

# Quantum entanglement in coupled lossy waveguides

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**Abstract:** We investigate the viability of coupled waveguides as basic units of quantum circuits. We study entanglement when the waveguides are fed in by light produced by a down-converter working either in low gain limit or under large gain. We present explicit analytical results for the measure of entanglement in terms of the logarithmic negativity for a variety of input states. We also address the effect of loss on entanglement dynamics of waveguide modes. Our results indicate that the waveguide structures are reasonably robust against the effect of loss and thus quite appropriate for quantum architectures as well as for the study of coherent phenomena like random walks. Our analysis is based on realistic structures used currently.

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## 1. Introduction

Discrete optical systems like coupled waveguides are known to be extremely efficient in manipulating the flow of light and have been investigated extensively in the last two decades [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Many key quantum effects like quantum interference, entanglement and quantum walk has been investigated in these systems [3, 4, 5, 10]. For example, using coherent beam Peretes et al. [4] have observed quantum walk effects in a system consisting of large number of waveguides. In another experiment, Bromberg et al. investigated the quantum correlations in GaAs waveguide arrays [5] using two-photon input states. In particular, they considered both the separable and entangled two photon state and observed various features as-

sociated with quantum interference. In addition, the coupled waveguide arrays have been used to study the discrete analogue of the Talbot effect [8]. The entanglement between the waveguide modes and behavior of nonclassical light in coupled waveguides has been studied [9, 11]. In a recent experiment Politi et. al. [12] have shown how a CNOT gate can be implemented on a single Silicon chip using coupled silica waveguides, thus showing possible application of waveguides in quantum computation. They also observed Hong-Ou-Mandel two photon interference in these coupled waveguides. In a following experiment [13] coupled silica waveguides was used to generate a multimode interferometer on an integrated chip. It was further shown that these interferometers can be used to generate arbitrary quantum circuits. They also showed that two and four photon entangled states similar to NOON states [14] can be generated on the silicon chip. All these studies have hence given a new impetus to the field of quantum information processing and quantum optics with waveguides. In particular, for effective use of these waveguide circuits in quantum computation and communication tasks sustainability of generated entanglement is very important [15]. In light of this, it is imperative to study entanglement in waveguides using quantitative measures for entanglement. This is the main purpose of the present study. Moreover, in practice the waveguides are not completely lossless. Thus an immediate question of interest would be how does this loss affects the entanglement in the waveguide modes? It is well known that entanglement is quite susceptible to decoherence [16] and thus the above question bears immense interest in context to quantum information processing using waveguides. Further it is important to understand the role of loss in coherent phenomena like quantum random walk [4, 17, 18].

In this paper we investigate these in a simple system of two single mode waveguides, which are coupled through the overlap of evanescent fields. This simple system serves as a unit or the basic element for constructing a quantum circuit [19]. The input light to the coupled waveguide system is usually produced by the parametric down-conversion process. At high gain the parametric down-conversion process produces a squeezed state of light while single photon states are produced at low gain. Behavior of photon number states such as the single photon state and the NOON state have also been investigated in these systems [12, 13, 19]. We thus consider a variety of nonclassical input states like squeezed states and photon number states which have been extensively investigated in couple waveguide system and study their respective entanglement dynamics. We quantify the evolution of entanglement in terms of logarithmic negativity and present explicit analytical results for both squeezed and number state inputs. We further investigate the question of possible effects of loss on the entanglement dynamics in waveguides by considering lossy waveguide modes. We find that in this case, for both number state inputs as well as squeezed state inputs, entanglement shows considerable robustness against loss.

The organization of the paper is as follows: In section 2, we describe the model and derive analytical result for the field modes of the coupled waveguide system. In section 3, we study the evolution of entanglement for two classes of photon number states, (A) separable single photon state  $|1, 1\rangle$  and (B) entangled two-photon NOON state. We quantify the degree of entanglement of these states by using the logarithmic negativity. In section 4, we then study the time evolution of entanglement by evaluating the logarithmic negativity for two classes of squeezed input states (A) separable two mode squeezed state and (B) entangled two mode squeezed state. The effect of loss in waveguides on the entanglement dynamics is then discussed in section 5. Finally we summarize our results in section VI with a future outlook.

## 2. The model

We consider a system with two single mode waveguides, coupled through nearest-neighbor interaction as shown in Fig. 1. Let  $a$  and  $b$  be the field operators for the modes in each waveguide. These obey bosonic commutation relations  $[a, a^\dagger] = 1$ ;  $(a \rightarrow b)$ . The Hamiltonian describing

the evanescent coupling between the waveguide mode in such a system of two coupled waveguides can be derived using the coupled mode theory [20]. The coupling among the waveguides is incorporated in this framework by treating it as a perturbation to the mode amplitudes. It is assumed that the presence of the second waveguide perturbs the medium outside the first waveguide. This creates a source of polarization outside the first waveguide, which thereby leads to modification of the amplitude of the mode in it. Further, the amplitude of the modes in each waveguide is assumed to be a slowly varying function of the propagation distance. Moreover, in this perturbative approach the coupling does not effect the propagation constant or transverse spatial distribution of the waveguide modes. The field of the first waveguide has a similar effect on the second waveguide. Under these assumptions, the field mode of the composite structure are governed by the Helmholtz equation which gives two coupled first order differential equations which can be solved to obtain the time evolution of field modes in the coupled waveguide structure. The corresponding description for the nonclassical light can be studied by quantizing the field amplitudes as has been done in the work of Lai et. al. [21]. Following an approach similar to that developed by Lai et. al., we can write the corresponding quantum mechanical Hamiltonian for the coupled waveguide as

$$H = \hbar\omega(a^\dagger a + b^\dagger b) + \hbar J(a^\dagger b + b^\dagger a), \quad (1)$$

where the first two terms correspond to the free energy of the waveguide modes and the last

