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On the transmission of data packets through fiber-optic cables of uniform index

https://doi.org/10.1515/joc-2021-0031
Received January 31, 2021; accepted April 15, 2021;
published online May 10, 2021

Abstract: The treatment of Maxwell equations show that propagating wave of packets in fiber-optic cables is dispersive, propagating in groups, such that group velocity along certain curves in the frequency-phase velocity diagrams vanishes. It is suggested that stalling of wave groups is responsible, for bursting propagation observed in experimental measurements, causing some delay in transmission. The dispersion equations developed here, are different from those given in texts that are based on "weakly guiding approximation". The queue of such data packets arriving at a router station is found to have a "raised tail" distribution unlike that of Poisson arrivals. For accounting the property, a Mittag–Leffler function distribution (MLFD) of probability is used following a modification of that for a Poisson process, the tail raising is shown to cause delay in transmission, and its estimate is analysed based on the theory.

Keywords: burst; delay; dispersion; fiber-optic data transmission; Mittag-Leffler function distribution.

1 Introduction

The transmission of data packets in optical fibers takes place as pulses of optical wave guided through the core of the fiber. A theory for such waves is presented in Bhadra and Ghatak [1] for uniform refractive indices of glass. As the indices of the glass fiber and that of the cladding of the cable hardly differ, the electromagnetic waves in the composite cable are assumed at the very outset to be governed by decoupled wave equations. It appears that this assumption called the "weakly guiding approximation" is an over simplification, altering the dispersion phenomenon in the cable. The exact problem of transmission is treated in Stratton [2] (chapter 9, section 3.15); where the exact dispersion equation is developed for the dispersed transmission of the waves. That equation is treated in this paper for the approximate case of nearly equal refractive indices of the glass fiber and its cladding, for proper derivation of the dispersion equation. The new equation is numerically treated and it is shown that the wave phenomenon takes place in groups due to dispersion of the waves. It is also shown that the group velocity vanishes in the frequency-phase velocity diagram along certain curves that would cause sudden spikes of waves at those frequencies. This phenomenon suggests that dispersion is a contributory factor for sudden bursts observed in high resolution experimental packet transmission of data (Willinger, Paxson, Taqqu [3], Thompson, Miller, Wilder [4], and Kim, Won [5]).

The "bursty" flush flows of the data packets makes the transmission process to be additionally stochastic; as a consequence queuing theory methods come in to use for the transmission process, beginning with the basic $M/M/1$ queue in which the arrival and service rates are assumed uniform for all densities of flow. The inadequacy of the Markovian models became apparent in the high resolution packet transport experiments, which showed "raised" tails of the arrivals distribution as compared to the rapidly falling "thin tails" of the Markovian models (Willinger, Paxson, and Taqqu [3]). Also it is found in the measurements that there is no natural length of a "burst", but possesses self similarity from milliseconds to minutes to hours of duration (Leland, Taqqu, Willinger, and Wilson [6]). Thus Markovian traffic is ruled out by Koh and Kim [7], and instead the Pareto/M/1/K model is proposed to explain the tail raising effect. The Pareto distribution is defined in terms of two parameters; a shape parameter α and a location parameter β (Koh and Kim [8]). Moreover it is yet to be proven that a Pareto distribution tends to a Poisson process in the limit when α and β approach certain values as would be necessary when the traffic becomes uniform at low activity levels. In this respect a useful process based on the Mittag–Leffler function distribution (MLFD) is suggested by Chakraborty and Ong [9] for incorporating the "tail raising" effect. The model was first developed by Conway and Maxwell [10] for modeling state dependent service rates, by a modification of the Poisson process. In the present paper this model is presented for the "bursty"

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transmission that shows some delay in the transport of data packets due to the bursts.

2 Modal analysis of the guided waves

It is assumed that the optical fiber is homogeneous and uniform along its length having refractive index n_1, with a cladding of refractive index n_2 ($<n_1$). A pulse of digitally coded information passing through it can in general be decomposed in to a number of modes of propagating waves by Fourier's theorem (Born and Wolf [11], p. 19), and the transmission in general is multimodal, the fiber acting as a wave guide of the waves. A unimodal transmission consisting of only the fundamental least frequency mode is often employed with advantage.

