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Geometry on the quantum Heisenberg manifold

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Abstract

A class of C^*-algebras called quantum Heisenberg manifolds were introduced by Rieffel in (Comm. Math. Phys. 122 (1989) 531) as strict deformation quantization of Heisenberg manifolds. Using the ergodic action of Heisenberg group we construct a family of spectral triples. It is shown that associated Kasparov modules are homotopic. We also show that they induce cohomologous elements in entire cyclic cohomology. The space of Connes–deRham forms have been explicitly calculated. Then we characterize torsionless/unitary connections and show that there does not exist a connection that is simultaneously torsionless and unitary. Explicit examples of connections are produced with negative scalar curvature. This part illustrates computations involving some of the concepts introduced in Frohlich et al. (Comm. Math. Phys. 203 (1999) 119), for which to the best of our knowledge no infinite-dimensional example is known other that the noncommutative torus.

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1. Introduction

Let

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

be the Heisenberg group of three by three upper triangular real matrices with ones on the diagonal. For a positive integer c, let H_c be the subgroup of G obtained when x, y, cz are integers. The Heisenberg manifold M_c is the quotient G/H_c. Nonzero Poisson brackets on M_c invariant under left translation by G are parametrized by two real parameters μ, v with $\mu^2 + v^2 \neq 0$ [12]. For each positive integer c and real numbers μ, v, Rieffel constructed a C^*-algebra $A^{c,\hbar}_{\mu,v}$ as an example of deformation quantization along a Poisson bracket [12]. These algebras have further been studied in [1,2,13]. It was also remarked in [12] that it should be possible to construct example of noncommutative geometry as expounded in [6] in these algebras also. It is known [12] that Heisenberg group acts ergodically on $A^{c,\hbar}_{\mu,v}$ and $A^{c,\hbar}_{\mu,v}$ accommodates a unique invariant tracial state τ. Using the group action we construct a family of spectral triples. It is shown that they induce same element in K-homology. We also show that the associated Kasparov module is nontrivial. This has been achieved by constructing explicitly the pairing with a unitary. We also compute the space of forms as described in [6,7]. Then we characterize torsionless and unitary connections. As an immediate corollary it follows that a torsionless unitary connection cannot exist. For a family of unitary connections we compute Ricci curvature and scalar curvature as introduced in [7]. This family has nontrivial curvature. The construction of the canonical completely positive semigroup (heat semigroup in analogy with the classical case) and its stochastic dilation will be treated else where.

Organization of the paper is as follows. In Section 2, after introducing the algebra we compute the GNS space of τ using a crucial result of Weaver [13]. In the next section, following a general principle of construction of spectral triple on a C^*-dynamical system with dynamics governed by a Lie group, we construct spectral triples and compute the hypertrace [5,10] associated with the spectral triple. In Section 4, we compute the space of forms [6, Chapter V]. There are not too many instances of this computation in the literature. In Section 5, after briefly recalling the notions introduced in [7], we compute the space of L^2-forms and characterize torsionless/unitary connections. In the next section we compute Ricci curvature and scalar curvature for a concrete family of unitary connections. In Section 7, we show that the spectral triples we consider give rise to same Kasparov element and that they have nontrivial Chern character.

2. The quantum Heisenberg algebra

For $x \in \mathbb{R}$, we will denote $e^{2\pi i x}$ by $e(x)$.

Definition 1. For any positive integer c let S^c denote the space of infinitely differentiable functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ that satisfy the following two conditions:

(a) $\Phi(x+k, y, p) = e(ckpy)\Phi(x, y, p)$ for all $k \in \mathbb{Z}$, and

(b) for every partial differential operator $\widetilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$ and every polynomial P on $\mathbb{Z}$, the function $P(p)(\widetilde{X}\Phi)(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset K of $\mathbb{R} \times \mathbb{T}$.

For each $\hbar, \mu, v \in \mathbb{R}, \mu^2 + v^2 \neq 0$, let $\mathscr{A}_\hbar^\infty$ denote the space S^c equipped with product and involution defined, respectively, by

$$(\Phi \bigstar \Psi)(x, y, p)$$

$$= \sum_q \Phi(x - \hbar(q-p)\mu, y - \hbar(q-p)v, q)\Psi(x - \hbar q\mu, y - \hbar q v, p - q), \tag{2.1}$$

$$\Phi^*(x, y, p) = \bar{\Phi}(x, y, -p). \tag{2.2}$$

Let π be the representation of $\mathscr{A}_\hbar^\infty$ on $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ given by

$$(\pi(\Phi)\xi)(x, y, p) = \sum_q \Phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)v, q)\xi(x, y, p-q). \tag{2.3}$$

Then π gives a faithful representation of the involutive algebra $\mathscr{A}_\hbar^\infty$. The norm closure of $\pi(\mathscr{A}_\hbar^\infty)$, to be denoted by $\mathscr{A}_{\mu,v}^{c,\hbar}$ is called the quantum Heisenberg manifold. Let $N_\hbar$ denote the weak closure of $\pi(\mathscr{A}_\hbar^\infty)$.

We will identify $\mathscr{A}_\hbar^\infty$ with $\pi(\mathscr{A}_\hbar^\infty)$ without any mention. Since we are going to work with fixed parameters $c, \mu, v, \hbar$ we will drop them altogether and denote $\mathscr{A}_{\mu,v}^{c,\hbar}$ simply by $\mathscr{A}_\hbar$, the subscript merely distinguishes the Heisenberg algebra from a general algebra.

Action of the Heisenberg group: To define the group action we parametrize the points of Heisenberg group by $\mathbb{R}^3$. This is legitimate because as a topological space Heisenberg group is isomorphic with $\mathbb{R}^3$. For $\Phi \in S^c, (r, s, t) \in \mathbb{R}^3$

$$(L_{(r,s,t)}\phi)(x, y, p) = e(p(t + cs(x - r)))\phi(x - r, y - s, p) \tag{2.4}$$

extends to an ergodic action of the Heisenberg group on $\mathscr{A}_{\mu,v}^{c,\hbar}$.

The trace: $\tau : \mathscr{A}_\hbar^\infty \to \mathbb{C}$, given by $\tau(\phi) = \int_0^1 \int_{\mathbb{T}} \phi(x, y, 0)\, dx\, dy$ extends to a faithful normal tracial state on $N_\hbar$. τ is invariant under the Heisenberg group action. So, the group action can be lifted to $L^2(\mathscr{A}_\hbar^\infty)$. We will denote the action at the Hilbert space level by the same symbol.

Theorem 2 (Weaver). *Let* $\mathscr{H} = L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ *and* V_f, W_k, X_r *be the operators defined by*

$$(V_f\xi)(x,y,p) = f(x,y)\xi(x,y,p),$$

$$(W_k\xi)(x,y,p) = e(-ck(p^2\hbar v + py))\xi(x+k,y,p),$$

$$(X_r\xi)(x,y,p) = \xi(x - 2\hbar r\mu, y - 2\hbar r v, p + r).$$

Let $T \in \mathscr{B}(\mathscr{H})$. *Then* $T \in N_\hbar$ *iff* T *commutes with the operators* V_f, W_k, X_r *for all* f *in* $L^\infty(\mathbb{R} \times \mathbb{T})$, *and* $k, r \in \mathbb{Z}$.

Lemma 3. *Let* $S^c_{\infty,\infty,1}$ *be the space of all functions* $\psi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ *satisfying the following conditions* (i) ψ *is measurable*, (ii) $\psi_n = \sup_{x \in \mathbb{R}, y \in \mathbb{T}} |\psi(x,y,n)|$ *is an* l_1 *sequence*, *and* (iii) $\psi(x+k,y,p) = e(ckyp)\psi(x,y,p)$ *for all* $k \in \mathbb{Z}\}$. *Then*, *for* $\phi \in S^c_{\infty,\infty,1}$, $\pi(\phi)$ *defined by* (2.3) *gives a bounded operator on* $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$.

Proof. Let $\widetilde{\phi} : \mathbb{Z} \to \mathbb{R}_+$ be defined by $\widetilde{\phi}(n) = \sup_{x \in \mathbb{R}, y \in \mathbb{T}} |\phi(x,y,n)|$. Then

$$|(\pi(\phi)\xi)(x,y,p)| \leqslant (\widetilde{\phi} \bigstar |\xi(x,y,.)|)(p),$$

where $\bigstar$ denotes convolution on $\mathbb{Z}$ and $|\xi(x,y,.)|$ is the function $p \mapsto |\xi(x,y,p)|$. By Young's inequality we have

$$||(\pi(\phi)\xi)(x,y,.)||_{l_2} \leqslant ||\widetilde{\phi} \bigstar |\xi(x,y,.)| \, ||_{l_2} \leqslant ||\widetilde{\phi}||_{l_1} ||\xi(x,y,.)||_{l_2}$$

consequently $||\pi(\phi)|| \leqslant ||\phi||_{\infty,\infty,1}$, where $||\phi||_{\infty,\infty,1} = ||\widetilde{\phi}||_{l_1}$. □

Remark 4. (i) Product and involution defined by (2.1), and (2.2) turns $S^c_{\infty,\infty,1}$ into an involutive algebra.

(ii) $\phi \mapsto ||\phi||_{\infty,\infty,1}$ is a $*$-algebra norm.

Lemma 5. $\pi(S^c_{\infty,\infty,1}) \subseteq N_\hbar$.

Proof. Follows from Weaver's characterization of $N_\hbar$. □

Proposition 6. $L^2(\mathscr{A}^\infty_\hbar, \tau)$ *is unitarily equivalent with* $L^2(\mathbb{T} \times \mathbb{T} \times \mathbb{Z}) \cong L^2([0,1] \times [0,1] \times \mathbb{Z})$.

Proof. For $\phi \in S^c_{\infty,\infty,1}$, $\Gamma\phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ given by

$$\Gamma\phi(x,y,p) = \begin{cases} e(-cxyp)\phi(x,y,p) & \text{for } y < 1, \\ \phi(x,y,p) & \text{for } y = 1, \end{cases}$$

satisfies $\Gamma\phi(x+k,y,p)=\Gamma\phi(x,y,p)$. Also note that

$$\begin{aligned}\tau(\phi^*\bigstar\phi) &= \int_0^1\int_{\mathbb{T}}\sum_q |\phi(x-\hbar q\mu, y-\hbar q\nu, -q)|^2\,dx\,dy\\ &= \int_0^1\int_{\mathbb{T}}\sum_q |\phi(x,y,q)|^2\,dx\,dy,\end{aligned}$$

and therefore $\tau(\phi^*\bigstar\phi)=||\Gamma\phi||^2$, i.e, $\Gamma: L^2(\mathscr{A}_\hbar^\infty,\tau)\to L^2(\mathbb{T}^2\times\mathbb{Z})$ is an isometry. To see that Γ is a unitary observe that

(i) $N_\hbar\subseteq L^2(\mathscr{A}_\hbar^\infty,\tau)$, since τ is normal;
(ii) $\phi_{m,n,k}$ defined by

$$\phi_{m,n,k}(x,y,p)=\begin{cases} e(cxyp)e(mx+ny)\delta_{kp} & \text{for } 0\leqslant y\leqslant 1\\ \delta_{kp}e(mx) & \text{for } y=1\end{cases}$$

is an element of $S^c_{\infty,\infty,1}\subseteq N_\hbar$;
(iii) $\{\Gamma\phi_{m,n,k}\}_{m,n,k\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T}^2\times\mathbb{Z})$. □

Remark 7. $\phi\mapsto\phi|_{[0,1]\times\mathbb{T}\times\mathbb{Z}}$ gives an unitary isomorphism.