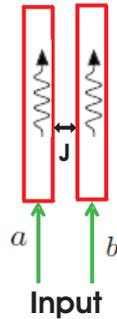


Fig. 1. (Color online) Schematic diagram of a coupled waveguide system. The parameter  $J$  gives the coupling between the waveguide modes.

two terms account for the evanescent coupling between the waveguide modes with  $J$  as the coupling strength. The coupling  $J$  depend on the distance between the waveguides. The input to the coupled waveguide system can be in a separable or an entangled state. Let  $\gamma$  be the loss rates of the modes  $a$  and  $b$ . The loss  $\gamma$  arises from the loss in the material of the waveguide. Table I below gives the experimental values of coupling parameter  $J$  and loss  $\gamma$  for different waveguide systems.

Table 1. Approximate values of some of the parameters used in waveguide structures [22, 23, 24]. The loss, usually quoted in  $dB/cm$ , for different waveguides is converted to frequency units used in this paper by using the formula,  $10 \text{ Log}(\frac{P_{out}}{P_{in}}) \equiv 10 \text{ Log}(e^{-2\gamma/c})$ , where  $P_{in}$  is the input power,  $P_{out}$  is the power after traveling unit length.

| Waveguide Type                | Coupling parameter $J$ ( $sec^{-1}$ )       | Loss $\gamma$ ( $sec^{-1}$ ) | $\gamma/J$ |
|-------------------------------|---|------------------------------|------------|
| Lithium Niobate ( $LiNbO_3$ ) | $1.83 \times 10^{10} - 4.92 \times 10^{10}$ | $3 \times 10^9$              | 1/7-1/20   |
| AlGaAs                        | $2.46 \times 10^{11}$                       | $2.7 \times 10^{10}$         | 1/10       |
| Silica                        | $1.53 \times 10^{11}$                       | $3 \times 10^9$              | 1/50       |

As known the silica waveguides have very little intrinsic loss and should be preferable in many applications. Nevertheless the loss is to be included as this could be detrimental in long propagation for example in the study of quantum random walks. Since the two waveguides are identical, we have taken the loss rate of both the modes to be the same. We can model the loss in waveguides in the framework of system-reservoir interaction well known in quantum optics and is given by,

$$\mathcal{L}\rho = -\gamma(\hat{a}^\dagger \hat{a}\rho - 2\hat{a}\rho\hat{a}^\dagger + \rho\hat{a}^\dagger\hat{a}) - \gamma(\hat{b}^\dagger \hat{b}\rho - 2\hat{b}\rho\hat{b}^\dagger + \rho\hat{b}^\dagger\hat{b}), \quad (2)$$

where  $\rho$  is the density operator corresponding to the system consisting of fields in the modes a and b. The dynamical evolution of any measurable  $\langle O \rangle$  in the coupled waveguide system is then governed by the quantum-Liouville equation of motion given by,

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}\rho \quad (3)$$

where  $\langle \dot{O} \rangle = \text{Tr}\{O\dot{\rho}\}$ , the commutator gives the unitary time evolution of the system under the influence of coupling and the last term account for the loss. Note that in absence of loss (lossless waveguides) the time evolution of the field operators can be evaluated using the Heisenberg equation of motion and is given by,

$$a(t) = a(0)\cos(Jt) - ib(0)\sin(Jt); \quad (a \rightarrow b). \quad (4)$$

Next we will study the entanglement characteristics of photon number and squeezed input states as they propagate through the waveguides. To keep the analysis simple in the next few sections we consider the case of lossless waveguide modes ( $\gamma = 0$ ). We defer the discussion of loss on entanglement to Sec V.

### 3. Evolution of entanglement for input states at single photon level

In this section we study the dynamics of entanglement for photon number input state. We quantify the entanglement of the system by studying the time evolution for the logarithmic negativity [25]. For a bipartite system described by the density matrix  $\rho$  the logarithmic negativity is

$$E_{\mathcal{N}}(t) = \log_2 \|\rho^T\|, \quad \|\rho^T\| = (2N(\rho) + 1), \quad (5)$$

where  $\rho^T$  is the partial transpose of  $\rho$  and the symbol  $\|\cdot\|$  denotes the trace norm. Also  $N(\rho)$  is the absolute value of the sum of all the negative eigenvalues of the partial transpose of  $\rho$ . The log negativity is a non-negative quantity and a non-zero value of  $E_{\mathcal{N}}$  would mean that the state is entangled.

### 3.1. Separable photon number state as an input

We first consider the case when there is no loss and hence we set  $\gamma = 0$ . We assume that the input is in a separable state. Further, for studying the entanglement dynamics for photon number states we first consider the case of a single photon input in each waveguide. Thus the initial state is  $|\psi(0)\rangle = |1, 1\rangle$ . Using Eq. (4) we can show that a single photon input state given by  $|\psi(0)\rangle$  evolves into a state :  $|\psi(t)\rangle \rightarrow \alpha_1|2, 0\rangle + \beta_1|1, 1\rangle + \delta_1|0, 2\rangle$ . The coefficients  $\alpha_1$ ,  $\beta_1$  and  $\delta_1$  are given by :

$$\alpha_1 \equiv -i \sin(2Jt)/\sqrt{2}, \quad \beta_1 \equiv \cos(2Jt), \quad \delta_1 \equiv -i \sin(2Jt)/\sqrt{2}. \quad (6)$$

