u$. Then, as in (3.13), there exist constants μ_1 and μ_2 such that

$$\begin{aligned}&\int_{\Omega_1}|\nabla u|^p-\left(\frac{p-1}{p}\right)^p\int_{\Omega_1}\left|\frac{\nabla E}{E}\right|^p u^p\\ &\qquad\geqslant\frac{\mu_1}{M^2}\int_{\Omega_1^+}\left|\frac{\nabla E}{E}\right|^{p-2}E^{p-1}|\nabla v^{p/2}|^2+\frac{\mu_2}{M^p}\int_{\Omega_1^-}E^{p-1}|\nabla v|^p,\end{aligned}\tag{3.17}$$

where

$$\Omega_1^+=\left\{\frac{E}{v}\frac{|\nabla v|}{|\nabla E|}\leqslant M\right\}\quad\text{and}\quad\Omega_1^-=\left\{\frac{E}{v}\frac{|\nabla v|}{|\nabla E|}\geqslant M\right\}.$$

Now regularize E and v by E_ε, v_ε such that $E_\varepsilon, v_\varepsilon \in C^\infty(\Omega)$; $E_\varepsilon > 0$, $v_\varepsilon > 0$ and, as $\varepsilon \to 0$, $E_\varepsilon \to E$ almost everywhere and $v_\varepsilon \to v$ in $C^1(\bar{\Omega}_1)$. As $\varepsilon \to 0$,

$$L_p E_\varepsilon = \delta_0 + o(1). \tag{3.18}$$

Now choose $M > 1$ such that M^2 is a regular value of

$$\frac{E_\varepsilon^2}{|\nabla E_\varepsilon|^2} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}.$$

By perturbing Ω_1 to Ω_ε such that $\Omega_\varepsilon \to \Omega_1$, $0 \in \Omega_\varepsilon^+$, and $\partial\Omega_\varepsilon$ is transversal to Γ, where

$$\begin{aligned}
\Omega_\varepsilon^+ &= \{x \in \bar{\Omega}_\varepsilon : E_\varepsilon^2 |\nabla v_\varepsilon|^2 < M^2 |\nabla E_\varepsilon|^2 v_\varepsilon^2\},\\
\Omega_\varepsilon^- &= \{x \in \bar{\Omega}_\varepsilon : E_\varepsilon^2 |\nabla v_\varepsilon|^2 > M^2 |\nabla E_\varepsilon|^2 v_\varepsilon^2\},\\
\Gamma &= \{x \in \bar{\Omega}_\varepsilon : E_\varepsilon^2 |\nabla v_\varepsilon|^2 = M^2 |\nabla E_\varepsilon|^2 v_\varepsilon^2\},\\
m_\varepsilon &= \inf_{\bar{\Omega}_\varepsilon} E_\varepsilon.
\end{aligned}$$

Let ν^+ and ν^- denote the unit outward normals to $\partial\Omega_\varepsilon^+$ and $\partial\Omega_\varepsilon^-$, respectively, with respect to the common boundary Γ. Then $\nu^+ = -\nu^-$. From (3.17) and (3.18) we now have

$$\begin{aligned}
\int_{\Omega_\varepsilon^+} &\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^{p-2} E_\varepsilon^{p-1} |\nabla v_\varepsilon^{p/2}|^2\\
&\geqslant \frac{1}{4}\sum_{i=1}^k \int_{\Omega_\varepsilon^+} \eta_i^2\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^p E_\varepsilon^{p-1} v_\varepsilon^p\\
&\quad + \frac{1}{2}\sum_{i=1}^k \int_\Gamma \eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right) |\nabla E_\varepsilon|^{p-2}\langle \nabla E_\varepsilon, \nu^+\rangle v_\varepsilon^p\\
&\quad + \frac{1}{2}\sum_{i=1}^k \int_{\partial\Omega_\varepsilon^+ \cap \partial\Omega_\varepsilon} \eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right) |\nabla E_\varepsilon|^{p-2}\langle \nabla E_\varepsilon, \nu\rangle v_\varepsilon^p + o(1).
\end{aligned}\tag{3.19}$$