Corollary 8. *Let M_{yp} be the multiplication operator on $\mathscr{H}=L^2(\mathbb{T}^2\times\mathbb{Z})$ explicitly given by $(M_{yp}f)(x,y,p)=ypf(x,y,p)$ on its natural domain. If we consider $\mathscr{A}_\hbar^\infty$ as a subalgebra of $\mathscr{B}(\mathscr{H})$ by the left regular representation then $[M_{yp},\mathscr{A}_\hbar^\infty]\subseteq\mathscr{B}(\mathscr{H})$.*

Proof. Note that for $\phi\in\mathscr{A}_\hbar^\infty$, $(M_{yp}\phi)(x,y,p)=yp\phi(x,y,p)$ gives an element in $S^c_{\infty,\infty,1}$, and hence a bounded operator. Now for $\psi\in\mathscr{A}_\hbar^\infty$,

$$\begin{aligned}&[M_{yp},\phi]\psi(x,y,p)\\ &\quad=\sum_q (yp-(y-\hbar q\nu)(p-q))\phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q)\\ &\qquad\times\psi(x-\hbar q\mu, y-\hbar q\nu, p-q)\\ &\quad=\sum_q q(y-\hbar(q-p)\nu)\phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q)\\ &\qquad\times\psi(x-\hbar q\mu, y-\hbar q\nu, p-q)\\ &\quad=(M_{yp}(\phi)\bigstar\psi)(x,y,p).\end{aligned}$$

This completes the proof. □

3. A class of spectral triples

Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system with G an n-dimensional Lie group, and τ a G-invariant trace on $\mathcal{A}$. Let $\mathcal{A}^\infty$ be the space of smooth vectors, $\mathcal{K} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N$ where $N = 2^{\lfloor n/2 \rfloor}$. Fix any basis $X_1, X_2, \ldots, X_n$ of $L(G)$ the Lie algebra of G. Since G acts as a strongly continuous unitary group on $\mathcal{H} = L^2(\mathcal{A}, \tau)$ we can form self-adjoint operators d_{X_i} on $\mathcal{H}$. Let us define $D : \mathcal{K} \to \mathcal{K}$ by $D = \sum_i d_{X_i} \otimes \gamma_i$, where $\gamma_1, \ldots \gamma_n$ are self-adjoint matrices in $M_N(\mathbb{C})$ such that $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$. The operator D along with $\mathcal{A}^\infty$ and $\mathcal{K}$ should be a candidate for a spectral triple. For such a D, clearly one has $[D, \mathcal{A}^\infty] \subseteq \mathcal{A}^\infty \otimes M_N(\mathbb{C})$.

Proposition 9. *For the quantum Heisenberg manifold, if we identify the Lie algebra of Heisenberg group with the Lie algebra of upper triangular matrices, then D as described above is a self-adjoint operator with compact resolvent with the following choice of X_i's:*

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \in \mathbb{R}$ is greater than one.

Proof. Domain of the operator D consists of all those square integrable functions f defined on $[0,1] \times [0,1] \times \mathbb{Z}$ that satisfy the boundary conditions (i) $f(x,0,p) = f(x,1,p)$, (ii) $f(1,y,p) = e(cpy)f(0,y,p)$, (iii) $pf, \frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are square integrable. On this domain D is defined by

$$D(f \otimes u) = \sum_{j=1}^{3} id_j(f) \otimes \sigma_j(u),$$

where

$$id_1(f) = -i\frac{\partial f}{\partial x},$$
$$id_2(f) = -2\pi cpxf(x,y,p) - i\frac{\partial f}{\partial y},$$
$$id_3(f) = -2\pi pc\alpha f(x,y,p),$$

and σ_j's are the Pauli spin matrices.

Let $\eta : L^2([0,1] \times [0,1] \times \mathbb{Z}) \to L^2([0,1] \times [0,1] \times \mathbb{Z})$ be the unitary given by

$$\eta(f)(x,y,p) = \begin{cases} e(-cxyp)f(x,y,p) & \text{for } y < 1, \\ f(x,y,p) & \text{for } y = 1. \end{cases}$$

Then domain of the operator $D' = (\eta \times I_2)D(\eta \otimes I_2)^{-1}$ is given by all those square integrable functions f that satisfy the boundary conditions, namely (i) $f(0,y,p) = f(1,y,p)$, (ii) $f(x,0,p) = f(x,1,p)$, and (iii) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, pf$ are square integrable. On this domain D' is prescribed by

$$D'(f \otimes u) = \sum_{j=1}^{3} i d_j'(f) \otimes \sigma_j(u),$$

where

$$d_1'(f)(x,y,p) = -2\pi icypf(x,y,p) - \frac{\partial f}{\partial x}(x,y,p),$$
$$d_2'(f)(x,y,p) = -\frac{\partial f}{\partial y}(x,y,p),$$
$$d_3'(f)(x,y,p) = 2\pi ipc\alpha f(x,y,p).$$

Note that, on $Dom(D')$, $D' = T + S$ where $Dom(T) = Dom(D') \subseteq Dom(S)$ and T, S given, respectively, by

$$T = -i\frac{\partial}{\partial x} \otimes \sigma_1 - i\frac{\partial}{\partial y} \otimes \sigma_2 - 2\pi c\alpha M_p \otimes \sigma_3, \quad S = 2\pi c M_{yp} \otimes \sigma_1.$$

These are self-adjoint operators on their respective domains. Also observe that T has compact resolvents. Our conclusion follows from the Rellich lemma since S is relatively bounded with respect to T with relative bound less than $\frac{1}{\alpha} < 1$. $\square$

Theorem 10. *Let* $\mathscr{H} = L^2(\mathscr{A}_\hbar^\infty, \tau) \otimes \mathbb{C}^2$, $\mathscr{A}_\hbar^\infty$ *with its diagonal action becomes a subalgebra of* $\mathscr{B}(\mathscr{H})$. $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ *is an odd spectral triple of dimension* 3.

Proof. The fact that $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ is a spectral triple follows from the previous proposition and the remark preceding that. We only have to show $|D|^{-3} \in \mathscr{L}^{(1,\infty)}$, the ideal of Dixmier traceable operators [6]. For that observe:

(i) Via Fourier transform T can be identified with the operator T' on $L^2(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \otimes \mathbb{C}^2$ given by

$$\frac{T'}{2\pi} = N_1 \otimes \sigma_1 + N_2 \otimes \sigma_2 - 2\pi c\alpha N_3 \otimes \sigma_3,$$

where N_i is the number operator on the appropriate copy of $\mathbb{Z}$. Then T'^2 is nothing but $N_1^2 + N_2^2 + 4\pi^2 c^2 \alpha^2 N_3^2$. Using the fact that the volume of the ball of radius r in $\mathbb{R}^3$ grows like r^3 we get $\mu_n(T^{-1}|_{kerT^\perp}) = \mu_n(T'^{-1}|_{kerT'^\perp}) = O(1/n^{1/3})$, where μ_n stands for the n^{th} singular value.

(ii) S is relatively bounded with relative bound less than $\frac{1}{\alpha} < 1$, hence we have $||S(T+i)^{-1}|| \leqslant \frac{1}{\alpha}$ and $||(1 + S(T+i)^{-1})^{-1}|| \leqslant \frac{\alpha}{\alpha - 1}$.

(iii) $\mu_n(AB) \leqslant \mu_n(A)||B||$, for bounded operators A, B.

Applying (i)–(iii) to $(D'+i)^{-1} = (T+i)^{-1}(1+S(T+i)^{-1})^{-1}$ we get the desired conclusion for D' and hence for D. $\square$

Corollary 11. *Let T, S, D, D' be as in the proof of Proposition* 9. *Let us denote by D_0 the operator $(\eta \otimes I_2)^{-1} T(\eta \otimes I_2)$. Then $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)$ is an odd spectral triple of dimension* 3.

Proof. We only have to show $[D_0, \mathscr{A}_\hbar^\infty] \subseteq \mathscr{B}(\mathscr{H})$. Let $B = (\eta \otimes I_2)^{-1} S(\eta \otimes I_2)$. Then since $\eta \otimes I_2$ commutes with S, we have $B = S$. By Corollary 8, $[B, \mathscr{A}_\hbar^\infty] \subseteq \mathscr{B}(\mathscr{H})$. Now the previous theorem along with $D = D_0 + B$ completes the proof. $\square$

Remark 12. Similarly taking $D_t = D_0 + tB$ one can show that $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_t)$ forms an odd spectral triple of dimension 3, for $t \in [0, 1]$.

Remark 13. D and D_0 constructed above depends on α.

Proposition 14. *If $\{1, \hbar\mu, \hbar\nu\}$ is rationally independent, then the positive linear functional on $\mathscr{A}_\hbar \otimes M_2(\mathbb{C})$ given by $\int : a \mapsto tr_\omega a|D|^{-3}$ coincides with $\frac{1}{2}(tr_\omega |D|^{-3})\tau \otimes tr$, where tr_ω is a Dixmier trace* [6].

Proof. Observe that $D^2 = \left(\begin{smallmatrix} X_1 & 0 \\ 0 & X_2 \end{smallmatrix}\right)$, where

$$X_1 = -\left(d_1^2 + d_2^2 + \left(d_3 + \frac{1}{2\alpha}\right)^2 - \frac{1}{4\alpha^2}\right),$$

$$X_2 = -\left(d_1^2 + d_2^2 + \left(d_3 - \frac{1}{2\alpha}\right)^2 - \frac{1}{4\alpha^2}\right).$$

It is easily seen that:

(i) compactness of resolvents of D^2 implies that for X_1, X_2,

(ii) eigenvalues of X_1, X_2 have similar asymptotic behaviour.