The density matrix corresponding to the state  $|\psi(t)\rangle$  can be written as :  $\rho = |\psi(t)\rangle\langle\psi(t)|$ . Thus, using Eq. (5), we can write the log negativity  $E_{\mathcal{N}}$  for the state  $|\psi(t)\rangle$  as:

$$E_{\mathcal{N}} = \log_2(1 + 2N(\rho)) = \log_2(1 + 2|(\alpha_1\beta_1 + \alpha_1\delta_1 + \delta_1\beta_1)|). \quad (7)$$

We show the time evolution of  $E_{\mathcal{N}}$  for the single photon input state  $|1, 1\rangle$  in the red curve of Fig. 2 (a). We would like to emphasize that the values of  $\theta$  studied here are very similar to the ones employed in the recent experiments [5, 12]. At time  $t = 0$ , we begin with a separable input state and thus the value of log negativity is  $E_{\mathcal{N}} = 0$ . The entanglement quantified by the log negativity increases with time and attains a maximum value of 1.58 for  $\theta \simeq 0.15$ . In this case the single photon state evolves into a maximally entangled state given by:  $|\psi_m\rangle \equiv e^{-i\pi/2}(|2, 0\rangle + |0, 2\rangle) + |1, 1\rangle/\sqrt{3}$ . Further, for  $\theta = 1/4$ , we get an analog of the well known Hong–Ou–Mandel interference [26]. Note that in this case the logarithmic negativity  $E_{\mathcal{N}}$  attains a value of 1 which is less than the corresponding value of  $E_{\mathcal{N}}$  for the maximally entangled state  $|\psi_m\rangle$ . In addition, for  $\theta = 1/2$ , we find that  $E_{\mathcal{N}}$  vanishes and the state at this point is  $e^{i\pi}|1, 1\rangle$ . At later times, we see a periodic behavior which can be attributed to the inter-waveguide coupling  $J$ . We next consider the case where we have two photons in one waveguide and none in the other input. Thus the initial state can be written as :  $|\varphi(0)\rangle = |2, 0\rangle$ . Again using Eq. (4) we find that the  $|\varphi(0)\rangle$  evolves into a state given by :  $|\varphi(t)\rangle \rightarrow \alpha_2|2, 0\rangle + \beta_2|1, 1\rangle + \delta_2|0, 2\rangle$ . The coefficients  $\alpha_2$ ,  $\beta_2$  and  $\delta_2$  are given by :

$$\alpha_2 \equiv \cos(Jt)^2, \quad \beta_2 \equiv -\sqrt{2}i \cos(Jt) \sin(Jt), \quad \delta_2 \equiv -\sin(Jt)^2. \quad (8)$$

Using a similar procedure as discussed above we can evaluate the log negativity  $E_{\mathcal{N}}$  for the state  $|\varphi(t)\rangle$ . We show the result for the log negativity in black curve of Fig. 2 (a). In this case we find that the log negativity increases and attains a maximum value of 1.54. After reaching the maximum value the log negativity decreases and eventually becomes equal to zero. Thus the state becomes disentangled at this point of time. At later times we see a periodic behavior and the system gets entangled and disentangled periodically. Clearly the entanglement dynamics of the states  $|\psi(0)\rangle$  and  $|\varphi(0)\rangle$  are different. Unlike the earlier case for the  $|1, 1\rangle$  input state, we don't see any interference effects in this case [5].

### 3.2. NOON state as an input

Next we consider the entangled state prepared in a  $N$  photon NOON state [14] as our initial state :

$$|\psi_{in}\rangle = (|N, 0\rangle + |0, N\rangle)/\sqrt{2} = ((a(0))^N + (b(0))^N|0, 0\rangle)/\sqrt{2N!}, \quad (9)$$

Using Eq. (4) we can show that the input state given by  $|\psi_{in}\rangle$  evolves into a state :

$$|\psi_{out}\rangle = (((a(t))^N + (b(t))^N|0, 0\rangle)/\sqrt{2N!}), \quad (10)$$

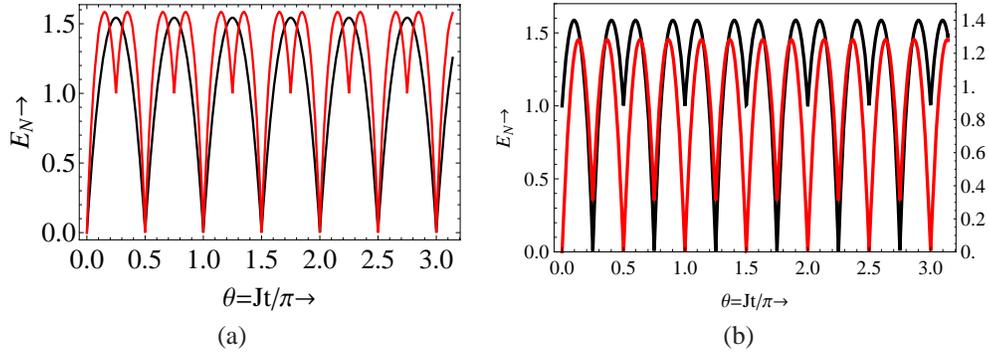


Fig. 2. (a) Time evolution of log negativity for the separable input state. The red curve shows the result for  $|1, 1\rangle$  state while the black curve shows the result for the  $|2, 0\rangle$  state. (b) The behavior of log negativity for the NOON state. The black curve shows the result for two photon NOON state while the red curve shows the result  $(E_N - 1)$  for the four photon NOON state

where  $a(t)$  and  $b(t)$  are given by Eq. (4). Using Eq. (4) in the above equation, we get

$$|\psi_{out}\rangle = \left(\sum \beta_k |k, N-k\rangle\right), \quad \beta_k = \alpha_k + \alpha_{N-k}, \quad \alpha_k = (C(N, k))^{1/2} (\cos(Jt))^k (-i \sin(Jt))^{N-k} \quad (11)$$