Now from (3.3) we have

$$\begin{aligned}
s\eta_i' &= (h_1 + h_1 h_2 + \cdots + h_1 h_i)\eta_i\\
&\geqslant \eta_i^2(s).
\end{aligned}\tag{3.20}$$

This gives us

$$\begin{aligned}
\int_\Gamma &\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon, \nu^+\rangle v_\varepsilon^p\\
&= -\int_\Gamma \eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon, \nu^-\rangle v_\varepsilon^p\\
&= \int_{\partial\Omega_\varepsilon^- \cap \partial\Omega_\varepsilon} \eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon, \nu\rangle v_\varepsilon^p\\
&\qquad - \int_{\Omega_\varepsilon^-} |\nabla E_\varepsilon|^{p-2}\left\langle \nabla E_\varepsilon, \nabla\left(\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)v_\varepsilon^p\right)\right\rangle + o(1)
\end{aligned}$$

$$\geqslant -\int_{\partial\Omega_\varepsilon^-\cap\partial\Omega}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nu\rangle v_\varepsilon^p + \int_{\Omega_\varepsilon^-}\eta_i^2\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\frac{|\nabla E_\varepsilon|^p}{E_\varepsilon}v_\varepsilon^p - p\int_{\Omega_\varepsilon^-}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nabla v_\varepsilon\rangle v_\varepsilon^{p-1}+o(1).$$

Substituting this into (3.19), we obtain

$$\int_{\Omega_\varepsilon^+}\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^{p-2}E_\varepsilon^{p-1}|\nabla v_\varepsilon^{p/2}|^2 = \frac{1}{4}\sum_{i=1}^k\int_{\Omega_\varepsilon}\eta_i^2\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^{p-2}E_\varepsilon^{p-1}v_\varepsilon^p + \frac{1}{2}\sum_{i=1}^k\int_{\partial\Omega_\varepsilon}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nu\rangle v_\varepsilon^p - \tfrac{1}{2}p\sum_{i=1}^k\int_{\Omega_\varepsilon^-}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nabla v_\varepsilon\rangle v_\varepsilon^{p-1}+o(1).$$

Hence,

$$\begin{aligned}
&\frac{\mu_1}{M^2}\int_{\Omega_\varepsilon^+}\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^{p-2}E_\varepsilon^{p-1}|\nabla v_\varepsilon^{p/2}|^2 + \frac{\mu_2}{M^p}\int_{\Omega_\varepsilon^-}E_\varepsilon^{p-1}|\nabla v_\varepsilon|^p + o(1)\\
&\geqslant \frac{\mu_1}{4M^2}\sum_{i=1}^k\int_{\Omega_\varepsilon^-}\eta_i^2\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^p E_\varepsilon^{p-1}v_\varepsilon^p\\
&\quad + \frac{p\mu_1}{2M^2}\sum_{i=1}^k\int_{\partial\Omega_\varepsilon}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nu\rangle v_\varepsilon^p\\
&\quad + \int_{\Omega_\varepsilon^-}\left\{\frac{\mu_2}{M^p}E_\varepsilon^{p-1}|\nabla v_\varepsilon|^p - \frac{p\mu_1}{2M^2}\left(\sum_{i=1}^k\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nabla v_\varepsilon\rangle v_\varepsilon^{p-1}\right\}\\
&\geqslant \frac{\mu_1}{4M^2}\sum_{i=1}^k\int_{\Omega_\varepsilon}\eta_i^2\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^p E_\varepsilon^{p-1}v_\varepsilon^p\\
&\quad + \frac{\mu_1}{4M^2}\sum_{i=1}^k\int_{\partial\Omega_\varepsilon}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nu\rangle v_\varepsilon^p\\
&\quad + \int_{\Omega_\varepsilon^-}\left\{\frac{\mu_2}{M^p}-\frac{k\mu_1}{2M^{p+1}}\right\}E_\varepsilon^{p-1}|\nabla v_\varepsilon|^p\\
&\geqslant \frac{\mu_1}{4M^2}\sum_{i=1}^k\int_{\Omega_\varepsilon}\eta_i^2\left(\frac{m_\varepsilon}{E_\varepsilon}\right)\left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^p E_\varepsilon^{p-1}v_\varepsilon^p\\
&\quad + \frac{\mu_1}{4M^2}\sum_{i=1}^k\int_{\partial\Omega_\varepsilon}\eta_i\left(\frac{m_\varepsilon}{E_\varepsilon}\right)|\nabla E_\varepsilon|^{p-2}\langle\nabla E_\varepsilon,\nu\rangle v_\varepsilon^p,
\end{aligned}$$