Therefore $X_1^{-3/2}, X_2^{-3/2} \in \mathscr{L}^{(1,\infty)}$ and $tr_\omega\, aX_1^{-3/2} = tr_\omega\, aX_2^{-3/2}$ for any $a \in \mathscr{B}(L^2(\mathscr{A}_\hbar))$. Consider the unitary group on $\mathscr{H} \cong L^2([0,1] \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2$ given by

$$U_t(x \otimes y \otimes e_p \otimes z) = e(pt)(x \otimes y \otimes e_p \otimes z).$$

Then $U_t D = D U_t$ and

$$\int A = tr_\omega\, U_t A U_t^* |D|^{-3} = tr_\omega\left(\left(\int_0^1 U_t A U_t^*\, dt\right)|D|^{-3}\right) = \int (A)_0,$$

where

$$A = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \mapsto (A)_0 = \begin{pmatrix} (\psi_{11})_0 & (\psi_{12})_0 \\ (\psi_{21})_0 & (\psi_{22})_0 \end{pmatrix}$$

is the completely positive map given by $(\psi)_0(x,y,p) = \delta_{p0}\psi(x,y,p)$ for $\psi \in S^c$. Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ commutes with $|D|^{-3}$, we get

$$\begin{aligned} \int A &= tr_\omega (a_{11})_0 X_1^{-3/2} + tr_\omega (a_{22})_0 X_2^{-3/2} \\ &= tr_\omega ((a_{11})_0 + (a_{22})_0) X_1^{-3/2}. \end{aligned}$$

Consider the homomorphism $\Phi : C(\mathbb{T}^2) \to \mathscr{A}_\hbar$ given by $\Phi(f)(x,y,p) = \delta_{p0} f(x,y)$. Now by Riesz representation theorem for $\int \circ (\Phi \otimes I_2) : C(\mathbb{T}^2) \to \mathbb{C}$, we get a measure λ on $\mathbb{T}^2$ such that $tr_\omega\, 2(\psi)_0 X_1^{-3/2} = \int (\psi)_0(x,y,0)\, d\lambda$ implying

$$\int A = \frac{1}{2} \int ((a_{11})_0 + (a_{22})_0)\, d\lambda. \qquad (3.1)$$

In the next lemma we show λ is proportional to Lebesgue measure. That will prove that $\int$ is proportional with $\tau \otimes tr$ and the proportionality constant is obtained by evaluating both sides on I. □

Lemma 15. *If $\{1, \hbar\mu, \hbar v\}$ is rationally independent then λ as obtained in the previous proposition is proportional to Lebesgue measure.*

Proof. It is known [5,10] that for a spectral triple $(\mathscr{A}, \mathscr{H}, \mathscr{D})$ with $|\mathscr{D}|^{-p} \in \mathscr{L}^{(1,\infty)}$ for some $p, a \mapsto tr_\omega\, a|\mathscr{D}|^{-p}$ is a trace on the algebra. This along with (3.1) gives

$$\int (\phi \star \psi)(x,y,0)\, d\lambda(x,y) = \int (\psi \star \phi)(x,y,0)\, d\lambda(x,y), \quad \forall \phi, \psi \in S^c. \qquad (3.2)$$

Taking $\phi(x,y,p) = e(c[x]yp) f(x - [x]) g(y) \delta_{1p}$ where $g : \mathbb{T} \to \mathbb{C}, f : [0,1] \to \mathbb{C}$ are smooth functions with $supp(f) \subseteq [\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$ and $\psi = \phi^*$ we get from (3.2)

$$\int |\phi(x + \hbar\mu, y + \hbar v, 1)|^2\, d\lambda(x,y) = \int |\phi \circ \gamma(x + \hbar\mu, y + \hbar v, 1)|^2 \lambda(x,y), \quad (3.3)$$

where $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ is given by $\gamma(x,y) = (x - 2\hbar\mu, y - 2\hbar v)$. The hypothesis of linear independence of $(1, \hbar\mu, \hbar v)$ over the rationals implies that γ-orbits are dense. This along with (3.3) proves the lemma. □

Remark 16. In the rest of the paper $\int$ will denote $\frac{1}{2} \tau \otimes tr$.

4. Space of forms

Lemma 17. *Let $\mathscr{A}$ be a dense subalgebra of a unital C^* algebra $\bar{\mathscr{A}}$ closed under holomorphic function calculus, then $\mathscr{A}$ is simple provided $\bar{\mathscr{A}}$ is so.*

Proof. Let $J \subseteq \mathscr{A}$ be an ideal. Then $\bar{J} = \bar{\mathscr{A}}$, since $\bar{\mathscr{A}}$ is simple. There exists $x \in J$ such that $||x - I|| < 1$. Then $x^{-1} \in \bar{\mathscr{A}}$, hence in $\mathscr{A}$ because $\mathscr{A}$ is closed under holomorphic function calculus. Therefore $1 = xx^{-1} \in J$. $\square$

Assumption 18. Henceforth, we will assume $\{1, \hbar\mu, \hbar\nu\}$ is rationally independent. In that case $\mathscr{A}_\hbar$ is simple [12], hence so is $\mathscr{A}_\hbar^\infty$.

Definition 19 (Connes). Let $(\mathscr{A}, \mathscr{H}, D)$ be a spectral triple. Let

$$\Omega^k(\mathscr{A}) = \left\{ \sum_{i=1}^{N} a_0^i \delta a_1^i \ldots \delta a_k^i \mid n \in \mathbb{N}, a_j^i \in \mathscr{A} \right\}, \Omega^\bullet(\mathscr{A}) = \bigoplus_0^\infty \Omega^k(\mathscr{A})$$

be the unital graded algebra of universal forms. Here δ is an abstract linear operator satisfying $\delta^2 = 0, \delta(ab) = \delta(a)b + a\delta(b)$. $\Omega^\bullet(\mathscr{A})$ becomes a *-algebra under the involution $(\delta a)^* = -\delta(a^*)$ for all $a \in \mathscr{A}$. Let $\pi : \Omega^\bullet(\mathscr{A}) \to \mathscr{B}(\mathscr{H})$ be the *-representation given by $\pi(a) = a, \pi(\delta a) = [D, a]$ Let $J_k = \ker \pi|_{\Omega^k(\mathscr{A})}$. The unital graded differential *-algebra of differential forms $\Omega_D^\bullet(\mathscr{A})$ is defined by

$$\Omega_D^\bullet(\mathscr{A}) = \bigoplus_0^\infty \Omega_D^k(\mathscr{A}), \quad \Omega_D^k(\mathscr{A}) = \Omega^k(\mathscr{A})/(J_k + \delta J_{k-1}) \cong \pi(\Omega^k(\mathscr{A}))/\pi(\delta J_{k-1}).$$

Let us introduce some notations before we proceed further. Let $\phi \in S^c$, then $[D, \phi] = \sum \delta_i(\phi) \otimes \sigma_i$ where $\delta_j(\phi) = id_j(\phi)$ (see proof of Proposition 9 for d_j) but looked upon as derivation on $\mathscr{A}_\hbar^\infty$. Also note that $[\delta_1, \delta_3] = [\delta_2, \delta_3] = 0, [\delta_1, \delta_2] = \delta_3$. In the sequel, we will need a special class of elements of $\mathscr{A}_\hbar^\infty$ whose symbols are given by $\phi_{m,n}(x, y, p) = e(mx + ny)\delta_{p0}$.

Lemma 20. *Let $\mathscr{A}$ be a unital simple algebra, $M \subseteq \overbrace{\mathscr{A} \oplus \ldots \oplus \mathscr{A}}^{n\ times}$ a sub $\mathscr{A}$–$\mathscr{A}$ bimodule. Suppose there exists $a_{ij}, 1 \leqslant n, 1 \leqslant j \leqslant i$ such that* (i) $a_{ii} \neq 0$, (ii) $b_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0) \in M$.

Then M is isomorphic to $\cong \underbrace{\mathscr{A} \oplus \ldots \oplus \mathscr{A}}_{n\ times}$ as an $\mathscr{A}$–$\mathscr{A}$ bimodule.

Proof. By induction on n,

For $n = 1, 0 \neq M$ is an ideal in $\mathscr{A}$, hence $M = \mathscr{A}$. Let $\pi : M \to \mathscr{A}$ be $\pi(a_1, \ldots, a_n) = a_n$.

Then by hypothesis, $\pi(M)$ is a nontrivial ideal in $\mathscr{A}$ and hence equals $\mathscr{A}$. So, we have a split short exact sequence

$$0 \to \ker(\pi) \to M \to \mathscr{A} \to 0.$$

Therefore $M = ker(\pi) \oplus Im\,\pi = ker(\pi) \oplus \mathscr{A} = \underbrace{\mathscr{A} \oplus \ldots \oplus \mathscr{A}}_{n \text{ times}}$. In the last equality we have used induction hypothesis for $ker(\pi)$. $\square$

Proposition 21. (i)

$$\begin{aligned}\Omega^1_D(\mathscr{A}^\infty_\hbar) &= \left\{\sum a_i \otimes \sigma_i \mid a_i \in \mathscr{A}^\infty_\hbar, \sigma_i\text{'s are spin matrices}\right\} \\ &= \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar.\end{aligned}$$

(ii) $\pi(\Omega^k(\mathscr{A}^\infty_\hbar)) = \mathscr{A}^\infty_\hbar \otimes M_2(\mathbb{C}) = \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar$ *for* $k \geqslant 2$.

Proof. $\Omega^1_D(\mathscr{A}^\infty_\hbar) = \pi(\Omega^1(\mathscr{A}^\infty_\hbar)) \subseteq$ RHS.

Let $\phi_{mn}(x,y,p) = \delta_{p0} e(mx+ny)$ and $\phi \in S^c$ be such that $\phi(x,y,p) = \delta_{p1}\phi(x,y,p)$. Then applying the previous lemma to $[D,\phi_{01}], [D,\phi_{10}], [D,\phi] \in \pi(\Omega^1(\mathscr{A}))$ we get result (i).

For (ii) use (i) along with $\Omega^k(\mathscr{A}^\infty_\hbar) = \underbrace{\Omega^1(\mathscr{A}^\infty_\hbar) \otimes_{\mathscr{A}^\infty_\hbar} \ldots \otimes_{\mathscr{A}^\infty_\hbar} \Omega^1(\mathscr{A}^\infty_\hbar)}_{k \text{ times}}$. $\square$

Proposition 22. (i) $\pi(\delta J_1) = \mathscr{A}^\infty_\hbar$.

(ii) $\Omega^2_D(\mathscr{A}^\infty_\hbar) = \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar \oplus \mathscr{A}^\infty_\hbar$.

Proof. (i) Let $\omega = \sum a_i \delta(b_i) \in J_1$. Then $\pi(\omega) = \sum a_i \delta_j(b_i)\sigma_j = 0$ gives $\sum a_i \delta_j(b_i) = 0, \forall j$.

$$\begin{aligned}\pi(\delta\omega) &= \sum_i \left(\sum_j \delta_j(a_i)\sigma_j\right)\left(\sum_k \delta_k(b_i)\sigma_k\right) \\ &= \sum_i \left(\sum_j \delta_j(a_i)\delta_j(b_i)\right) \otimes I_2 \\ &\quad + \sum_i \left(\sum_{j<k} (\delta_j(a_i)\delta_k(b_i) - \delta_k(a_i)\delta_j(b_i))\sigma_j\sigma_k\right), \qquad (4.1)\end{aligned}$$

$$\begin{aligned}\sum_i [\delta_j, \delta_k](a_i b_i) &= \sum_i \delta_j(\delta_k(a_i)b_i) - \delta_k(\delta_j(a_i)b_i) \quad \left[\text{Since } \sum a_i\delta_j(b_i) = 0, \forall j\right] \\ &= \sum_i [\delta_j, \delta_k](a_i)b_i + \sum_i (\delta_k(a_i)\delta_j(b_i) - \delta_j(a_i)\delta_k(b_i)). \qquad (4.2)\end{aligned}$$

Also note

$$\begin{aligned}\sum_i [\delta_j,\delta_k](a_ib_i) &= \sum_i [\delta_j,\delta_k](a_i)b_i + \sum_i a_i[\delta_j,\delta_k](b_i)\\ &= \sum_i [\delta_j,\delta_k](a_i)b_i. \end{aligned} \tag{4.3}$$

Comparing right-hand side of (4.2) and (4.3) we see that the second term on the right-hand side of (4.1) vanishes, thus proving $\pi(\delta J_1)\subseteq\mathscr{A}_\hbar^\infty$. For equality in view of Lemma 20 it is enough to note that $\omega = 2\phi_{02}\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\in J_1$, $\pi(\delta\omega) = 2\phi_{03}\otimes I_2\neq 0$.