where  $C(N, k)$  is the Binomial coefficient given by:  $C(N, k) = N! / (N-k)! k!$ . The density matrix corresponding to the state  $|\psi_{out}\rangle$  can be written as:  $\rho_{out} = \sum \beta_k \beta_m^* |k, N-k\rangle \langle m, N-m|$ . Taking the partial transpose of  $\rho_{out}$ , we get  $\rho_{out}^T = \sum \beta_k \beta_m^* |k, N-m\rangle \langle m, N-k|$ . Further it can be proved that  $(\rho_{out}^T)^2$  is a diagonal matrix and the eigenvalues of  $(\rho_{out}^T)^2$  is of the form:  $|\beta_k|^2 |\beta_m|^2$ . Thus the negative eigenvalues of  $\rho_{out}^T$  are of the form  $|\beta_k| |\beta_m|$  ( $k \neq m$ ) and the log negativity  $E_{\mathcal{N}}$  can be written as:

$$E_{\mathcal{N}} = \log_2(1 + 2N(\rho)) = \log_2\left(1 + 2 \sum_{k \neq m} |\beta_k| |\beta_m|\right).$$

We can use the above equation to study entanglement dynamics for the multi-photon NOON state. We first consider the entangled state prepared in a two photon NOON state as our initial state. As shown in the black curve of Fig. 2 (b), the value of  $E_{\mathcal{N}}$  at time  $t = 0$  is equal to 1 which indicates entanglement. The log negativity  $E_{\mathcal{N}}$  in the black curve of Fig. 2 (b) shows a behavior that is similar to the result for the  $|1, 1\rangle$  state shown in the red curve of Fig. 2 (a). Also note the shift of  $\pi/4$  between the corresponding results in Figs. 2 (a) and 2 (b). As in the case of single photon input state  $|1, 1\rangle$ , the initial state evolves into a maximally entangled state corresponding to a value of  $E_{\mathcal{N}}$  which is equal to 1.58. In addition, for  $\theta = 1/2$ , we again see a signature of quantum interference such that the probability of getting the single photons at each of the output port vanishes [5]. The logarithmic negativity  $E_{\mathcal{N}}$  at this point is equal to 1. At later times the entanglement shows an oscillatory behavior and the system gets periodically entangled and disentangled. The red curve in Fig. 2 (b) shows the negativity  $(E_N - 1)$  for the four photon NOON state. As earlier, the value of  $E_{\mathcal{N}}$  at time  $t = 0$  is equal to 1 which indicates entanglement. The curve for four photon NOON state also shows quantum interference effect. Further, the logarithmic negativity never becomes zero in this case and hence the initially entangled state remains entangled for later times.

## 4. Evolution of entanglement for squeezed input states

### 4.1. Separable two mode squeezed state as an input

We next study the generation and evolution of entanglement for the case of squeezed input states. For this purpose we first consider a separable squeezed input state coupled to the modes  $a$  and  $b$  of the waveguide given by:  $|\zeta\rangle = |\zeta_a\rangle \otimes |\zeta_b\rangle$ , where  $|\zeta_a\rangle$  ( $|\zeta_b\rangle$ ) are single mode squeezed states defined as,  $|\zeta_a\rangle = \exp(\frac{r}{2}\{a^{\dagger 2} - a^2\})|0\rangle$ ; ( $a \rightarrow b$ ) where  $r$  is taken to be real. Here  $r$  is proportional to the gain of the down-converter. Such a separable two mode state can be obtained as the field at the output ports of a 50-50 beam splitter whose input ports are fed by light from a down-converter. It is well known that a two mode squeezed state like  $|\zeta\rangle$  can be completely characterized by its first and second statistical moments given by the first moment:  $(\langle x_1 \rangle, \langle p_1 \rangle, \langle x_2 \rangle, \langle p_2 \rangle)$  and the covariance matrix  $\sigma$ . The squeezed vacuum state falls under the class of Gaussian states. It is to be noted that evolution of Gaussian states has been studied for many different model Hamiltonians [27, 28]. We focus on the practical case of propagation of light produced by a down converter in coupled waveguides which currently are used in quantum architectures and quantum random walks. Note that since the first statistical moments can be arbitrarily adjusted by local unitary operations, it does not affect any property related to entanglement or mixedness and thus the behavior of the covariance matrix  $\sigma$  is all important for the study of entanglement. The measure of entanglement for a Gaussian state is best characterized by the logarithmic negativity  $E_{\mathcal{N}}$ , a quantity evaluated in terms of the symplectic eigenvalues of the covariance matrix  $\sigma$  [29, 30]. The elements of the covariance matrix  $\sigma$  are given in terms of conjugate observables,  $x$  and  $p$  in the form,

$$\sigma = \begin{bmatrix} \alpha & \mu \\ \mu^T & \beta \end{bmatrix}; \quad (12)$$

where  $\alpha$ ,  $\beta$  and  $\mu$  are  $2 \times 2$  matrices given by

$$\alpha = \begin{bmatrix} \langle x_1^2 \rangle & \langle \frac{x_1 p_1 + p_1 x_1}{2} \rangle \\ \langle \frac{x_1 p_1 + p_1 x_1}{2} \rangle & \langle p_1^2 \rangle \end{bmatrix} = \beta(1 \rightarrow 2); \quad \mu = \begin{bmatrix} \langle \frac{x_1 x_2 + x_2 x_1}{2} \rangle & \langle \frac{x_1 p_2 + p_2 x_1}{2} \rangle \\ \langle \frac{x_2 p_1 + p_1 x_2}{2} \rangle & \langle \frac{p_1 p_2 + p_2 p_1}{2} \rangle \end{bmatrix}. \quad (13)$$