provided that $M \geqslant (pk\mu_1)/2\mu_2$. Now, from (3.17) and the above inequality, given that $M \geqslant (k\mu_1)/2\mu_2$, we have

$$\begin{aligned}\int_{\Omega_1} |\nabla u|^p - \left(\frac{p-1}{p}\right)^p \int_{\Omega_1} \left|\frac{\nabla E}{E}\right|^p u^p + o(1) \\ \geqslant \frac{\mu_1}{M^2} \int_{\Omega_\varepsilon^+} \left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^{p-2} E_\varepsilon^{p-1} |\nabla v_\varepsilon^{p/2}|^2 + \frac{\mu_2}{M^p} \int_{\Omega_\varepsilon^-} E_\varepsilon^{p-1} |\nabla v_\varepsilon|^p \\ \geqslant \frac{\mu_1}{4M^2} \sum_{i=1}^{k} \int_{\Omega_\varepsilon} \eta_i^2 \left(\frac{m_\varepsilon}{E_\varepsilon}\right) \left|\frac{\nabla E_\varepsilon}{E_\varepsilon}\right|^p E_\varepsilon^{p-1} v_\varepsilon^p \\ + \frac{\mu_1}{4M^2} \sum_{i=1}^{k} \int_{\partial\Omega_\varepsilon} \eta_i \left(\frac{m_\varepsilon}{E_\varepsilon}\right) |\nabla E_\varepsilon|^{p-2} \langle \nabla E_\varepsilon, \nu\rangle v_\varepsilon^p.\end{aligned}$$

Now letting $\varepsilon \to 0$ and using the fact that $v|_{\partial\Omega_1} = 0$, we obtain the desired inequality (2.14).

In order to prove theorem 2.2, we need to obtain an analogous lemma to lemma 3.1 for the Heisenberg group.

LEMMA 3.3. *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain and $R > 0$ such that $f < R$ in $\bar{\Omega}$. Let $1 < p \leqslant n+2$ and L_p and E_p be defined as in (2.20) and (2.22), respectively. There then exists a constant $C_p \in \mathbb{R}$ such that the following conditions hold.*

(i) *We set*

$$L_p E_p = C_p \delta_0. \tag{3.21}$$

(ii) *In $\mathbb{H}^n \setminus (0)$, if $p < n+2$ and in $\Omega \setminus (0)$, if $p = n+2$, we see that $W_p = E_p^{(p-1)/p}$ satisfies*

$$L_p W_p + \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla_{\mathbb{H}} E_p}{E_p}\right|^p \frac{W_p^{p-1}}{|z|^{p-2}} = 0. \tag{3.22}$$