(ii) Suppose $\phi\in S^c$ satisfies $\phi(x,y,p) = \delta_{1p}\phi(x,y,p)$. Let $\omega_1 = \delta(\phi_{10})\delta(\phi_{01})$, $\omega_2 = \delta(\phi_{10})\delta(\phi)$, $\omega_3 = \delta(\phi_{01})\delta(\phi)$. Now Lemma 20 together with (i) implies the result. □

Lemma 23. $\pi(\delta J_2) = \{\sum a_j\otimes\sigma_j | a_j\in\mathscr{A}_\hbar^\infty\} = \mathscr{A}_\hbar^\infty\oplus\mathscr{A}_\hbar^\infty\oplus\mathscr{A}_\hbar^\infty$.

Proof. Let $\omega = \sum a_i\delta(b_i)\delta(c_i)\in J_2$,

$$\begin{aligned} 0 = \pi(\omega) &= \sum a_i\left(\sum_j \delta_j(b_i)\sigma_j\right)\left(\sum_j \delta_k(c_i)\sigma_k\right)\\ &= a_i\delta_j(b_i)\delta_j(c_i) + \sum_{j<k} a_i(\delta_j(b_i)\delta_k(c_i) - \delta_k(b_i)\delta_j(c_i)\sigma_j\sigma_k. \end{aligned}$$

Comparing the coefficients of the various spin matrices we get

$$\sum a_i\delta_j(b_i)\delta_j(c_i) = 0, \tag{4.4}$$

$$\sum a_i(\delta_j(b_i)\delta_k(c_i) - \delta_k(b_i)\delta_j(c_i)) = 0, \quad \forall j\neq k, \tag{4.5}$$

from (4.5),

$$\begin{aligned} 0 &= \sum \delta_1(a_i(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))\\ &= \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))\\ &\quad + \sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)). \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))\\ &\quad = -\sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)). \end{aligned} \tag{4.6}$$

Similarly we get two more equalities. Let A be the coefficient of I_2 in $\pi(\delta\omega)$. Then

$$\begin{aligned}
\sqrt{-1}A &= \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) \\
&\quad + \sum \delta_2(a_i)(\delta_3(b_i)\delta_1(c_i) - \delta_1(b_i)\delta_3(c_i)) \\
&\quad + \sum \delta_3(a_i)(\delta_1(b_i)\delta_2(c_i) - \delta_2(b_i)\delta_1(c_i)) \\
&= - \Big(\sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) \\
&\quad + \sum a_i\delta_2(\delta_3(b_i)\delta_1(c_i) - \delta_1(b_i)\delta_3(c_i)) \\
&\quad + \sum a_i\delta_1(\delta_1(b_i)\delta_2(c_i) - \delta_2(b_i)\delta_1(c_i))\Big) \\
&= - \Big(\sum a_i([\delta_1,\delta_2](b_i)\delta_3(c_i) + \delta_2(b_i)[\delta_1,\delta_3](c_i)) \\
&\quad + \sum a_i([\delta_3,\delta_1](b_i)\delta_2(c_i) + \delta_3(b_i)[\delta_2,\delta_1](c_i)) \\
&\quad + \sum a_i([\delta_2,\delta_3](b_i)\delta_1(c_i) + \delta_1(b_i)[\delta_3,\delta_2](c_i))\Big) \\
&= 0.
\end{aligned}$$

Here second equality follows from (4.6) and the last equality follows from (4.5) since δ_j's form a Lie algebra. This shows,

$$\pi(\delta J_2) \subseteq \left\{ \sum_{j=1}^{3} a_j\sigma_j \mid a_j \in \mathscr{A}_\hbar^\infty \right\} \cong \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty. \tag{4.7}$$

Let $\phi \in S^c$ be such that $\phi(x,y,p) = \delta_{1p}\phi(x,y,p)$. Then,

$$\begin{aligned}
\omega_1 &= 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01}) \in J_2, \\
\omega_2 &= 2\phi_{20}\delta(\phi_{10})\delta(\phi_{10}) - \phi_{10}\delta(\phi_{20})\delta(\phi_{10}) \in J_2, \\
\omega_3 &= \phi_{02}\delta(\phi_{01})\delta(\phi) - \phi_{01}\delta(\phi_{02})\delta(\phi) \in J_2,
\end{aligned}$$

satisfies

$$\begin{aligned}
\pi(\delta\omega_1) &= 2\phi_{04}\sigma_2, \\
\pi(\delta\omega_2) &= 2\phi_{40}\sigma_1, \\
\pi(\delta\omega_3) &= 2\phi_{03}\delta_1(\phi)\sigma_1 + 2\phi_{03}\delta_2(\phi)\sigma_2 + 2\phi_{03}\delta_3(\phi)\sigma_3.
\end{aligned}$$

Therefore by Lemma 20 we get equality in (4.7). $\square$

Corollary 24. $\Omega_D^3(\mathscr{A}_\hbar^\infty) = \mathscr{A}_\hbar^\infty$.

Proof. Immediate from the previous lemma and Proposition 3.5(ii). $\square$

Lemma 25. (i) $\Omega_D^4(\mathscr{A}_\hbar^\infty) = 0$.
(ii) $\Omega_D^k(\mathscr{A}_\hbar^\infty) = 0, \forall k > 4$.

Proof. (i) It suffices to show $\pi(\delta J_3) = \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty$.
For that note

$$\omega_1 = 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01})\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01})\delta(\phi_{01}) \in J_3,$$

$$\omega_2 = 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01})\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01})\delta(\phi_{01}) \in J_3,$$

$$\omega_3 = 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01})\delta(\phi) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01})\delta(\phi) \in J_3,$$

$$\omega_4 = 2\phi_{02}\delta(\phi_{01})\delta(\phi_{10})\delta(\phi) - \phi_{01}\delta(\phi_{02})\delta(\phi_{10})\delta(\phi) \in J_3,$$

satisfies

$$\pi(\delta\omega_1) = 2\phi_{05} \otimes I_2,$$

$$\pi(\delta\omega_2) = 2\phi_{14}\sigma_2\sigma_1,$$

$$\pi(\delta\omega_3) = 2\phi_{04}\delta_2(\phi) \otimes I_2 + 2\phi_{04}\delta_1(\phi)\sigma_2\sigma_1 + 2\phi_{04}\delta_3(\phi)\sigma_2\sigma_3,$$

$$\pi(\delta\omega_4) = 2\phi_{13}\delta_1(\phi)I_2 + 2\phi_{13}\delta_2(\phi)\sigma_1\sigma_2 + 2\phi_{13}\delta_3(\phi)\sigma_1\sigma_3.$$

Now an application of Lemma 20 completes the proof.
(ii) The same argument as in (i) does the job with the following choice: $\omega_i{}' = \omega_i \underbrace{\delta(\phi_{01})\ldots\delta(\phi_{01})}_{(k-4)\text{ times}}, \ i = 1, \ldots, 4.$ $\square$

5. Torsionless and unitary connections

Definition 26Fröhlich et al. [7]
(i) $\int$ determines a semi-definite sesquilinear form on $\Omega^\bullet(\mathscr{A}_\hbar^\infty)$ by setting

$$(\omega, \eta) = \int \pi(\omega)\pi(\eta)^* \quad \forall \omega, \eta \in \Omega^\bullet(\mathscr{A}_\hbar^\infty).$$

(ii) Let

$$K_k = \{\omega \in \Omega^k(\mathscr{A}_\hbar^\infty) | (\omega, \omega) = 0\}, \quad K = \bigoplus_{k=0}^{\infty} K_k$$

$K, K+\delta K$ are two sided *-ideals, the later is closed under differential.

$$\widetilde{\Omega}^{\bullet}(\mathscr{A}_\hbar^\infty) = \bigoplus_{k=0}^{\infty} \widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty), \quad \widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty) = \Omega^k(\mathscr{A}_\hbar^\infty)/K_k.$$

(iii) $\widetilde{\mathscr{H}}^k$ denotes the Hilbert space completion of $\widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty)$ with respect to the scalar product. $\widetilde{\mathscr{H}}^\bullet = \oplus_{k=0}^\infty \widetilde{\mathscr{H}}^k$, $\widetilde{\mathscr{H}}^k$ is to be interpreted as the space of square-integrable k-forms.

(iv) The algebra multiplication of $\Omega^\bullet(\mathscr{A}_\hbar^\infty)$ descends to a linear map

$$m : \widetilde{\Omega}^\bullet(\mathscr{A}_\hbar^\infty) \bigotimes_{\mathscr{A}_\hbar^\infty} \widetilde{\Omega}^\bullet(\mathscr{A}_\hbar^\infty) \to \widetilde{\Omega}^\bullet(\mathscr{A}_\hbar^\infty).$$

(v) The unital graded differential *-algebra of square-integrable differential forms is defined by

$$\widetilde{\Omega}_D^\bullet(\mathscr{A}_\hbar^\infty) = \bigoplus_{k=0}^{\infty} \widetilde{\Omega}_D^k(\mathscr{A}_\hbar^\infty), \quad \widetilde{\Omega}_D^k(\mathscr{A}_\hbar^\infty) = \widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty)/(K_k + \delta K_{k-1}).$$

(vi) $\delta : \Omega^\bullet(\mathscr{A}_\hbar^\infty) \to \Omega^{\bullet+1}(\mathscr{A}_\hbar^\infty)$ descends to a linear map $\delta : \widetilde{\Omega}_D^\bullet(\mathscr{A}_\hbar^\infty) \to \widetilde{\Omega}_D^{\bullet+1}(\mathscr{A}_\hbar^\infty)$.

(vii) A *connection* ∇ on a finitely generated projective $\mathscr{A}_\hbar^\infty$ module $\mathscr{E}$ is a $\mathbb{C}$ linear map

$$\nabla : \widetilde{\Omega}_D^\bullet(\mathscr{A}_\hbar^\infty) \otimes \mathscr{E} \to \widetilde{\Omega}_D^{\bullet+1}(\mathscr{A}_\hbar^\infty) \otimes \mathscr{E},$$

such that $\nabla(\omega s) = \delta(\omega)s + (-1)^k \omega \nabla(s)$ for all $\omega \in \widetilde{\Omega}_D^k(\mathscr{A}_\hbar^\infty)$ and all $s \in \widetilde{\Omega}_D^\bullet(\mathscr{A}_\hbar^\infty) \otimes \mathscr{E}$.

