# Scattering theory of distortion correction by phase conjugation

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The correction of wave distortions by the technique of optical phase conjugation is examined first on the basis of a newly derived integral equation for scattering of monochromatic scalar waves in the presence of a phase-conjugate mirror. The solution is developed in an iterative series, and the first- and second-order terms are analyzed and illustrated diagrammatically. A generalization of the integral equation is then presented, which takes into account the electromagnetic nature of light. It is also shown that, if the conjugate wave is generated without losses or gains and with a complete reversal of polarization, a total elimination of distortions may be achieved by this technique under circumstances that frequently occur in practice.

## 1. INTRODUCTION

One of the main applications envisaged for the rapidly developing technique of optical phase conjugation is the correction of distortions that are imparted on a wave field by its interaction with a scattering medium.[1] A number of experiments have been carried out that demonstrate the possibility of such a "healing" process,[2] but no satisfactory theory of this phenomenon has been developed so far. This is undoubtedly so because the physical processes that are involved are very complex, a fact that does not appear to be generally appreciated; they include two, conceptually distinct, scattering processes, namely, scattering of the original wave and of the conjugated wave, as well as a nonlinear interaction. The simple, intuitive arguments that have been put forward so far to explain the success of this technique have greatly oversimplified the problem. Only some special cases have been treated adequately until now, namely, corrections of distortions produced by weak scatterers[3,4] and corrections under conditions when the phase-conjugate wave is generated without losses or gains.[5]

In Ref. 5 we put forward an integral equation for scattering of monochromatic waves in the presence of a phase-conjugate mirror (PCM), from which the degree of corrections attainable by this technique can, in principle, be deduced. The equation was obtained within the framework of scalar wave theory by a plausibility argument, on the assumptions that the incident field contains no evanescent components, that the scatterer is nonabsorbing, and that the effects of the evanescent waves are negligible at the PCM—a condition that is likely to hold in most cases of practical interest. In the present paper some implications of this integral equation are studied, and a generalization of the equation is obtained that takes into account the electromagnetic nature of light.

In Section 2 we recall the (scalar) integral equation for the scattered conjugated field. In Section 3 we present a formal solution of this equation in a form of an iterative series. In Sections 4 and 5 we analyze the first- and second-order contributions to the conjugated field and illustrate the results diagrammatically. In Section 6 we examine the correction of distortions for the case when the conjugated wave is generated without losses or gains, and we show that under these circumstances a complete correction is obtained to all orders of scattering. In Section 7 we present a generalization of our basic integral equation within the framework of Maxwell's electromagnetic theory, under the assumption that the conjugated wave is generated with a complete reversal of polarization.[6] We find that if, in addition, there are no losses or gains on phase conjugation, a complete correction of distortions produced by nonabsorbing scatterers will again be achieved. In the concluding section (Section 8), the main assumptions implicit in our theory are summarized, and some possible generalizations are mentioned.

## 2. BASIC INTEGRAL EQUATION FOR THE SCATTERED FIELD IN THE PRESENCE OF A PHASE-CONJUGATE MIRROR

We consider a monochromatic scalar wave field

$$U^{(i)}(\mathbf{r}, t) = U^{(i)}(\mathbf{r})e^{-i\omega t} \tag{2.1}$$

that is incident upon a scattering medium occupying a finite volume $\mathcal{V}$ in free space. In Eq. (2.1), $\mathbf{r}$ denotes the position vector of a typical field point, $t$ denotes the time, and $\omega$ denotes a (real) frequency. The wave field is taken to be incident from a half-space $\mathcal{R}^-$ on one side of the scatterer [see Fig. 1(a)]. The scattering medium is assumed to be linear, time independent, and nonabsorbing. We denote by $n(\mathbf{r})$ the (real) distribution of the refractive index throughout the scattering volume.

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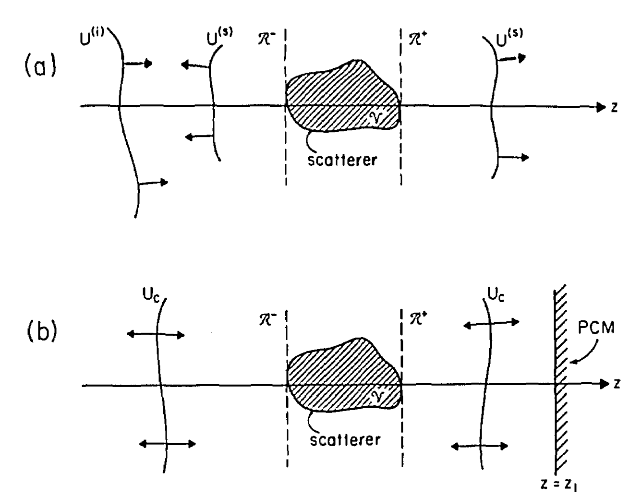


Fig. 1. Schematic diagram illustrating the notation. (a) In the absence of the PCM, the total field $U$ is given by the sum of the incident field $U^{(i)}$ and the scattered field $U^{(s)}$. (b) In the presence of the PCM in the plane $z = z_1$, the total field throughout the domain $z < z_1$ is denoted by $U_c$.