(iii) *Let $\omega_1 \in C^1(\bar{\Omega})$ and define ω_k inductively by*

$$\omega_k = h_k^{-1/2}\left(\frac{R}{f}\right) \omega_{k+1}. \tag{3.23}$$

Then

$$\begin{aligned}\int_\Omega \eta_{k-1}^{-1}\left(\frac{R}{f}\right) \left|\frac{\nabla_{\mathbb{H}} E_p}{E_p}\right|^{p-2} E_p^{p-1} \frac{|\nabla_{\mathbb{H}}(\omega_k)|^2}{|z|^{p-2}} \\ = \frac{1}{2} \int_\Omega \eta_k^{-1}\left(\frac{R}{f}\right) \left|\frac{\nabla_{\mathbb{H}} E_p}{E_p}\right|^{p-2} E_p^{p-1} \frac{|\nabla_{\mathbb{H}} \omega_{k+1}|^2}{|z|^{p-2}} \\ + \frac{1}{4} \int_\Omega \eta_k^2\left(\frac{R}{f}\right) \left|\frac{\nabla_{\mathbb{H}} E_p}{E_p}\right|^{p} E_p^{p-1} \frac{\omega_1}{|z|^{p-2}} \\ + \frac{1}{2} \int_{\partial\Omega} \eta_k\left(\frac{R}{f}\right) |\nabla_{\mathbb{H}} E_p|^{p-2} \langle \nabla_{\mathbb{H}} E_p, \nu_{\mathbb{H}}\rangle \frac{\omega_1^2}{|z|^{p-2}},\end{aligned} \tag{3.24}$$

where $\nu_{\mathbb{H}}$ is the normal corresponding to $\partial\Omega$ associated to the sub-Laplacian L_2.

Proof of lemma 3.3. By direct calculations, we have the following identities.

Let $f(x,y,t) = (|x|^2+|y|^2)^4 + t^2 = |z|^4 + t^2$. Then,

$$\left.\begin{aligned} X_j f &= 4(x_j|z|^2 + y_j t),\\ Y_j f &= 4(y_j|z|^2 - x_j t),\\ |\nabla_{\mathbb{H}} f|^2 &= 16|z|^2 f. \end{aligned}\right\} \tag{3.25}$$

Let $F_0 = f^{-n/2}$. Then, by direct calculation (see [19]), there exists a $c \in \mathbb{R}$ such that

$$L_2 F_0 = C\delta_0. \tag{3.26}$$

There exists a $C_1 = C_1(n,p,\mathbb{R})$ such that, from (3.25),

$$\left|\frac{\nabla_{\mathbb{H}} E_p}{|z|}\right|^{p-2} X_j E_j = C_1 f^{-(n/2)-1}(X_j f) = C_1 X_j(f^{-n/2}). \tag{3.27}$$

Hence, from (3.26) we have

$$L_p F_p = C_1 C\delta_0. \tag{3.28}$$

This proves (3.21).

For $(x,y,t) \neq 0$ we have $0 < E_p \in C^\infty$. Hence,

$$\nabla_{\mathbb{H}} W_p = \left(\frac{p-1}{p}\right)\frac{W_p}{E_p}\nabla_{\mathbb{H}} E_p,$$

$$\left|\frac{\nabla_{\mathbb{H}} W_p}{|z|}\right|^{p-2} X_j W_p = \left(\frac{p-1}{p}\right)^{p-1} E_p^{(p-1)/p}\left|\frac{\nabla_{\mathbb{H}} E_p}{|z|}\right|^{p-2} X_j E_p.$$

Hence,

$$\begin{aligned} -L_p W_p &= \left(\frac{p-1}{p}\right)^{p} E_p^{(-2p+1)/p+p}\frac{|\nabla_{\mathbb{H}} E_p|^p}{|E_p|^p|z|^{p-2}} + \left(\frac{p-1}{p}\right)^{p-1} E_p^{-(p-1)/p} L_p E_p\\ &= \left(\frac{p-1}{p}\right)\left|\frac{\nabla_{\mathbb{H}} E_p}{E_p}\right|^p \frac{W_p^{p-1}}{|z|^{p-2}}. \end{aligned}$$

This proves (3.22). □

From (3.21), the proof of (3.24) follows exactly as in lemma 3.1 and hence we omit its proof. This proves the lemma. □