Here  $x_1, x_2$  and  $p_1, p_2$  are given in terms of the normalized bosonic annihilation (creation) operators  $a(a^\dagger)$ ,  $b(b^\dagger)$  associated with the modes  $a$  and  $b$  respectively,

$$x_1 = \frac{(a + a^\dagger)}{\sqrt{2}} \quad (x_2 : a \rightarrow b), \quad p_1 = \frac{(a - a^\dagger)}{\sqrt{2}i} \quad (p_2 : a \rightarrow b); \quad (14)$$

The observables,  $x_j, p_j$  satisfy the canonical commutation relation  $[x_k, p_j] = i\delta_{kj}$ . The condition for entanglement of a Gaussian state like  $|\zeta\rangle$  is derived from the PPT criterion [30], according to which the smallest symplectic eigenvalue  $\tilde{\nu}_<$  of the transpose of matrix  $\sigma$  should satisfy,

$$\tilde{\nu}_< < \frac{1}{2}; \quad \tilde{\nu}_< = \min[\tilde{\nu}_+, \tilde{\nu}_-]; \quad (15)$$

where  $\tilde{\nu}_\pm$  is given by,

$$\tilde{\nu}_\pm = \sqrt{\frac{\tilde{\Delta}(\sigma) \pm \sqrt{\tilde{\Delta}(\sigma)^2 - 4\text{Det}\sigma}}{2}}; \quad (16)$$

where  $\tilde{\Delta}(\sigma) = \Delta(\tilde{\sigma}) = \text{Det}(\alpha) + \text{Det}(\beta) - 2\text{Det}(\mu)$ . Thus according to the condition (15) when  $\tilde{v}_< \geq 1/2$  a Gaussian state become separable. The corresponding quantification of entanglement is given by the logarithmic negativity  $E_{\mathcal{N}}$  [25, 31] defined as,

$$E_{\mathcal{N}}(t) = \max[0, -\ln\{2\tilde{v}_<(t)\}]; \quad (17)$$

which constitute an upper bound to the *distillable entanglement* of any Gaussian state [31]. On evaluating the covariance matrix  $\sigma$  for the state  $|\zeta\rangle$  for  $\gamma = 0$  (no loss), using equation (3), (4) and (14) we find,

$$\alpha = \beta = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}; \quad \mu = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}; \quad (18)$$

where  $c, d, e$  are given by

$$c = \frac{1}{2}\{\cosh(2r) + \sinh(2r)\cos(2Jt)\} \quad (19)$$

$$d = \frac{1}{2}\{\cosh(2r) - \sinh(2r)\cos(2Jt)\}; \quad e = -\frac{1}{2}\sinh(2r)\sin(2Jt). \quad (20)$$

The corresponding symplectic eigenvalues  $\tilde{v}_{\pm}$  are then given by

$$\tilde{v}_{\pm} = \sqrt{cd} \pm e. \quad (21)$$

One can clearly see from equations (17), (19), (20) and (21) the dependence of logarithmic negativity  $E_{\mathcal{N}}$  on coupling strength  $J$  between the waveguides and the squeezing parameter  $r$ . In figure 3, we plot the logarithmic negativity as a function of scaled time,  $\theta = Jt$  for the state  $|\zeta\rangle$ . Here  $t$  is related to the length  $l$  of the waveguide and its refractive index  $n$  by  $t = nl/v$ ,  $v$  being the velocity of light. We see from figure (3) that as  $|\zeta\rangle$  is separable at  $t = 0$ ,  $E_{\mathcal{N}} = 0$  initially but as  $Jt$  increases, it oscillates periodically between a non-zero and zero value. Thus the initially separable state  $|\zeta\rangle$  becomes periodically entangled and disentangled as its propagates through the waveguide. We attribute this periodic generation of entanglement to the coupling  $J$  among the waveguides. We further find that  $\tilde{v}_< = 1/2$  at certain points along the waveguide given by  $2\theta = (k+1)\pi, k = 0, 1, 2, 3, \dots$ . Note that at this points  $E_{\mathcal{N}}$  vanishes and  $|\zeta\rangle$  becomes separable. At all other points the state  $|\zeta\rangle \neq |\zeta_a\rangle \otimes |\zeta_b\rangle$ . We see that  $E_{\mathcal{N}}$  is maximum and has a value equal to the amount of squeezing  $2r$  at the points given by  $2\theta = (k+1)\pi/2$ . Hence at this points the initial separable state  $|\zeta\rangle$  becomes maximally entangled and is given by  $|\zeta\rangle \equiv \exp\{e^{i\pi}r(a^\dagger b^\dagger + ab)\}|00\rangle$ .

#### 4.2. Entangled two mode squeezed state as an input

Let us now study the dynamical evolution of a two mode squeezed state  $|\xi\rangle = \exp[r(a^\dagger b^\dagger - ab)]|00\rangle$  as an input to the waveguide. As before we consider  $r$  to be real. To quantify the entanglement of the state  $|\xi\rangle$  we need to evaluate the logarithmic negativity  $E_{\mathcal{N}}$ . Thus we first evaluate the covariance matrix  $\sigma$  for the state  $|\xi\rangle$  using equations (3) with  $\gamma = 0$ , (4) and (14). We find  $\sigma$  to be

$$\sigma = \begin{bmatrix} f & g & h & 0 \\ g & f & 0 & -h \\ h & 0 & f & g \\ 0 & -h & g & f \end{bmatrix} \quad (22)$$

where  $f, g$  and  $h$  are given by,

$$f = \frac{1}{2}\cosh(2r); \quad g = -\frac{1}{2}\sinh(2r)\sin(2Jt); \quad h = \frac{1}{2}\sinh(2r)\cos(2Jt). \quad (23)$$