(viii) The *curvature* of a connection ∇ on $\mathscr{E}$ is given by

$$R(\nabla) = -\nabla^2 : \mathscr{E} \to \widetilde{\Omega}_D^2(\mathscr{A}_\hbar^\infty) \bigotimes_{\mathscr{A}_\hbar^\infty} \mathscr{E}.$$

Remark 27. Each $\omega \in \widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty)$ determines two operators $m_L(\omega), m_R(\omega)$ from $\widetilde{\Omega}^n(\mathscr{A}_\hbar^\infty)$ to $\widetilde{\Omega}^{n+k}(\mathscr{A}_\hbar^\infty)$ given by $m_L(\omega)(\eta) = m(\omega \otimes \eta), m_R(\omega)(\eta) = m(\eta \otimes \omega)$. These operators extend to bounded linear operators $m_L(\omega), m_R(\omega) : \widetilde{\mathscr{H}}^n \to \widetilde{\mathscr{H}}^{n+k}$ for all n.

Proposition 28. *For* $k \geqslant 2$ *we have*

(i) $\widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty) = \mathscr{A}_\hbar^\infty \otimes M_2(\mathbb{C}) \cong \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty \oplus \mathscr{A}_\hbar^\infty$,

(ii) $\widetilde{\mathscr{H}}^k = L^2(\mathscr{A}_\hbar^\infty, \tau) \otimes \mathbb{C}^4$,

(iii) $\widetilde{\Omega}_D^k(\mathscr{A}_\hbar^\infty) = \Omega_D^k(\mathscr{A}_\hbar^\infty)$.

Proof. (i) Faithfulness of $A \mapsto \int A$, defined on $\pi(\Omega^\bullet(\mathscr{A}_\hbar^\infty)) = \mathscr{A}_\hbar^\infty \otimes M_2(\mathbb{C})$ gives $J_k = K_k$.

Hence $\widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty) = \Omega^k(\mathscr{A}_\hbar^\infty)/ker(\pi) \cong \pi(\Omega^k(\mathscr{A}_\hbar^\infty)) = \mathscr{A}_\hbar^\infty \otimes M_2(\mathbb{C})$.

(ii) Follows from (i) and Proposition 14.

(iii)This follows from (i) and the definitions. □

Remark 29. Since $\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)$ is free with 3 generators, we can and will identify $\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)\otimes_{\mathscr{A}^\infty_\hbar}\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)$ with $\mathscr{A}^\infty_\hbar\otimes M_3(\mathbb{C})$ and a connection ∇ is specified by its value on the generators.

Definition 30. A connection $\nabla:\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)\to\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)\otimes_{\mathscr{A}^\infty_\hbar}\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)$ is called torsionless if $T(\nabla)=\delta-m\circ\nabla:\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)\to\widetilde{\Omega}^2_D(\mathscr{A}^\infty_\hbar)$ vanishes.

Proposition 31. *A connection is torsionless iff its values on the generators* $\sigma_1,\sigma_2,\sigma_3$ *are given by*

$$\nabla(\sigma_1)=\begin{pmatrix}\square & a & b\\ a & \square & c\\ b & c & \square\end{pmatrix},\quad \nabla(\sigma_2)=\begin{pmatrix}\square & d & e\\ d & \square & f\\ e & f & \square\end{pmatrix},\quad \nabla(\sigma_3)=\begin{pmatrix}\square & p-1 & q\\ p & \square & r\\ q & r & \square\end{pmatrix},$$

where all the matrix entries are from $\mathscr{A}^\infty_\hbar$ *with restrictions on them as indicated above and* $\square$ *denotes an unrestricted entry.*

Proof. Note that

$$\begin{aligned}\delta\left(\sum_{i,j}a_i\delta_j(b_i)\sigma_j\right)=-\sqrt{-1}\Bigg(&\sum_i(\delta_1(a_i)\delta_2(b_i)-\delta_2(a_i)\delta_1(b_i))\sigma_3\\&+\sum_i(\delta_2(a_i)\delta_3(b_i)-\delta_3(a_i)\delta_2(b_i))\sigma_1\\&+\sum_i(\delta_3(a_i)\delta_1(b_i)-\delta_1(a_i)\delta_3(b_i))\sigma_2\Bigg),\end{aligned}$$

$$\begin{aligned}m\circ\nabla\left(\sum_{i,j}a_i\delta_j(b_i)\sigma_j\right)&=m\left(\sum_{i,j}\delta(a_i\delta_j(b_i))\otimes\sigma_j\right)+\sum_{i,j}a_i\delta_j(b_i)m\circ\nabla(\sigma_j)\\&=m\left(\sum_{i,j,k}\delta_k(a_i\delta_j(b_i))\sigma_k\otimes\sigma_j\right)+\sum_{i,j}a_i\delta_j(b_i)m\circ\nabla(\sigma_j).\end{aligned}$$

Torsion of ∇ vanishes iff $(\delta-m\circ\nabla)(\sum a_i\delta_j(b_i)\sigma_j)\equiv 0$, or equivalently,

$$\begin{aligned}&\sum_i(\delta_j(a_i)\delta_k(b_i)-\delta_k(a_i)\delta_j(b_i))\\&\quad=\sum_i(\delta_j(a_i\delta_k(b_i))-\delta_k(a_i\delta_j(b_i)))+\sum_{i,l}a_i\delta_l(b_i)(m\circ\nabla(\sigma_l))_n\end{aligned}$$

whenever $j \neq k$ and n satisfies $\sigma_j \sigma_k \sigma_n = \sqrt{-1}$. This happens iff

$$0 = \sum_i a_i[\delta_j, \delta_k](b_i) + \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_n$$

whenever $j \neq k$ and n satisfies $\sigma_j \sigma_k \sigma_n = \sqrt{-1}$.

Using the Lie algebra relations between the δ_j's we get equivalence of the above system of equations with

$$0 = \sum_i a_i \delta_3(b_i) + \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_3,$$
$$0 = \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_2,$$
$$0 = \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_1.$$

Taking $b_i = \phi_{01}, a_i = 1$ we get $\delta_1(b_i) = \delta_3(b_i) = 0, \delta_2(b_i) = b_i$. Substituting these in the above relations we get $(m \circ \nabla(\sigma_2))_j = 0$ for $j = 1, 2, 3$. Similarly taking $b_i = \phi_{10}, a_i = 1$ we get $(m \circ \nabla(\sigma_1))_j = 0$ for $j = 1, 2, 3$. Substituting these values in the above equations we get

$$\sum_i a_i \delta_3(b_i)(m \circ \nabla(\sigma_3))_1 = 0,$$
$$\sum_i a_i \delta_3(b_i)(m \circ \nabla(\sigma_3))_2 = \sum_i a_i \delta_3(b_i)(1 + (m \circ \nabla(\sigma_3))_3 = 0.$$

Note that $J = \{\sum a_i \delta_3(b_i) | n \in \mathbb{N}, a_1, \ldots, a_i, b_1, \ldots, b_i \in \mathscr{A}_\hbar^\infty\}$ is a nontrivial ideal in $\mathscr{A}_\hbar^\infty$ and hence it equals $\mathscr{A}_\hbar^\infty$. Therefore $(m \circ \nabla(\sigma_3))_3 = -1$ and $(m \circ \nabla(\sigma_3))_1 = (m \circ \nabla(\sigma_3))_2 = 0$. Now the result follows from the anticommutation relation between the spin matrices. □

Definition 32. A connection on a finitely generated projective $\mathscr{A}_\hbar^\infty$ module $\mathscr{E}$, endowed with an $\mathscr{A}_\hbar^\infty$ valued inner product $\langle \cdot, \cdot \rangle$ is called unitary if

$$\delta \langle s, t \rangle = \langle \nabla s, t \rangle - \langle s, \nabla t \rangle, \forall s, t \in \mathscr{E},$$

where the right-hand side of this equation is defined by $\langle \omega \otimes s, t \rangle = \omega \langle s, t \rangle$, and $\langle s, \eta \otimes t \rangle = \langle s, t \rangle \eta^*$.

Proposition 33. *A connection* ∇ *on* $\widetilde{\Omega}^1_D(\mathscr{A}_\hbar^\infty)$ *is unitary iff its values on the generators* $\sigma_1, \sigma_2, \sigma_3$ *are given by*

$$\nabla(\sigma_1) = \begin{pmatrix} X & Y & Z \\ Y & U & P \\ Z & V & Q \end{pmatrix}, \qquad \nabla(\sigma_2) = \begin{pmatrix} Y & U & V \\ U & R & S \\ P & S & F \end{pmatrix},$$

$$\nabla(\sigma_3) = \begin{pmatrix} Z & P & Q \\ V & S & F \\ Q & F & G \end{pmatrix},$$

where all the matrix entries are self-adjoint elements of $\mathscr{A}_\hbar^\infty$.

Proof. Taking $s = a_i\sigma_i, t = b_j\sigma_j$ in the defining condition of a unitary connection we get

$$\delta(\delta_{ij}a_ib_j^*) = a_i(\langle \nabla(\sigma_i), \sigma_j \rangle - \langle , \sigma_i, \nabla(\sigma_j) \rangle)b_j^* + \delta_{ij}(\delta(a_i)b_j^* - a_i(\delta(b_j))^*) \tag{5.1}$$

implying that $\langle \nabla(\sigma_i), \sigma_j \rangle = \langle \sigma_i, \nabla(\sigma_j) \rangle$, which means the jth row of $\nabla(\sigma_i)$ is the star of the ith column of $\nabla(\sigma_j)$. This completes the proof. □

Corollary 34. *A connection* ∇ *cannot simultaneously be torsionless and unitary.*

Proof. If possible let ∇ be one such. Comparing the forms of $\nabla(\sigma_j), j = 1, 2, 3$ in Propositions (5.6) and (5.8) we get that $V = c = P$ and also $V - P = -1$. This leads to a contradiction. □

6. Connections with nontrivial scalar curvature

Definition 35 (Frohlich et al. [7, Theorem 2.9]). There is a sesquilinear map

$$\langle \cdot, \cdot \rangle_D : \widetilde{\Omega}_D^k(\mathscr{A}_\hbar^\infty) \otimes \widetilde{\Omega}_D^k(\mathscr{A}_\hbar^\infty) \to N_\hbar$$

satisfying $(x, \langle \omega, \eta \rangle_D) = \int x\eta\omega^*$, for all $x \in \mathscr{A}_\hbar$.

In the following proposition we identify $\widetilde{\Omega}^k(\mathscr{A}_\hbar^\infty)$ with $\mathscr{A}_\hbar^\infty \otimes M_2(\mathbb{C})$.

Proposition 36. $\langle \omega, \eta \rangle_D = \frac{1}{2}(I \otimes tr)(\omega\eta^*)$.