Proof of theorem 2.2. Let $1 < p \leqslant n+2$ and E_p, Ω and R be as defined in (2.22) with the condition that $f < R$ in $\bar{\Omega}$. For the sake of notational simplification we set $E = E_p$ and $\nabla_{\mathbb{H}} = \nabla$. Let $0 \leqslant u \in C_0^1(\mathbb{H}^n)$ if $p < n+2$ and $u \in C_0^1(\Omega)$ if $p = n+2$. Let $v = E^{-(p-1)/p}u$. Then $v \geqslant 0$ and $v(0) = 0$. Hence,

$$\frac{\nabla u}{u} = \left(\frac{p-1}{p}\right)\frac{\nabla E}{E} + \frac{\nabla v}{v}.$$

Then, from (3.12)–(3.14) we have

$$|\nabla u|^p \geqslant \left(\frac{p-1}{p}\right)^p u^p\left|\frac{\nabla E}{E}\right|^p\left\{1 - \frac{p^2}{p-1}\frac{E}{|\nabla E|^2}\left\langle\nabla E, \frac{\nabla v}{v}\right\rangle + B\left(\frac{p}{p-1}\frac{E}{|\nabla E|}\frac{\nabla v}{v}\right)\right\}.$$

Hence, for $p < n+2$,

$$\begin{aligned}\int_{\mathbb{H}^n} \frac{|\nabla u|^p}{|z|^{p-2}} - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^n} \left|\frac{\nabla E}{E}\right|^p \frac{u^p}{|z|^{p-2}} \\ \geqslant -\left(\frac{p-1}{p}\right)^{p-1} \int_{\mathbb{H}^n} \frac{|\nabla E|^{p-2}}{|z|^{p-2}} \langle \nabla E, \nabla v^p\rangle \\ + \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^n} \frac{u^p}{|z|^{p-2}} \left|\frac{\nabla E}{E}\right|^p B\left(\frac{p}{p-1}\frac{E}{|\nabla E|}\frac{\nabla v}{v}\right) \\ \geqslant 0. \end{aligned} \tag{3.29}$$

If $p = n+2$, we can replace $\mathbb{H}^n$ by Ω in the above inequality to obtain

$$\int_{\Omega} \frac{|\nabla u|^p}{|z|^{p-2}} - \left(\frac{p-1}{p}\right)^p \int_{\Omega} \left|\frac{\nabla E}{E}\right|^p \frac{u^p}{|z|^{p-2}} \geqslant 0. \tag{3.30}$$

From (3.25) and (2.22) we have

$$\left(\frac{p-1}{p}\right)^p \left|\frac{\nabla E}{E}\right|^p \frac{1}{|z|^{p-2}} = \begin{cases} \left(\dfrac{n+2-p}{2p}\right)^p \dfrac{|z|^2}{f^{p/2}} & \text{if } 1 < p < n+2, \\ \left(\dfrac{p-1}{p}\right)^p \dfrac{|z|^2}{(\log(R/f))^p f^{p/2}} & \text{if } p = n+2. \end{cases} \tag{3.31}$$

Substituting this in (3.30) gives (2.24). Next we claim that

$$\left(\frac{n+2-p}{2p}\right)^p$$

is the best constant in (2.24) if $p < n+2$ and that

$$\left(\frac{p-1}{p}\right)^p$$

is the best constant if $p = n+2$.

Let $1 < p < n+2$ and $\varepsilon > 0$, $R > 0$. Define W by

$$W = \begin{cases} \varepsilon^{-(n+2-p)/2p} & \text{if } f \leqslant \varepsilon, \\ f^{-(n+2-p)/2p} & \text{if } \varepsilon \leqslant f \leqslant R, \\ R^{-(n+2-p)/2p}\left(2 - \dfrac{f}{R}\right) & \text{if } R \leqslant f \leqslant 2R, \\ 0 & \text{if } f \geqslant 2R. \end{cases}$$