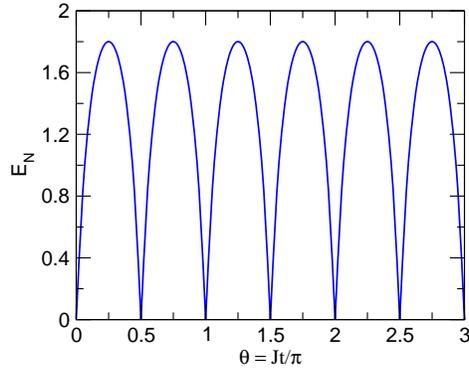


Fig. 3. Plot of the time dependent logarithmic negativity  $E_{\mathcal{N}}$  for the state  $|\xi\rangle$ . Here amount of squeezing is taken to be  $r = 0.9$ .

The corresponding symplectic eigenvalues  $\tilde{v}_{\pm}$  is then given by,

$$\tilde{v}_{\pm} = \sqrt{(f+g)(f-g)} \pm h; \quad (24)$$

The logarithmic negativity  $E_{\mathcal{N}}$  can then be evaluated using equations (15), (17) and (24). From equations (23) and (24) the dependence of  $E_{\mathcal{N}}$  on the squeezing  $r$  and the coupling  $J$  between the waveguides is clearly visible. From equation (24) we find that  $E_{\mathcal{N}} = 0$  i.e. entanglement become zero when,  $2\theta = (k+1)\pi/2$  as then  $\tilde{v}_{<} = 1/2$  and thus the initially entangled state  $|\xi\rangle$  becomes separable, i.e  $|\xi\rangle = \exp\{\frac{r}{2}e^{i\pi}(a^{\dagger 2} + a^2)\}|0\rangle \otimes \exp\{\frac{r}{2}e^{i\pi}(b^{\dagger 2} + b^2)\}|0\rangle$ . The behavior of time evolution of logarithmic negativity for the initial two mode squeezed state is found to be similar to figure (3). In this case though the points of zero entanglement is shifted by  $\pi/4$  with respect to that for the initial separable state  $|\xi\rangle$ . The oscillatory behavior of entanglement is as discussed before, due to the coupling  $J$  among the waveguides. Each time the states get separable the presence of coupling leads to interaction among the modes of the waveguides and creates back the entanglement. Further we find that the logarithmic negativity  $E_{\mathcal{N}}$  reaches maximum at times given by the points  $2\theta = (k+1)\pi$  and is equal to  $2r$ . Thus at this points the state  $|\xi\rangle$  regains its initial form.

## 5. Lossy waveguides

In this section we study the entanglement dynamics of lossy waveguides ( $\gamma \neq 0$ ). The loss  $\gamma$  arises from the loss in the material of the waveguide. In this case the dynamical evolution of the waveguide modes is governed by the full quantum-Liouville equation (3). We next consider the cases of both photon number state and squeezed states at the input of the waveguide and discuss the influence of the loss on their respective entanglement evolution.

### 5.1. Effect of loss on entanglement for photon number states

As discussed above, we first study the effect of loss on the entanglement dynamics of the waveguide modes for photon number input states. For this purpose we consider a single photon input state  $|1, 1\rangle$  as the initial state. In this case we can analytically solve the quantum-Liouville equation described in (3). To proceed further, we work in the interaction picture in which the density matrix is  $\tilde{\rho}(t) = e^{iJt(a^{\dagger}b+b^{\dagger}a)}\rho(t)e^{-iJt(a^{\dagger}b+b^{\dagger}a)}$ . Then in the interaction picture we can

write Eq. (3) as

$$\frac{\partial \tilde{\rho}(t)}{\partial t} = -\gamma(\tilde{a}^\dagger \tilde{a} \tilde{\rho} - 2\tilde{a} \tilde{\rho} \tilde{a}^\dagger + \tilde{\rho} \tilde{a}^\dagger \tilde{a}) - \gamma(\tilde{b}^\dagger \tilde{b} \tilde{\rho} - 2\tilde{b} \tilde{\rho} \tilde{b}^\dagger + \tilde{\rho} \tilde{b}^\dagger \tilde{b}), \quad (25)$$

where  $\tilde{a}$  and  $\tilde{b}$  are given by

$$\tilde{a} = a \cos(Jt) - ib \sin(Jt) \quad (\tilde{b} : a \rightarrow b). \quad (26)$$

Using Eq. (26), we can rewrite Eq. (25) as

$$\frac{\partial \tilde{\rho}(t)}{\partial t} = -\gamma(a^\dagger a \tilde{\rho} - 2a \tilde{\rho} a^\dagger + \tilde{\rho} a^\dagger a) - \gamma(b^\dagger b \tilde{\rho} - 2b \tilde{\rho} b^\dagger + \tilde{\rho} b^\dagger b). \quad (27)$$

For the separable input state  $|1, 1\rangle$ , the solution for the density matrix (27) can be written as [32] :

$$\tilde{\rho}(t) = e^{-4\gamma t} \{ (e^{2\gamma t} - 1)^2 |0, 0\rangle \langle 0, 0| + (e^{2\gamma t} - 1) (|1, 0\rangle \langle 1, 0| + |0, 1\rangle \langle 0, 1|) + |1, 1\rangle \langle 1, 1| \}. \quad (28)$$

Further, we can write  $\rho(t)$  in terms  $\tilde{\rho}(t)$  using the following equation:

$$\rho(t) = e^{-iJt(a^\dagger b + b^\dagger a)} \tilde{\rho}(t) e^{iJt(a^\dagger b + b^\dagger a)}. \quad (29)$$