Optical wave propagation is governed by Maxwell equations in terms of electric intensity **E** and magnetic intensity **H**. An account of modal analysis of EM wave propagation in a circular cylinder of radius a embedded in an infinite homogeneous medium is given in Stratton [2]. Such a model fits to portray fiber-optic transmission because of the wave guide action through the core of the cable, and the dispersion equation for propagation is

$$\left[\frac{\mu_1}{u}\frac{J_n'(u)}{J_n(u)}-\frac{\mu_2}{w}\frac{H_n^{(1)'}(w)}{H_n^{(1)}(w)}\right]\times\left[\frac{k_1^2}{\mu_1 u}\frac{J_n'(u)}{J_n(u)}-\frac{k_2^2}{\mu_2 w}\frac{H_n^{(1)'}(w)}{H_n^{(1)}(w)}\right] = n^2k^2\left(\frac{1}{w^2}-\frac{1}{u^2}\right)^2 \tag{1}$$

where $i=\sqrt{-1}$, and $n=0, \pm1, \pm2, \cdots, \pm\infty$ are the modal numbers; but as Eq. (1) is even in n, one need consider only the positive values of n. μ_1,μ_2 are magnetic permeability of glass and cladding materials respectively, while $k_1=\omega/c_1, k_2=\omega/c_2$ in which ω = angular frequency, and c_1,c_2 are the velocity of propagation of light in the two respective materials. The two functions $J_n(\cdot)$ and $H_n^{(1)}(\cdot)$ denote the usual Bessel and Hankel function of the first kind. The arguments u and w appearing in the two functions are defined by the relations

$$u = ak\sqrt{\frac{c_p^2}{c_1^2}-1},\ v = ak\sqrt{1-\frac{c_p^2}{c_2^2}},\ w = iv \tag{2}$$

where k = wave number and $c_p = \omega/k$ = phase velocity. For modal propagation as a guided wave it is required that $c_1 < c_p < c_2$. The refractive indices n_1 and n_2 are related to c_1 and c_2 by the equations $n_1 = c/c_1, n_2 = c/c_2$, where c is the velocity of light in vacuum. For $c_1 < c_2$ to hold, one must have $n_1 > n_2$.

There are two important cases for Eq. (1), when $n = 0$ (fundamental mode propagation) and when $n_1 \approx n_2$, or $c_1 \approx c_2$. In the latter case $(w^2-u^2)^2$ is of second order smallness and can be neglected. Thus in both the cases the right hand side of Eq. (1) can be taken as zero, and there results two dispersion equations of transmission represented by the two factors in square brackets. In as much as $c_1 \approx c_2$ by assumption, $k_1 \approx k_2$, and one may also assume $\mu_1 \approx \mu_2$. Under these simplifications, the dispersion equation becomes

$$\frac{1}{u}\frac{J_n'(u)}{J_n(u)}-\frac{1}{w}\frac{H_n^{(1)'}(w)}{H_n^{(1)}(w)} = 0 \tag{3}$$

or, using the well-known properties of Bessel functions (Abramowitz and Stegun [12], pp. 361, 375), one gets

$$\frac{1}{u}\frac{J_{n+1}(u)}{J_n(u)}+\frac{1}{v}\frac{K_{n+1}(v)}{K_n(v)}-n\left(\frac{1}{u^2}+\frac{1}{v^2}\right) = 0 \tag{4}$$

where $K_n(\cdot)$ is the modified Bessel function.

Equation (4) can be expressed in terms of familiar variables employed in fiber-optic literature. Let,

$$f = \sqrt{u^2+v^2} = a\omega\sqrt{\frac{1}{c_1^2}-\frac{1}{c_2^2}} = \frac{a\omega}{c}\sqrt{n_1^2-n_2^2} \tag{5}$$

then f is proportional to the frequency of the propagating wave. Also let,

$$b = \frac{c^2/c_p^2-n_2^2}{n_1^2-n_2^2} = \frac{v^2}{f^2} \tag{6}$$

Thus b replaces the relative phase velocity c_p/c. The variables u and v in terms of f and b therefore become

$$v = \sqrt{b}f \quad \text{and} \quad u = \sqrt{1-b}f \tag{7}$$

Equation (4) therefore takes up the form

$$\frac{1}{\sqrt{1-b}f}\frac{J_{n+1}(\sqrt{1-b}f)}{J_n(\sqrt{1-b}f)}+\frac{1}{\sqrt{b}f}\frac{K_{n+1}(\sqrt{b}f)}{K_n(\sqrt{b}f)}-\frac{n}{b(1-b)f^2}=0 \tag{8}$$