Proof. Let $\omega = \omega_0 \otimes I_2 + \sum_{i=1}^3 \omega_i \otimes \sigma_i, \eta = \eta_0 \otimes I_2 + \sum_{i=1}^3 \eta_i \otimes \sigma_i$. Then $\frac{1}{2}(I \otimes tr)(\omega\eta^*) = \sum_{i=0}^3 \omega_i\eta_i^*$ and $(x, \sum_{i=0}^3 \omega_i\eta_i^*) = \sum \tau(x\eta_i\omega_i^*) = (x, \langle \omega, \eta \rangle_D)$ for all $x \in \mathscr{A}_\hbar$. This completes the proof since $\mathscr{A}_\hbar$ is dense in $\widetilde{\mathscr{H}}^0$. □

Remark 37. Let $\omega \in \Omega_D^1(\mathscr{A}_\hbar^\infty)$. Since $K + \delta K$ is an ideal in $\Omega_D^\bullet(\mathscr{A}_\hbar^\infty)$ we get two maps induced by $m : \widetilde{\Omega}^\bullet(\mathscr{A}_\hbar^\infty) \otimes_{\mathscr{A}_\hbar^\infty} \widetilde{\Omega}^\bullet(\mathscr{A}_\hbar^\infty) \to \widetilde{\Omega}^\bullet(\mathscr{A}_\hbar^\infty)$. These maps denoted by the same symbol

$m : \widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar) \otimes_{\mathscr{A}^\infty_\hbar} \widetilde{\Omega}^\bullet(\mathscr{A}^\infty_\hbar) \to \widetilde{\Omega}^{\bullet+1}(\mathscr{A}^\infty_\hbar)$, $m : \widetilde{\Omega}^\bullet(\mathscr{A}^\infty_\hbar) \otimes_{\mathscr{A}^\infty_\hbar} \widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar) \to \widetilde{\Omega}^{\bullet+1}(\mathscr{A}^\infty_\hbar)$ induce bounded maps $m_L(\omega), m_R(\omega) : \widetilde{\mathscr{H}}^k \to \widetilde{\mathscr{H}}^{k+1}$ as in Remark 27.

Since $\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)$ is a free bimodule with three generators the curvature $R(\nabla)$ of a connection ∇, $R(\nabla) = -\nabla^2 : \widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar) \to \widetilde{\Omega}^2_D(\mathscr{A}^\infty_\hbar) \otimes_{\mathscr{A}^\infty_\hbar} \widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)$ is given by a 3×3 matrix $((R_{ij}))$ with entries in $\widetilde{\Omega}^2_D(\mathscr{A}^\infty_\hbar)$. Let $P_{\delta K_1} : \widetilde{\mathscr{H}}^2 \to \widetilde{\mathscr{H}}^1$ be the projection onto closure of $\pi(\delta K_1) \subseteq \widetilde{\Omega}^2_D(\mathscr{A}^\infty_\hbar)$, and $R^\perp_{ij} = (I - P_{\delta K_1})(R_{ij})$. Let e_1, e_2, e_3 be the canonical basis of $\widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar)$. If we denote by $Ric_j = \sum_i m_L(e_i)^\dagger (R^\perp_{ij}) \in \widetilde{\mathscr{H}}^1$ then Ricci curvature of ∇ is given by

$$Ric(\nabla) = \sum_j Ric_j \otimes e_j \in \widetilde{\mathscr{H}}^1 \otimes_{\mathscr{A}^\infty_\hbar} \widetilde{\Omega}^1_D(\mathscr{A}^\infty_\hbar),$$

where $\dagger$ denotes Hilbert space adjoint. Finally, the scalar curvature $r(\nabla)$ of ∇ is given by

$$r(\nabla) = \sum_i m_R(e_i^*)^\dagger (Ric_i) \in \widetilde{\mathscr{H}}^0 = L^2(\mathscr{A}^\infty_\hbar).$$

Proposition 38. *Let* $f, g : \mathbb{T} \to \mathbb{R}$ be smooth maps. We visualize them as elements of S^c in the following way, $f(x,y,p) = \delta_{0p} f(x)$, $g(x,y,p) = \delta_{0p} g(y)$. Let ∇ be the connection given by $\nabla(\sigma_1) = f'\delta(g)\sigma_1 + g'\delta(f)\sigma_2, \nabla(\sigma_2) = g'\delta(f)\sigma_1, \nabla(\sigma_3) = 0$, then $r(\nabla)$ is $-2f'^2 g'^2$.

Proof. It is clear that the derivative functions f', g' also can be visualized as elements of S^c exactly in the same way as f and g. By direct computation one gets

$$\nabla^2(\sigma_1) = -R_{11}\sigma_1 - R_{12}\sigma_2, \quad \nabla^2(\sigma_2) = -R_{21}\sigma_1, \quad \nabla^2(\sigma_3) = 0,$$

where

$$R_{11} = f'' g \sigma_3, \quad R_{12} = \sqrt{-1}(f'^2 g'^2 - g'' f')\sigma_3,$$

$$R_{21} = -\sqrt{-1}(g'' f' + f'^2 g'^2)\sigma_3,$$

and the other R_{ij}'s are zero.

Then

$$Ric_1 = -f'' g \sigma_2 - (g'' f' + f'^2 g'^2)\sigma_1, \quad Ric_2 = (g'' f' - f'^2 g'^2)\sigma_2$$

implying the desired conclusion $r(\nabla) = -2f'^2 g'^2$. $\square$

Remark 39. (i) All the above notions of Ricci curvature, scalar curvature was introduced in [7]. This is one infinite-dimensional example where one can have connections with nontrivial scalar curvature (see also [4]).

(ii) Note that our choice of the spectral triple depends on a parameter α. However, for the connections we have considered the scalar curvature does not depend on the parameter α.

7. Nontriviality of the Chern character associated with the spectral triples

The spectral triple we constructed depends on a real parameter α. In this section we show that the Kasparov module associated with the spectral triple are homotopic [3,6]. We also argue that they give nontrivial elements in $K^1(\mathscr{A}_\hbar)$ by explicitly computing pairing with some unitary in the algebra representing elements of $K_1(\mathscr{A}_\hbar)$.

Lemma 40. *Let A be a self-adjoint operator with a bounded inverse and B a symmetric operator with* $Dom(A) \subseteq Dom(B)$ *on some Hilbert space* $\mathscr{H}$. *Also suppose that* $||Bu|| \leqslant a||Au||$, $\forall u \in Dom(A)$. *Then* $|A|^{-p}B|A|^{-(1-p)} \in \mathscr{B}(\mathscr{H})$ *and* $|| \, |A|^{-p}B|A|^{-(1-p)}|| \leqslant a$ *for* $0 \leqslant p \leqslant 1$.

Proof. Essentially the argument in [11, p. 33], gives a proof. $\square$

Lemma 41. *Let* A, B *be as above with* $a<1$. *Let* $A_t = A + tB, t \in [0,1]$. *Then the assignment* $t \mapsto \tan^{-1}(A_t)$ *gives a norm continuous function.*

Proof. Let us denote $|A|^{-1/2}B|A|^{-1/2}$ by C. Then by the previous lemma $||C|| \leqslant a$. We also have $|||A|(A-\lambda)^{-1}| \leqslant 1$ for $\lambda \in i\mathbb{R}$.

$$\begin{aligned} A_t - \lambda &= (A-\lambda) + t|A|^{1/2}C|A|^{1/2} \\ &= |A|^{1/2}((A-\lambda)|A|^{-1} + tC)|A|^{1/2} \\ &= |A|^{1/2}(1 + tC(A-\lambda)^{-1}|A|)(A-\lambda)|A|^{-1}|A|^{1/2}. \end{aligned}$$

Now note $||tC(A-\lambda)^{-1}|A| \, || \leqslant a < 1$ for $0 \leqslant t \leqslant 1$. Therefore

$$(A_t-\lambda)^{-1} = |A|^{-1/2}|A|(A-\lambda)^{-1}(1+tC(A-\lambda)^{-1}|A|)^{-1}|A|^{-1/2}.$$

So, if we denote by $R_t(\lambda) = (A_t - \lambda)^{-1}$ and $F(\lambda) = |A|(A-\lambda)^{-1}$ then the above equality becomes

$$\begin{aligned} R_t(\lambda) &= |A|^{-1/2}|A|R_0(\lambda)(1+tC|A|F(\lambda))^{-1}|A|^{-1/2} \\ &= R_0(\lambda) + |A|^{-1/2}F(\lambda)\sum_{n=1}^{\infty}(-tCF(\lambda))^n|A|^{-1/2}. \qquad (7.1) \end{aligned}$$

Let $\lambda \in \mathbb{R}, t, s \in [0,1], u, v \in D(A)$. Observe (i)

$$\begin{aligned}
&\left\| \sum_{n=1}^{\infty} (-tCF(i\lambda))^n |A|^{-1/2} u - \sum_{n=1}^{\infty} (-sCF(i\lambda))^n |A|^{-1/2} u \right\| \\
&\quad \leqslant \sum_{n=0}^{\infty} ||(t^{n+1} - s^{n+1})CF(i\lambda)||^n ||C|| \, ||F(i\lambda)|A|^{-1/2}u|| \\
&\quad \leqslant \sum_{n=0}^{\infty} |(t^{n+1} - s^{n+1})|a^n a||F(i\lambda)|A|^{-1/2}u|| \\
&\quad \leqslant |(t-s)| \sum_{n=0}^{\infty} n + 1 a^{n+1} ||F(i\lambda)|A|^{-1/2}u|| \\
&\quad \leqslant |(t-s)| \frac{a}{(1-a)^2} ||F(i\lambda)|A|^{-1/2}u||,
\end{aligned}$$

(ii)

$$\begin{aligned}
\int_0^{\infty} ||F(i\lambda)|A|^{-1/2}u||^2 d\lambda &\leqslant \int_0^{\infty} \langle (A^2 + \lambda^2)^{-1} u, |A|u \rangle d\lambda \\
&= \frac{1}{2} \int_0^{\infty} \langle (A^2 + \xi)^{-1} u, |A|u \rangle \frac{d\xi}{\sqrt{\xi}} \\
&= \frac{1}{2} \pi \langle A^2 {-1/2} u, |A|u \rangle = \frac{\pi}{2} ||u||^2.
\end{aligned}$$

(iii) Using (7.1), (i), (ii) we get

$$\begin{aligned}
&\int_0^{\infty} |\langle (R_t(i\lambda) - R_s(i\lambda))u, v \rangle| d\lambda \\
&\quad \leqslant \int_0^{\infty} |(t-s)| \frac{a}{(1-a)^2} ||F(i\lambda)|A|^{-1/2}u|| \, ||F(-i\lambda)|A|^{-1/2}v|| d\lambda \\
&\quad \leqslant |(t-s)| \frac{a}{(1-a)^2} \left(\int_0^{\infty} ||F(i\lambda)|A|^{-1/2}u||^2 d\lambda \right)^{1/2} \left(\int_0^{\infty} ||F(-i\lambda)|A|^{-1/2}v||^2 \, d\lambda \right)^{1/2} \\
&\quad \leqslant |(t-s)| \frac{a}{(1-a)^2} \frac{\pi}{2} ||u|| \, ||v||.
\end{aligned}$$

This shows $\lim_{s \to t} || \int_0^{\infty} (R_t(i\lambda) - R_s(i\lambda)) \, d\lambda|| = 0$. Similarly one can show $\lim_{s \to t} || \int_0^{\infty} (R_t(-i\lambda) - R_s(-i\lambda)) \, d\lambda|| = 0$. Now the result follows once we observe $\tan^{-1} A_t = \int_0^{\infty} (R_t(i\lambda) + R_t(-i\lambda)) \, d\lambda$. $\square$

Lemma 42. *Let A, B be as above except now we do not require A to be invertible. Instead we assume A to have discrete spectrum. Then there exists $\kappa \geqslant 0$ such that $t \mapsto \tan^{-1}(A_t + \kappa)$ is norm continuous.*

Proof. Without loss of generality, we can assume 0 is an eigenvalue of A. Otherwise we are done by the previous lemma. Choose $2 \leqslant n \in \mathbb{N}$ such that $b = a\frac{n}{n-1} < 1$. Choose $\kappa > 0$ such that,

(i) smallest positive eigenvalue of A is greater than κ,

(ii) if β is the biggest negative eigenvalue then $\beta < n\kappa$.