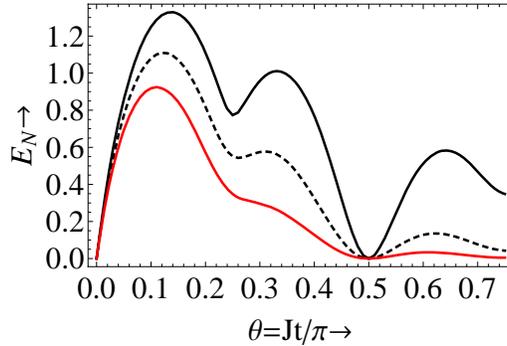


Fig. 4. Time evolution of the logarithmic negativity  $E_{\mathcal{N}}$  in presence of loss of the waveguide modes for the initial separable input state  $|1, 1\rangle$ . The decay rates of the modes are given by  $\gamma/J = 0.1$  (solid black),  $\gamma/J = 0.2$  (broken black) and  $\gamma/J = 0.3$  (red).

The above equation gives the time evolution of the density matrix corresponding to the single photon state  $|1, 1\rangle$ . Following a similar approach as discussed in section 3, we can evaluate the log negativity for the lossy waveguide case. But the resulting expressions are lengthy and do not exhibit a simple structure. Thus we only give the numerical results for the lossy waveguide case. In Fig. (4), we show the decay of entanglement, as a function of scaled time for the state  $|1, 1\rangle$ . Note that the range of  $\gamma/J$  values studied here are similar to the numerical values used in the experiments [22, 23]. For example, the coupling parameter  $J$  for the lithium niobate waveguide lie between  $1.83 \times 10^{10} \text{ sec}^{-1}$  and  $4.92 \times 10^{10} \text{ sec}^{-1}$ . The loss parameter for these waveguides is close to  $3 \times 10^9 \text{ sec}^{-1}$  [22] which corresponds to a value of  $\gamma/J$  between  $1/7$  and  $1/20$ . For AlGaAs waveguides the loss  $\gamma$  is close to  $2.7 \times 10^{10} \text{ sec}^{-1}$  [23]. The coupling parameter  $J$  for these waveguides is about  $2.46 \times 10^{11} \text{ sec}^{-1}$ . Thus the  $\gamma/J$  value for these waveguides is of the order of  $1/10$ . It is worth mentioning that the  $\gamma/J$  value for silica waveguides is significantly lower than the corresponding values for the lithium niobate and AlGaAs waveguides.

This means that even a small loss would add up to a significant decoherence in these complex quantum systems. From Fig. (4) we find that for the lossy waveguide case the entanglement between the waveguide modes decrease with time. In addition, we find that increasing the value of  $\gamma/J$  makes the waveguide modes more fragile, as is evident from Fig. (4). However, we find that the decrease in entanglement is not substantial. Our results indicate that the waveguide system can sustain the entanglement even for the higher decay rates. Thus the coupled waveguide system can be used as an efficient tool for the study of basic quantum optical effects. In addition, the persistence of entanglement suggests that the coupled waveguide system can be used effectively for various applications in quantum information processing [13]. For example, the single photon entanglement described here is a key step for the successful implementation of the CNOT gate [12]. We also studied the behavior of log negativity for the entangled initial state in the form of two photon NOON state. In this case also we found that the entanglement quantified by  $E_{\mathcal{N}}$  shows a considerable robustness against the decoherence effect.

### 5.2. Effect of loss on entanglement for squeezed input states

For the initially separable two mode squeezed state  $|\zeta\rangle$ , we find that elements of the covariance matrix  $\sigma$  in presence of loss become dependent on the decay rate  $\gamma$  and is given by,

$$\sigma = \begin{bmatrix} c' & 0 & 0 & e' \\ 0 & d' & e' & 0 \\ 0 & e' & c' & 0 \\ e' & 0 & 0 & d' \end{bmatrix}; \quad (30)$$

where  $c', d', e'$  are given by

$$c' = \frac{1}{2}\{1 + 2e^{-2\gamma t} \sinh^2(r) + e^{-2\gamma t} \sinh(2r) \cos(2Jt)\} \quad (31)$$

$$d' = \frac{1}{2}\{1 + 2e^{-2\gamma t} \sinh^2(r) - e^{-2\gamma t} \sinh(2r) \cos(2Jt)\}; \quad e' = -\frac{1}{2}e^{-2\gamma t} \sinh(2r) \sin(2Jt). \quad (32)$$

The corresponding symplectic eigenvalue  $\tilde{\nu}$  of the covariance matrix is then found to be  $\tilde{\nu}_{\pm} = \sqrt{c'd'} \pm e'$ . On substituting  $\tilde{\nu}_{\pm}$  in equations (15) and using (17) we get the logarithmic negativity-for lossy waveguides.

To study the dependence of entanglement on loss of the waveguide modes we plot the logarithmic negativity  $E_{\mathcal{N}}$  for different decay rates  $\gamma/J$  in figure (5). As for the case of single photon states we focus on the range of  $\theta$  important from the experiment point of view. We see new features in the entanglement dynamics as an effect of the loss. We see from figure (5) that in presence of loss the maximum value of entanglement for the state  $|\zeta\rangle$  reduces in comparison to the case of lossless waveguides. However it is important to note that this decrease is not substantial. We further find that with increase in decay rate, the entanglement maximum decreases but does not show considerable reduction (the maximum changes by only 0.4 as the decay rate becomes three times). Thus we see that entanglement is quite robust against decoherence in this coupled waveguide systems. The robustness of entanglement dynamics is an artifact of coherent coupling among the waveguide modes. These findings hence suggest that coupled waveguide can be used as an effective quantum circuit for use in quantum information computations. Further we see another new feature in entanglement in figure (5). We find that there exist an interval of  $\theta$  during which the state  $|\zeta\rangle$  remains separable. Note that in absence of loss the state  $|\zeta\rangle$  becomes separable momentarily and entanglement starts to build up instantaneously once it becomes zero (see figure 3.) Thus this feature that entanglement remains zero for certain interval of time arises solely due to loss.