Then

$$\begin{aligned}\int_{\mathbb{H}^n} \frac{|\nabla W|^p}{|z|^{p-2}} &= \left(\frac{2(n+2-p)}{p}\right)^p \int_{\varepsilon \leqslant f \leqslant R} |z|^2 f^{-(n+2)/2} \\ &\quad + O\left(R^{-(n+2+p)/2} \int_{R \leqslant f \leqslant 2R} |z|^2 f^{p/2}\right) \\ &= \left(\frac{2(n+2-p)}{p}\right)^p \int_{\varepsilon \leqslant f \leqslant R} |z|^2 f^{-(n+2)/2} + O(1)\end{aligned}$$

$$\int_{\mathbb{H}^n} \frac{|W|^p|z|^2}{f^{p/2}} = O\biggl(\varepsilon^{-(n+2-p)/2}\int_{f\leqslant\varepsilon}\frac{|z|^2}{f^{p/2}} + R^{-(n+2-p)/2}\int_{R\leqslant f\leqslant 2R}\frac{|z|^2(2R-f)^p}{f^{p/2}}\biggr) + \int_{\varepsilon\leqslant f\leqslant R}|z|^2 f^{-(n+2)/2}$$
$$= O(1) + \int_{\varepsilon\leqslant f\leqslant R}|z|^2 f^{-(n+2)/2}.$$

Since

$$\lim_{\substack{\varepsilon\to 0,\\ R\to\infty}}\int_{\varepsilon\leqslant f\leqslant R}|z|^2 f^{-(n+2)/2} = \infty,$$

we get

$$\lim_{\substack{\varepsilon\to 0,\\ R\to\infty}}\int_{\mathbb{H}^n}\frac{|\nabla W|^p}{|z|^p}\biggl(\int_{\mathbb{H}^n}\frac{|z|^2|W|^p}{f^{p/2}}\biggr)^{-1} = \biggl(\frac{2(n+2-p)}{p}\biggr)^p. \tag{3.32}$$

Similar truncation also proves the result for $p = n+2$. This proves the claim. Let $2\leqslant p\leqslant n+2$. Then from (3.12) and (3.29), there exists an M_1 such that

$$\int_\Omega\frac{|\nabla u|^p}{|z|^{p-2}} - \biggl(\frac{p-1}{p}\biggr)^p\int_\Omega\biggl|\frac{\nabla E}{E}\biggr|^p\frac{|u|^p}{|z|^{p-2}} \geqslant M_1\int_\Omega\biggl|\frac{\nabla E}{E}\biggr|^{p-2}E^{p-1}\frac{|\nabla v^{p/2}|^2}{|z|^{p-2}}. \tag{3.33}$$

Since $u|_{\partial\Omega} = 0$, as in theorem 2.1, from (3.24) we have

$$\int_\Omega\frac{|\nabla u|^p}{|z|^{p-2}} - \biggl(\frac{p-1}{p}\biggr)^p\int_\Omega\biggl|\frac{\nabla E}{E}\biggr|^p\frac{|u|^p}{|z|^{p-2}} \geqslant \frac{M_1}{4}\sum_{i=1}^\infty\int_\Omega\eta_i^2\biggl(\frac{R}{f}\biggr)\biggl|\frac{\nabla E}{E}\biggr|^p\frac{|u|^p}{|z|^{p-2}}$$
$$= \frac{M_1}{4}\sum_{i=1}^\infty\int_\Omega\frac{\eta_i^2(R/f)|z|^2|u|^p}{(|z|^4+t^2)^{p/2}}.$$

This proves (2.25).