Let $\widetilde{A} = A + \kappa, \widetilde{A}_t = \widetilde{A} + tB$. Then by choice of κ

(i) $\widetilde{A}$ is an invertible self-adjoint operator.

(ii) $||B\widetilde{A}^{-1}|| \leqslant a||A(A+\kappa)^{-1}|| \leqslant a\frac{n}{n-1} < 1$.

That is B is relatively bounded with respect to $\widetilde{A}$ with relative bound $b < 1$. Now an application of the previous result to the pair $\widetilde{A}, B$ does the job. □

Combining these two we get

Proposition 43. *Let A, B be operators on the Hilbert space $\mathscr{H}$ such that*

(i) *A is self-adjoint with compact resolvent.*

(ii) *B is symmetric with $Dom(A) \subseteq Dom(B)$, and relatively bounded with respect to A with relative bound less than* 1.

Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying, $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = -1$ *such that $t \mapsto f(A + tB)$ is norm continuous.*

Proof. If A is invertible then by Lemma 41 $f(x) = \frac{2}{\pi}\tan^{-1}(x)$ serves the purpose. In the other case by Lemma 42 $f(x) = \frac{2}{\pi}\tan^{-1}(x+\kappa)$ does the job. □

Let the Hilbert space $\mathscr{H}$ and the operators D_0, B, D be as in Corollary 11.

Corollary 44. *The Kasparov module associated with $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ is operatorial homotopic with $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)$.*

Proof. Let $D_t = D_0 + tB$ for $t \in [0,1]$. Then $D = D_1$ and as in remark (3.4) $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_t)$ are spectral triples. Let f be the function obtained from the previous proposition for the pair D_0, B. Then $((\mathscr{A}_\hbar, \mathscr{H}, f(D_t)))_{t\in[0,1]}$ gives the desired homotopy. □

As remarked earlier the operator D_0 depends on a real parameter $\alpha > 1$. Now we will make that explicit and denote D_0 by $D_0^{(\alpha)}$.

Proposition 45. *The Kasparov modules associated with $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0^{(\alpha)})$ are operatorially homotopic for $\alpha > 1$.*

Proof. By Proposition 6, $\mathscr{H} = L^2(\mathbb{T} \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2$. Let B be the operator $-2\pi c M_p \otimes \sigma_3$. Here p denotes the $\mathbb{Z}$ variable in the L^2 space. Then B is self-adjoint with $D(D_0^{(\alpha)}) \subseteq D(B)$. Also B is relatively bounded with respect to $D_0^{(\alpha)}$ with relative bound less than $\frac{1}{\alpha}<1$. Let $D_t^{(\alpha)} = D_0^{(\alpha)} + tB$ for $t \in [0,1]$. Then $D_t^{(\alpha)} = D_0^{(\alpha+t)}$. Let f be the function obtained from Proposition 5.4 for the pair $D_0^{(\alpha)}, B$. Then from the norm continuity of $t \mapsto f(D_0^{(\alpha+t)})$ we see the Kasparov modules $((\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0^{(\alpha+t)}))_{t\in[0,1]}$ are homotopic. Since α is arbitrary this completes the proof. $\square$

Remark 46. Proposition 45 and Corollary 44 together imply the Kasparov module associates with the spectral triple $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ is independent of α.

In the next proposition we show $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ has nontrivial chern character.

Proposition 47. *The Kasparov module associated with $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ gives a nontrivial element in $K^1(\mathscr{A}_\hbar)$.*

Proof. By Corollary 44 $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ and $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)$ give rise to same element $[(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)] \in K^1(\mathscr{A}_\hbar)$. Let $\phi \in \mathscr{A}_\hbar^\infty$ be the unitary whose symbol in S^c is given by $\phi(x,y,p) = \delta_{0p} e^{2\pi i y}$. This gives an element $[\phi] \in K_1(\mathscr{A}_\hbar)$. It suffices to show $\langle [\phi], [(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)] \rangle \neq 0$ where the pairing $\langle \cdot, \cdot \rangle : K_1(\mathscr{A}_\hbar) \times K^1(\mathscr{A}_\hbar) \to \mathbb{Z}$ is the one coming from the Kasparov product. ϕ acts on $L^2(\mathscr{A}_\hbar) \otimes \mathbb{C}^2 \cong L^2([0,1] \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2$ as a composition of two commuting unitaries $U_1 = M_{e(y)} \otimes I_2, U_2 = M_{e(pv\hbar)} \otimes I_2$. Then note U_2 commutes with D_0. Let E be the projection $E = I(D_0 \geqslant 0)$. U_2 also commutes with E. Now by Proposition 2 [6, p. 289] EU_1U_2E is a Fredholm operator and $\langle [\phi], [(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)] \rangle = Index(EU_1U_2E) = Index(EU_1E)$, last equality holds because U_2 commutes with E. Now $Index(EU_1E) \neq 0$ because this is the index pairing of the Dirac operator on $\mathbb{T}^3$ with the unitary U_1. $\square$

8. Invariance of Chern character in entire cyclic cohomology

Now we will show that Chern character associated with the spectral triples considered above is same. We begin with a general proposition of invariance of Chern character under relatively bounded perturbations, which is an adaptation of the arguments given in proposition (2.4) in [9].

Let $\mathscr{A}$ be a Banach algebra, and $(\mathscr{H}, \mathscr{D}_0)$ be an odd theta summable Fredholm module in the sense of [8] i.e, $\mathscr{H}$ is a Hilbert space, there is a continuous representation $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, $\mathscr{D}_0$ is an unbounded self-adjoint operator such that (i) $a \mapsto [\mathscr{D}_0, \pi(a)]$ defines a bounded derivation from $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$, (ii) for all $t>0$, $Tr \exp(-t\mathscr{D}_0^2)$ is finite. Suppose we are given another self-adjoint operator Δ such that $a \mapsto [\Delta, \pi(a)]$ defines bounded derivation and Δ is relatively bounded with respect to $\mathscr{D}_0$ with relative bound β strictly less than one. Then we have:

Lemma 48. *$(\mathscr{H}, \mathscr{D}_t = \mathscr{D}_0 + t\Delta)$ for $0 \leqslant t \leqslant 1$ define odd theta summable fredholm modules.*

Proof. Clearly $\mathscr{D}_t$ defines a self-adjoint operator and $a \mapsto [\mathscr{D}_t, \pi(a)]$ defines a bounded derivation. It only remains to show that $Tr \exp(-s\mathscr{D}_t^2)$ is finite for all $s > 0$. For that note for bounded operators B_1, B_2, with B_1 compact, we have

$$\mu_n(B_1 B_2) \leqslant \mu_n(B_1) ||B_2||, \tag{8.1}$$

where $\mu_n(\cdot)$ stands for the nth largest singular value. Letting $\mu_{n,t} =$ nth smallest singular value of $\mathscr{D}_t$, (8.1) along with the resolvent identity

$$(\mathscr{D}_t - i)^{-1} = (\mathscr{D}_0 - i)^{-1}(1 + t\Delta(\mathscr{D}_0 - i)^{-1})^{-1} \tag{8.2}$$

gives

$$(\mu_{n,0}^2 + 1)\left(\frac{\beta - 1}{\beta}\right)^2 \leqslant \mu_{n,t}^2 + 1. \tag{8.3}$$

Now we are done by the finiteness of $\sum \exp(-s\mu_{n,0}^2) = Tr \exp(-s\mathscr{D}_0^2)$. $\square$

Remark 49. From the proof of the previous lemma it also follows that $Tr \exp(-\mathscr{D}_t^2)$ is uniformly bounded.

Let $\widetilde{\mathscr{H}}$ be the $\mathbb{Z}/2$ graded Hilbert space given by $\widetilde{\mathscr{H}} = \mathscr{H}^+ \oplus \mathscr{H}^-$, where $\mathscr{H}^+ \cong \mathscr{H} \cong \mathscr{H}^-$. Let $\tilde{\pi}$ be the representation given by $\tilde{\pi} = \pi \oplus \pi$. Let $\widetilde{\mathscr{D}}_0 = \begin{pmatrix} 0 & i\mathscr{D}_0 \\ -i\mathscr{D}_0 & 0 \end{pmatrix}$, similarly define $\widetilde{\Delta}$ and $\widetilde{\mathscr{D}}_t$, then $\widetilde{\mathscr{D}}_t = \widetilde{\mathscr{D}}_0 + t\widetilde{\Delta}$. Let c_1 be the odd operator given by $c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then c_1 graded commutes with $\widetilde{\mathscr{D}}_t$'s and $\tilde{\pi}(\mathscr{A})$. Consider the multilinear maps $\langle \cdot, \cdots, \cdot \rangle_{t,n} : \mathscr{B}(\widetilde{\mathscr{H}})^{\otimes(n+1)} \to \mathbb{C}$ given by

$$\langle A_0, \ldots, A_n \rangle_{t,n} = \int_{\Delta_n} Str(c_1 A_0 e^{-s_0 \widetilde{\mathscr{D}}_t^2} A_1 e^{-s_1 \widetilde{\mathscr{D}}_t^2} \cdots A_n e^{-s_n \widetilde{\mathscr{D}}_t^2})\, d^n s,$$

where Δ_n denotes the n-simplex and the integration is with respect to the lebesgue measure on that simplex. Str stands for super trace, explicitly given by $Str(A) = TrA|_{\mathscr{H}^+} - TrA|_{\mathscr{H}^-}$. The Chern character of the theta summable Fredholm modules $(\mathscr{H}, \mathscr{D}_t)$ is given by the entire cyclic cocycles on $\mathscr{A}$ given by the formula

$$Ch^n(\mathscr{D}_t)(a_0, \ldots, a_n) = \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \ldots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n}.$$

Note that in the right-hand side a_i actually stands for $\tilde{\pi}(a_i)$. Our objective is to prove the following theorem.

Theorem 50. *The chern characters $Ch^\bullet(\mathscr{D}_t)$ associated with the Fredholm modules $(\mathscr{H}, \mathscr{D}_t)$ are cohomologous for $0 \leqslant t \leqslant 1$.*

For ease of reference let us recall some results (Lemmas (2.1), (2.2) from [9]).