Alternatively, replacing n by $-n$ and using the properties of Bessel functions, $J_{-n}(u) = (-1)^n J_n(u)$ and $K_{-n}(u) = K_n(u)$, one can write Eq. (11) in the form

$$\frac{1}{\sqrt{1-b}f}\frac{J_{n-1}(\sqrt{1-b}f)}{J_n(\sqrt{1-b}f)}-\frac{1}{\sqrt{b}f}\frac{K_{n-1}(\sqrt{b}f)}{K_n(\sqrt{b}f)}-\frac{n}{b(1-b)f^2}=0 \tag{9}$$

Equations (8) and (9) both somewhat differ from those stated in the literature (Bhadra and Ghatak [1], p. 21). In this study, Eq. (8) is treated as the dispersion equation for the optical wave propagation through the fiber.

For numerical study, typically $n_1 = 1.5$ and $n_2 = 1.48515$ are chosen. Higher modes generally speaking contribute little to a propagating pulse. For a qualitative view of the dispersion curves, the gravest mode $n = 0$ is selected. The next mode $n = 1$ yields similar results. As $0 < b < 1$, and $f > 0$. The zeros of the left hand side of Eq. (8) are first isolated in the domain $b = 0\,(0.1)\,1$ and $f = 0\,(0.25)\,10$. Next the zeros are refined by using the simple bisection method. The curves for the modes $n = 0$ and 1 are respectively shown in Figures 1 and 2.

3 Group velocity of propagation

In as much as the phase velocity determined by b depends on the frequency f, the optical waves through the circular cylindrical core propagate as group of waves with velocity $c_g = d\omega/dk$ (Born and Wolf [11], p. 21). This feature is observed in the experiments of Leland et al. [6]. It is also observed in that study that the waves shoot up for certain frequencies of propagation. Such bursts can occur if the group velocity c_g vanishes for some real values of f. For existence of such values, $d\omega/dk$ is set to zero by differentiating Eq. (8). Thus, for $n = 0$ one gets the equation

$$\frac{1}{u^3 J_0^2(u)}\left[u\left\{J_0^2(u) + J_1^2(u)\right\} - 2J_0(u)J_1(u)\right] - \frac{1}{v^3 K_0^2(v)}\left[v\left\{K_0^2(v) + K_1^2(v)\right\} - 2K_0(v)K_1(v)\right] = 0 \quad (10)$$

and for the case $n = 1$, the equation

$$\frac{1}{u^3 J_1^2(u)}\left[uJ_1^2(u) - \{2J_1(u) + uJ_0(u)\}J_2(u)\right] - \frac{1}{v^3 K_1^2(v)}\left[vK_1^2(v) - \{2K_1(v) + vK_0(v)\}K_2(v)\right] - \frac{2}{(1-b)v^3} = 0 \quad (11)$$

The zeros of Eqs. (10) and (11) follow certain trend for different values of f for the data used in the preceding section. These are shown in Figures 3 and 4, respectively, for the modes $n = 0$ and 1. The vanishing of the group velocity for certain values of f provides an explanation for bursts in transmission as stated earlier.

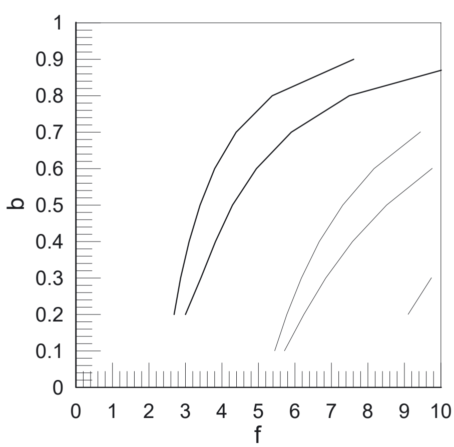

Figure 1: Dispersion curves ($n = 0$).