Following the same method as in theorem 2.1, (2.26) and (2.27) follow. This proves the theorem. □

4. Remarks and extensions

4.1. Open problem

Let E_P be as in (2.4) and let $w_p = E_p^{(p-1)/p}$. Then, in the sense of distributions, w_p satisfies

$$\left.\begin{aligned} L_p w_p - \biggl(\frac{p-1}{p}\biggr)^p\biggl|\frac{\nabla E_p}{E_p}\biggr|^p w_p^{p-1} &= 0 \quad \text{in } \Omega\setminus(0),\\ w_p|_{\partial\Omega} &= 0. \end{aligned}\right\} \tag{4.1}$$

In view of this, is $((p-1)/p)^p$ the best constant in the Hardy–Sobolev inequality (2.12)? For $p = 2$, using the regularity of E_2, is it possible to prove that $\frac{1}{4}$ is the best constant?

4.2. Interior and boundary Hardy–Sobolev-type inequalities

Here we consider extensions in which we make use of distributions that need not be a fundamental solution. For example, let L be the second-order elliptic operator in divergence form and let ∇_L be the associated gradient with respect to L. Let μ be a measure in Ω. Assume that there exist $E \in L^1_{\mathrm{loc}}(\Omega)$ such that

(i) $E \geqslant 0$ and $E \in C^1(\Omega \setminus \operatorname{supp}(\mu))$,

(ii) $E|_{\operatorname{supp}\mu} = \infty$,

(iii) $LE = \mu$.

Then we can obtain an analogous HS-type inequality by considering $v = E^{-1/2}u$, $u \in C^1_0(\Omega)$:

$$\int_\Omega |\nabla_L u|^2 - \frac{1}{4}\int_\Omega \left|\frac{\nabla_L E}{E}\right|^2 u^2 = \int_\Omega |\nabla_L v|^2 E \geqslant 0.$$

For example, we take $\mu = \sum_{i=1}^k \delta_{x_i}$, $x_i \in \Omega$, $L = -\Delta$ and

$$E = \sum_{i=1}^k \frac{c}{|x - x_i|^{n-2}}$$

for an appropriate constant c. This satisfies (i)–(iii). Then, for all $u \in C^1_0(\Omega)$, we have

$$\int_\Omega |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_\Omega \left|\sum_{i=1}^k \frac{x - x_j}{|x - x_i|^n}\right|^2 \left(\left|\sum_{i=1}^k \frac{1}{|x - x_i|^{n-2}}\right|^{-2}\right) u^2 \geqslant 0$$

and

$$\left(\frac{n-2}{2}\right)^2$$

is the best constant and is never achieved.

Next we can also combine the interior and boundary HS-type inequalities. For example, let Ω be a ball, $B(R)$, of radius R and let $A = ((\delta_{ij}))_{1\leqslant i,j\leqslant n}$ and $p = 2$, $n \geqslant 3$. Then

$$E_2 = C\left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}}\right),$$

$$\left|\frac{\nabla E_2}{E_2}\right| = \frac{n-2}{|x|(1 - (|x|/R)^{n-2})}.$$

Then the HS inequality (2.12) implies that, for all $u \in H^1_0(B(R))$,

$$\int_{B(R)} |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_{B(R)} \frac{u^2}{|x|^2(1 - (|x|/R)^{n-2})^2} \geqslant 0$$

and it is easy to show that

$$\left(\frac{n-2}{2}\right)^2$$

is the best constant and is never achieved. This inequality combines both the interior and the boundary HS inequalities.

4.3. Extension to compact manifolds

The analysis here is easily extended to the compact Riemannian manifold (M, g) of dimension n. For $1 < p \leqslant n$, $0 \in M$, let E_p be the fundamental solution of

$$\left.\begin{aligned} -\Delta_p E_p + E_p^{p-1} &= \delta_0 \quad \text{if } \partial M = \emptyset, \\ -\Delta_p E_p &= \delta_0 \quad \text{if } \partial M \neq \emptyset, \\ E_p &= 0 \quad\ \ \text{on } \partial M, \end{aligned}\right\} \tag{4.2}$$

where Δ_p is the analogous p-Laplacian generated by the metric g. Using this E_p as in theorem 2.1, we may obtain the corresponding HS-type inequality. The main point here is that if $\partial M = \emptyset$, then the zero on the right-hand side of the HS inequality will be replaced by a negative constant multiplied by the L_p norm of the function [2].