Lemma 51. (i) *If the operators* $A_j, G_j, j = 0, \dots, n$ *are bounded and at most* $(k+1)$ *of the* A_j's *are nonzero, then for* $0 < \varepsilon < (2e)^{-1}$

$$|\langle A_0\widetilde{\mathscr{D}}_t + G_0, \dots, A_n\widetilde{\mathscr{D}}_t + G_n \rangle_{t,n}|$$

$$\leqslant (2e\varepsilon)^{-(k+1)/2} \frac{\Gamma(1/2)^{k+1}}{\Gamma((2n-k+1)/2)} Tr e^{-(1-\varepsilon)\widetilde{\mathscr{D}}_t^2} \Pi_0^n(||A_j|| + ||G_j||).$$

(ii) *In each of the following cases we assume that the operators* A_i *are such that each term is well defined* $|A| = 0$ *if* A *is even*, $|A| = 1$ *if* A *is odd.*

(a) $\langle A_0, \dots, A_n \rangle_{t,n} = (-1)^{(|A_0|+\dots+|A_{j-1}|)(|A_j|+\dots+|A_n|)} \times \langle A_j, \dots, A_n, A_0, \dots, A_{j-1} \rangle_{t,n}$.

(b) $\langle A_0, \dots, A_n \rangle_{t,n} = \sum_0^n (-1)^{(|A_0|+\dots+|A_{j-1}|)(|A_j|+\dots+|A_n|)} \langle 1, A_j, \dots, A_n, A_0, \dots, A_{j-1} \rangle_{t,n+1}$.

(c) $\sum_0^n (-1)^{|A_0|+\dots|A_{j-1}|} \langle A_0, \dots, [\widetilde{\mathscr{D}}_t, A_j], \dots A_n \rangle_{t,n} = 0$.

(d) $\langle A_0, \dots, [\widetilde{\mathscr{D}}_t^2, A_j], \dots A_n \rangle_{t,n} = \langle A_0, \dots, A_{j-1}A_j, A_{j+1}, \dots, A_n \rangle_{t,n-1} - \langle A_0, \dots, A_{j-1}, A_jA_{j+1}, \dots, A_n \rangle_{t,n-1}$.

(e) $\frac{d}{dt}\langle A_0, \dots, A_n \rangle_{t,n} + \sum_0^n \langle A_0, \dots, A_j, [\widetilde{\mathscr{D}}_t, \dot{\widetilde{\mathscr{D}}}_t], A_{j+1}, \dots, A_n \rangle_{t,n+1} = 0$.

Proof of the Theorem. Let $A_0, A_1, \dots, A_n, G$ be bounded operators. Then,

(a)
$$|\langle A_0, \dots, A_j, G\widetilde{\Delta}, A_{j+1}, \dots, A_n \rangle_{t,n+1}|$$
$$= |\langle A_0, \dots, A_i, G\widetilde{\Delta}(\widetilde{\mathscr{D}}_t + i)^{-1}(\widetilde{\mathscr{D}}_t + i), A_{i+1}, \dots, A_n \rangle_{t,n+1}|$$
$$\leqslant 2(2e\varepsilon)^{-1/2} \frac{\beta}{\beta - 1} ||G|| \Pi_0^n ||A_i|| \frac{\Gamma(1/2)}{\Gamma((2n+1)/2)} Tr e^{-(1-\varepsilon)\widetilde{\mathscr{D}}_t^2}.$$

Therefore,

$$\widetilde{Ch}^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta})((a_0, \dots, a_n))$$
$$= \sum_0^n (-1)^j \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \dots, [\widetilde{\mathscr{D}}_t, a_j], \widetilde{\Delta}, [\widetilde{\mathscr{D}}_t, a_{j+1}], \dots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n+1}$$

defines an entire cochain.

(b) $\langle A_0, \dots, \widetilde{\Delta}A_j, \dots, A_n \rangle_{t,n} = \langle A_0, \dots, (\widetilde{\mathscr{D}}_t + i)(\widetilde{\mathscr{D}}_t + i)^{-1}\widetilde{\Delta}A_j, \dots, A_n \rangle_{t,n}$.

Left-hand side is well defined by (i) of Lemma 51 implying that the right-hand side is well defined too. Therefore,

$$\alpha^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta})((a_0, \dots, a_n)) = \sum_0^n \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \dots, [\widetilde{\Delta}, a_j], \dots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n}$$

defines an entire cochain.

(c) Again as in (b) it is easily seen that $\langle A_0, \ldots, A_j, \widetilde{\Delta}\widetilde{\mathscr{D}}_t, A_{j+1}, \ldots, A_n \rangle_{t,n+1}$, and $\langle A_0, \ldots, A_j, \widetilde{\mathscr{D}}_t\widetilde{\Delta}, A_{j+1}, \ldots, A_n \rangle_{t,n+1}$ make perfect sense. So, that we can talk about $\langle A_0, \ldots, A_j, [\widetilde{\mathscr{D}}_t, \widetilde{\Delta}], A_{j+1}, \ldots, A_n \rangle_{t,n+1}$ which is nothing but $\langle A_0, \ldots, A_j, [\widetilde{\mathscr{D}_t}, \dot{\widetilde{\mathscr{D}}}_t], A_{j+1}, \ldots, A_n \rangle_{t,n+1}$. Now we are in a position to apply (ii)(c) of Lemma 51 to the following choice:

$$A_j = \begin{cases} a_0, & \text{for } j = 0, \\ [\widetilde{\mathscr{D}}_t, a_j] & \text{for } j \leqslant k, \\ \widetilde{\Delta} & \text{for } j = k+1, \\ [\widetilde{\mathscr{D}}_t, a_{j-1}] & \text{for } j \geqslant k+2. \end{cases}$$

This gives

$$X_1 + X_2 + X_3 = 0, \tag{8.4}$$

where

$$X_1 = (-1)^k \langle [\widetilde{\mathscr{D}}_t, a_0], [\widetilde{\mathscr{D}}_t, a_1], \ldots, [\widetilde{\mathscr{D}}_t, a_k], \widetilde{\Delta}, [\widetilde{\mathscr{D}}_t, a_{k+1}], \ldots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n+1},$$

$$X_2 = \sum_{j<k} (-1)^{j+k-1} \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \ldots, [\widetilde{\mathscr{D}}_t^2, a_j], \ldots, [\widetilde{\mathscr{D}}_t, a_k], \widetilde{\Delta}, \ldots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n+1}$$
$$+ \sum_{j>k} (-1)^{k+j} \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \ldots, [\widetilde{\mathscr{D}}_t, a_k], \widetilde{\Delta}, \ldots [\widetilde{\mathscr{D}}_t^2, a_j], \ldots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n+1},$$

$$X_3 = \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \ldots, [\widetilde{\mathscr{D}}_t, a_k], [\widetilde{\mathscr{D}}_t, \widetilde{\Delta}], [\widetilde{\mathscr{D}}_t, a_{k+1}], \ldots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n+1}.$$

We now sum (8.4) over $0 \leqslant k \leqslant n$. By Lemma 51(ii)(b) we see after reordering terms that

$$\sum_k X_1 = -(B\widetilde{Ch}^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta}))((a_0, \ldots, a_n)). \tag{8.5}$$

Similarly, using Lemma 51(ii)(d),

$$\sum_k X_2 = -(b\widetilde{Ch}^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta}))((a_0, \ldots, a_n)) + \alpha^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta})((a_0, \ldots, a_n)). \tag{8.6}$$

Here b, B are the boundary operators in entire cyclic theory [9]. Combining (8.4)–(8.6) along with the expression for X_3 we get

$$\frac{dCh^*(\mathscr{D}_t)}{dt}(a_0, \ldots, a_n)$$
$$= \sum_k \langle a_0, [\widetilde{\mathscr{D}}_t, a_1], \ldots, [\widetilde{\mathscr{D}}_t, a_k], [\widetilde{\mathscr{D}}_t, \widetilde{\Delta}], [\widetilde{\mathscr{D}}_t, a_{k+1}], \ldots, [\widetilde{\mathscr{D}}_t, a_n] \rangle_{t,n+1}$$
$$+ \alpha^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta})((a_0, \ldots, a_n))$$
$$= (B + b)\widetilde{Ch}^*(\widetilde{\mathscr{D}}_t, \widetilde{\Delta})((a_0, \ldots, a_n)). \qquad \square$$

Let the Hilbert space $\mathscr{H}$ and the operators D_0, B, D be as in Corollary 3.3. $\mathscr{A}_\hbar^1$ defined as $\{a \in \mathscr{A}_\hbar | [D_0, a], [B, a] \in \mathscr{B}(\mathscr{H})\}$ becomes a Banach algebra with the norm $||a||_n = \max\{||a|| + ||[D_0, a]||, ||a|| + ||[B, a]||\}$. Let $\mathscr{D}_0 = D_0, \Delta = B$, then with these choice $\mathscr{A}_\hbar^1, \mathscr{H}, \mathscr{D}_0, \Delta$ satisfy all the hypothesis required for applying Theorem 50 by which we get

Corollary 52. *The Chern character associated with the spectral triples* $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$, *and* $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D_0)$ *are cohomologous.*

Remark 53. The spectral triple $(\mathscr{A}_\hbar^\infty, \mathscr{H}, D)$ depends on a real number $\alpha > 1$. An argument very similar to Proposition 45 will show that Chern character associated with this whole family of spectral triples is independent of α.

References

[1] B. Abadie, "Vector Bundles" over quantum Heisenberg manifolds, Algebraic Methods in Operator Theory, Birkhauser, Basel, 1994, pp. 307–315.

[2] B. Abadie, Generalized fixed-point algebras of certain actions on crossed products, Pacific J. Math. 171 (1) (1995) 1–21.

[3] B. Blackadar, K-Theory of Operator Algebras, in: MSRI Publications, Vol. 5, Springer, Berlin, 1986.

[4] P.S. Chakraborty, D. Goswami, K.B. Sinha, Probability and geometry on some noncommutative manifolds, J. Operator Theory 49 (2003) 187–203.

[5] F. Cipriani, D. Guido, S. Scarlatti, A remark on trace properties of K-cycles, J. Operator Theory 35 (1996) 179–189.

[6] A. Connes, Noncommutative Geometry, Academic Press, New York, 1994.

[7] J. Frohlich, O. Grandjean, A. Recknagel, Supersymmetric quantum theory and non-commutative geometry, Commun. Math. Phys. 203 (1999) 119–184.

[8] E. Getzler, The odd Chern character in cyclic homology and spectral flow, Topology 32 (1993) 489–507.

[9] E. Getzler, A. Szenes, On the Chern character of a Theta–Summable Fredholm module, J. Funct. Anal. 84 (1989) 343–357.

[10] J.M. Gracia-Bondia, J.C. Varilly, H. Figuera, Elements of Noncommutative Geometry, Birkhauser, Basel, 2000.

[11] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Vol. II, Academic Press, New York, 1978.
[12] M. Rieffel, Deformation quantization of Heisenberg manifolds, Comm. Math. Phys. 122 (1989) 531–562.
[13] N. Weaver, Sub-Riemannian metrics for quantum Heisenberg manifolds, J. Operator Theory 43 (2000) 223–242.