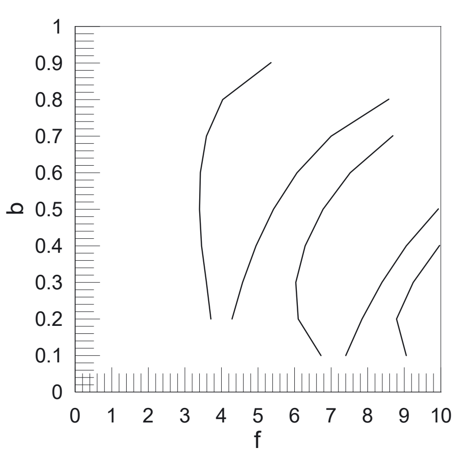

Figure 2: Dispersion curves ($n = 1$).

4 Distribution of transmission and delay

The transmission of data packets in the fiber-optic cable being nonuniform and bursty, the rate at which arrivals take place at a router station deviates from the Poisson distribution operating on the FCFS discipline. The statistically observed data show that the tail of the distribution is in fact "lifted up" as compared to the exponentially falling Poisson. To account for this phenomenon noting that a buffer is available at a router, the bursty transmission of the packets in the cable becomes stochastic such that there is a tendency

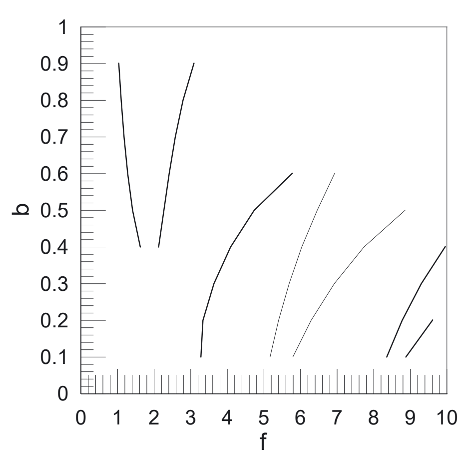

Figure 3: Zero group velocity curves ($n = 0$).

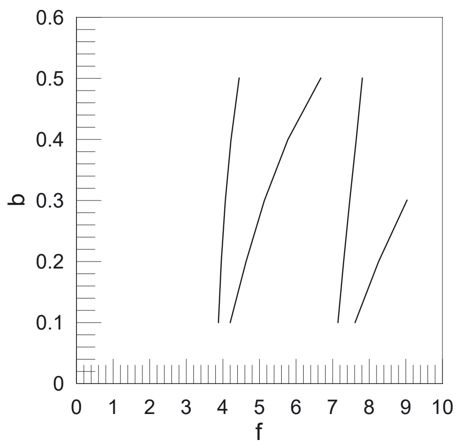

Figure 4: Zero group velocity curves ($n = 1$).

of instantaneous rise in their number in the wave guide. In order to build a suitable simple model of transmission, let on an average a certain constant fraction $\alpha < 1$ of a packet, in a given state of population n in the wave guide, breaks the FCFS discipline instantaneously to the next higher state $n + 1$. Now the number of ways breakaway depletion can take place in state n, is the permutation number

$$P(n\alpha, \alpha) = \frac{(n\alpha)!}{[(n-1)\alpha]!} = \frac{\Gamma(n\alpha + 1)}{\Gamma[(n-1)\alpha + 1]} \tag{12}$$

where $\Gamma(\cdot)$ is the Euler gamma function. The accretion to the next state $n + 1$ of the queue similarly can take place in $P((n+1)\alpha, \alpha)$ ways. The accretion rate of packets is suppose λ', then as in the theory of Poisson arrivals (Bose [13]), if $p_n(t)$ is the probability in the state n of the queue at time t, then it satisfies the differential-difference equation

$$\begin{aligned} p_n'(t) = &-\lambda p_n(t) + \lambda p_{n-1}(t) + \lambda'[P[(n+1)\alpha, \alpha]p_{n+1}(t) \\ &- P(n\alpha, \alpha)p_n(t)], \quad n \geq 1 \end{aligned} \tag{13}$$

Equation (13) reduces to the equation for the Poisson distribution when α and λ' vanish together.