To illustrate this, take $p = 2$ and $\partial\Omega = \phi$. The fundamental solution E_2 then exists and satisfies (4.2). For $u \in C^1(M)$, let $v = E_2^{-1/2}u$ and calculate $|\nabla u|^2$ as above, to obtain

$$\int_M |\nabla u|^2 = \frac{1}{4}\int_M \left|\frac{\nabla E}{E}\right|^2 u^2 + \frac{1}{2}\int_M \langle \nabla E \cdot \nabla v^2\rangle + \int_M |\nabla v|^2 E$$

and

$$\int_M \langle E \cdot \nabla v^2\rangle = v^2(0) - \int_M E v^2 = -\int_M E v^2 = -\int_M u^2.$$

Hence, we obtain an inequality and call it the HS-type inequality given by

$$\int_M |\nabla u|^2 - \frac{1}{4}\int_M \left|\frac{\nabla E}{E}\right|^2 u^2 + \frac{1}{2}\int_M u^2 = \int_M |\nabla v|^2 E \geqslant 0,$$

where $\frac{1}{4}$ and $\frac{1}{2}$ are the best constants. Hence, we get an extra term in the inequality since $1 \in C^1(M)$.

4.4. Extension to non-compact manifolds

Let (M, g) be an open Riemannian manifold without boundary. Again for $1 < p \leqslant n$, if there exists a fundamental solution E_p (as in the Euclidean case, see [2]) of the p-Laplacian, then we can obtain the analogous HS-type inequality by the method described in the theorems here. In particular, we can calculate the HS-type inequality for symmetric spaces. In order to illustrate this, we will give the example of an upper half-plane with the Poincaré metric (see [14] for details).

4.4.1. *HS-type inequality on the upper half-plane*

Let $H = \{z = x + \mathrm{i}y : y > 0\}$ denote the upper half-plane. The Poincaré metric on H is given by $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. Corresponding to this, the gradient ∇_H, the Laplace–Beltrami operator Δ_H and the fundamental solution E_H are respectively given by

$$\begin{aligned} \mathrm{d}^2(z,z') &= \frac{|z-z'|^2}{4yy'}, \\ \nabla_H\phi &= y^2\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right), \\ \Delta_H\phi &= y^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right), \\ E_H &= \log\frac{\mathrm{d}^2(z,z')}{1+\mathrm{d}^2(z,z')} \end{aligned}$$

(see [14] for details).

Let $R>0$, $e(z)=\mathrm{d}^2(z,i)$, $B_R=\{z : e(z)<R\}$ and define

$$E_R(z) = \log\left(\frac{R^2}{1+R^2}\right) - \log\left(\frac{e(z)}{1+e(z)}\right).$$

Then E_R satisfies

$$\begin{aligned} -\Delta_H E_R &= c\delta_i(t) && \text{in } B_R, \\ E_R &> 0 && \text{in } B_R, \\ E_R &= 0 && \text{on } \partial B_R, \end{aligned}$$

for some constant c.

By direct calculation we have

$$\left|\frac{\nabla_H E_R}{E_R}\right|^2 = \left(e(1+e)\log\left(\frac{R^2(1+e)}{(1+R^2)e}\right)\right)^{-1}$$

and the HS-type inequality is given by

$$\int_{B_R}|\nabla_H\phi|^2\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} - \frac{1}{4}\int_{B_R}\phi^2\left(e(1+e)\log\left(\frac{R^2(1+e)}{(1+R^2)e}\right)\right)^{-1}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \geqslant 0$$

and the equality holds if and only if $\phi=0$, where $\phi\in H_0^1(B_R)$. As in theorem 2.12, we can write the asymptotic expression on the right-hand side in the above inequality.

4.5. Eigenvalue problem for HS operators

In general the perturbed eigenvalue problem studied in [2, 3, 17] can be easily extended to the above HS-type operators coming from the fundamental solutions.

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