In figure (5b) we plot the long time behavior for entanglement of the state  $|\zeta\rangle$  with very small

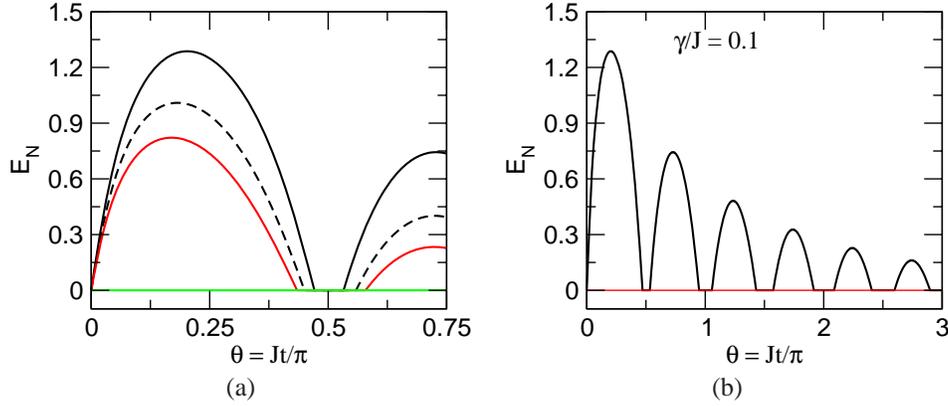


Fig. 5. (a) Time evolution of the logarithmic negativity  $E_{\mathcal{N}}$  in presence of loss of the waveguide modes for the input state  $|\zeta\rangle$ . The decay rates of the modes are given by  $\gamma/J = 0.1$  (solid black),  $\gamma/J = 0.2$  (broken black) and  $\gamma/J = 0.3$  (red). Here the squeezing is taken to be  $r = 0.9$ . The loss leads to new behavior in the entanglement. (b) Long time behavior of the logarithmic negativity in presence of loss of the waveguide modes for the state  $|\zeta\rangle$ .

decay rate of  $\gamma/J = 0.1$  and squeezing parameter  $r = 0.9$ . We see that entanglement decays slowly with increasing  $\theta$  as the magnitude of  $E_{\mathcal{N}}$  diminish successively with every oscillations. In addition periods of disentanglement arises repeatedly in its oscillations. We find that the length of this periods increases with increasing  $\theta$ . It is worth mentioning here that this kind of behavior has been predicted earlier for two qubit entanglement [33, 34].

Next we study the effect of the decay of waveguide mode on the entanglement dynamics of the initial entangled squeezed state  $|\xi\rangle$ . We find in this case the covariance matrix to be,

$$\sigma = \begin{bmatrix} f' & g' & h' & 0 \\ g' & f' & 0 & -h' \\ h' & 0 & f' & g' \\ 0 & -h' & g' & f' \end{bmatrix} \quad (33)$$

where  $f', g', h'$  are given by,

$$f' = \frac{1}{2} + e^{-2\gamma} \sinh^2(r); \quad g' = -\frac{1}{2} e^{-2\gamma} \sinh(2r) \sin(2Jt); \quad h' = \frac{1}{2} e^{-2\gamma} \sinh(2r) \cos(2Jt). \quad (34)$$

The symplectic eigenvalues  $\tilde{\nu}_{\pm}$  are found to be dependent on the decay rate of the waveguide modes and is given by:

$$\tilde{\nu}_{\pm} = \sqrt{m_+ m_-} \pm h', \quad m_{\pm}(t) = 1 - e^{-2\gamma} [1 - \{\cosh(2r) \pm \sinh(2r) \sin(2Jt)\}] \quad (35)$$

The corresponding measure of entanglement given by the logarithmic negativity  $E_{\mathcal{N}}$  can then be calculated by using equation (35), (15) and (17). The behavior of the time dependent logarithmic negativity  $E_{\mathcal{N}}$  for the state  $|\xi\rangle$  in presence of loss is found to be similar to that for the separable state  $|\zeta\rangle$ . For brevity we do not show the plot here. Thus as for the separable states, in case of initial entangled input states entanglement is found to be quite robust in the face of loss.

The loss in waveguides that we discussed in this section arises due to material properties like change in refractive index and absorption. On the other hand there can be decay of the

waveguide modes in the form of leakage to its surrounding also. It should be noted that leakage is inherently different from the evanescent coupling as the former can arise due to scattering and refraction due to refractive index difference at the waveguide boundaries. Thus the analysis of this section is also valid when the leakage is important as for example is the case when one couples channel waveguides to slab waveguides [3, 35].

## 6. Conclusion

To conclude, we investigated the time evolution of entanglement in a coupled waveguide system. We quantified the degree of entanglement between the waveguide modes in terms of logarithmic negativity. We have given explicit analytical results for logarithmic negativity in case of initially separable single photon states and for separable as well as entangled squeezed states. We have also addressed the question of decoherence in coupled waveguide systems by considering loss of waveguide modes. For the lossy waveguides we found that the entanglement shows considerable robustness even for substantial loss. Note that our results are based on experimental parameters and thus should be relevant for applications of waveguides in quantum information sciences. Our results serve as guide for experiments dealing with entanglement in waveguide structures. For efficient use of these waveguides, one should choose the waveguide parameter like  $\theta$  so that one is away from values where the entanglement is minimum.

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