Equation (13) was obtained earlier by Conway and Maxwell [10] in a different context. As the system stabilizes to a value p_n of $p_n(t)$ as $t \to \infty$, then $p_n'(t) \to 0$, and the equation reduces to the system of difference equations

$$[\rho + P(n\alpha, \alpha)]\, p_n = \rho p_{n-1} + P[(n+1)\alpha, \alpha]p_{n+1}, \quad n \geq 1 \tag{14}$$

where $\rho = \lambda/\lambda'$. For the case $n = 0$, there can be only one accretion, with no negative state. Hence the equation degenerates in to the equation $\rho p_0 = \alpha!\, p_1$. in which $P(\alpha, \alpha) = \alpha!$ Setting $n = 1, 2, 3, \ldots$, the solution of the system Eq. (14) is found to be

$$p_n = \frac{\rho^n p_0}{(n\alpha)!} \frac{1}{E_\alpha(\rho)} = \frac{\rho^n}{\Gamma(n\alpha + 1)} \frac{1}{E_\alpha(\rho)} \tag{15}$$

where

$$E_\alpha(\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(n\alpha + 1)} \tag{16}$$

The function $E_\alpha(\rho)$ is known as the Mittag–Leffler function and p_n defined by Eq. (15) is known as the MLFD (Conway and Maxwell [10]). For the case $\alpha \to 0$, the distribution reduces to the result $p_n = \rho^n(1 - \rho)$, if $0 < \rho < 1$. In general $p_n > 0$ as against $p_n = 0$ for Poisson arrivals, leading to the "raising of the tail" of the Poisson distribution in the arrival process.

The steady state queue length L_q for the system represented by Eq. (14) is given by the expectation value of the discrete distribution

$$L_q = \sum_{n=0}^{\infty} n p_n = \frac{1}{E_\alpha(\rho)} \sum_{n=1}^{\infty} \frac{n\rho^n}{\Gamma(n\alpha + 1)} = \frac{\rho E_\alpha'(\rho)}{E_\alpha(\rho)} \tag{17}$$

where the prime denotes differentiation with respect to ρ. The computation of $E_\alpha(\rho)$ and $E_\alpha'(\rho)$ for different numerical values of α and ρ is straight forward from their series expansions provided that $\rho < 1$. However, the case $\rho = \lambda/\lambda' \gg 1$ is more important as the nominal average rate of transmission λ is of the order of the phase velocity c_p of the transmitted pulsed waves, whereas the bursting rate λ' is unlikely to be that fast. Hence, using the asymptotic expansion of $E_\alpha(\rho)$ as given in Houblod, Mathai, and Saxena [14]

$$E_\alpha(\rho) \sim \frac{1}{\alpha} \rho^{1/\alpha}, \quad \text{for} \quad \rho \to \infty \tag{18}$$

the queue length L_q for large ρ also becomes

$$L_q \sim \frac{1}{\alpha} \rho^{1/\alpha} \tag{19}$$

Thus, for small values of α, the queue length rapidly increase as a power of ρ. In the theory of queues, the waiting time in the queue is defined as $W_q = L_q/\lambda$, which in the present context represents the delay in transmission. Hence Eq. (19) indicates that there can be significant delay in the transmission process.

5 Conclusion

Packet transmission of data through a uniform index optical fiber cable is studied in this paper by analyzing the propagation of the guided optical waves through the fiber and its cladding. Considering the exact solution of the Maxwell equations of electromagnetism for a problem of this type given in Stratton [2], it is found that the transmission in the cable takes place as superposed waves exhibiting dispersion in different modes. The dispersion equation for the modes obtained here are somewhat different from that obtained under the “weakly guiding approximation” given in texts (Bhadra and Ghatak [1]). Moreover, it is found that the dispersed group of waves in a transmitted pulse, in fact travel with velocity that vanishes for certain frequencies and phase velocity of propagation. It is argued that such halted groups stall the propagation to cause bursting character of transmission as observed in high resolution experiments. It makes the arrival process at router station more complex, deviating from the well-known Poisson process, which presumes more or less steady arrivals with an FCFS discipline in the queue. A modification, similar to that of Conway and Maxwell [10] in terma of the MLFD is developed here to estimate the delay in transmission in the cable caused by disruption of near steady propagation. The model analysis indicates that the delay in transmission can be significant on account of the bursts in the process of propagation of the data packets.

Acknowledgement: The author is thankful to the SN Bose National Center for Basic Sciences for supporting this research.
Author contributions: The author has accepted responsibility for the entire content of this submitted manuscript and approved submission.
Research funding: None declared.
Conflict of interest statement: The author declares no conflicts of interest regarding this article.

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