When the field interacts with the scatterer, a new field [with the time-dependent factor $\exp(-i\omega t)$ omitted from now on]

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r}) + U^{(s)}(\mathbf{r}) \tag{2.2}$$

is generated, where $U^{(s)}(\mathbf{r})$ represents the scattered field. It is well known that $U(\mathbf{r})$ satisfies the integral equation[7]

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r}) - \frac{1}{4\pi}\int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}')F(\mathbf{r}')U(\mathbf{r}')\mathrm{d}^3r', \tag{2.3}$$

where

$$G(\mathbf{r}, \mathbf{r}') = \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \tag{2.4}$$

($k = \omega/c$, $c$ being the speed of light in vacuum) is the outgoing free-space Green's function and

$$F(\mathbf{r}) = -k^2[n^2(\mathbf{r}) - 1] \tag{2.5}$$

represents the scattering potential.

Suppose now that a PCM,[8] assumed for simplicity to be infinite, is placed in a plane $z = z_1$ in the half-space $\mathcal{R}^+$ on the side opposite that from which the field $U^{(i)}(\mathbf{r})$ is incident [see Fig. 1(b)]. Were the scatterer absent, the PCM would replace the field distribution $U^{(i)}(\mathbf{r})|_{z=z_1}$ in that plane with the distribution $\mu U^{(i)*}(\mathbf{r})|_{z=z_1}$ that would give rise to an additional contribution to the total field in the half-space $z < z_1$. Here the asterisk denotes the complex conjugate and $\mu$ is a (generally complex) constant that accounts for the losses ($|\mu| < 1$) or gains ($|\mu| > 1$) that arise in the process of phase conjugation. The assumption of constant $\mu$ represents an idealization; in practice the PCM may respond differently to plane waves that propagate in different directions.

When both the scatterer and the PCM are present, a new field distribution is generated throughout the domain $z < z_1$ that we will denote by $U_c(\mathbf{r})$. It represents the total field after phase conjugation; for the sake of brevity, we will refer to it as the conjugate field.[9] If the incident field $U^{(i)}(\mathbf{r})$ contains no evanescent components, and if, in addition, the effects of the evanescent waves that might be created in the scattering process are negligible[10] at the PCM, then the field $U_c(\mathbf{r})$ satisfies the integral equation[5,11]

$$U_c(\mathbf{r}) = U_c^{(0)}(\mathbf{r}) - \frac{1}{4\pi}\int_{\mathcal{V}} \hat{G}_c(\mathbf{r}, \mathbf{r}')F(\mathbf{r}')U_c(\mathbf{r}')\mathrm{d}^3r', \tag{2.6}$$

where

$$U_c^{(0)}(\mathbf{r}) = U^{(i)}(\mathbf{r}) + \mu U^{(i)*}(\mathbf{r}) \tag{2.7}$$

and $\hat{G}_c(\mathbf{r}, \mathbf{r}')$ is an operator Green's function[12] that takes into account the presence of the PCM.

The operator Green's function $\hat{G}_c(\mathbf{r}, \mathbf{r}')$ was shown[5,11] to be expressible in terms of the following four quantities:

(1) The complex "reflection coefficient" $\mu$ of the PCM.
(2) The complex-conjugation operator $\hat{C}$, defined by the property that

$$\hat{C}f(\mathbf{r}) \equiv f^*(\mathbf{r}), \tag{2.8}$$

where $f(\mathbf{r})$ is an arbitrary function.
(3) The free-space Green's function $G(\mathbf{r}, \mathbf{r}')$.
(4) The function $G_>^{(H)}(\mathbf{r}, \mathbf{r}')$ associated with the "homogeneous" part of the free-space Green's function $G(\mathbf{r}, \mathbf{r}')$.

More specifically, the function $G_>^{(H)}(\mathbf{r}, \mathbf{r}')$ is related to the free-space Green's function $G(\mathbf{r}, \mathbf{r}')$ by the formula

$$G(\mathbf{r}, \mathbf{r}') = \Theta(z - z')G_>^{(H)}(\mathbf{r}, \mathbf{r}') + \Theta(z' - z)G_<^{(H)}(\mathbf{r}, \mathbf{r}') + G^{(I)}(\mathbf{r}, \mathbf{r}'), \tag{2.9}$$

where $G_>^{(H)}(\mathbf{r}, \mathbf{r}')$ and $G_<^{(H)}(\mathbf{r}, \mathbf{r}')$ represent the contributions to $G(\mathbf{r}, \mathbf{r}')$ from all homogeneous plane waves, $G^{(I)}(\mathbf{r}, \mathbf{r}')$ is the contribution from all inhomogeneous (evanescent) plane waves, and $\Theta(\zeta)$ is the unit step function, viz.,

$$\begin{aligned}\Theta(\zeta) &= 1 \quad \text{if } \zeta > 0,\\ &= 0 \quad \text{if } \zeta < 0.\end{aligned} \tag{2.10}$$

By using the Weyl representation of a spherical wave,[13] the functions $G_>^{(H)}(\mathbf{r}, \mathbf{r}')$, $G_<^{(H)}(\mathbf{r}, \mathbf{r}')$, and $G^{(I)}(\mathbf{r}, \mathbf{r}')$ may be shown to be expressible explicitly as

$$G_>^{(H)}(\mathbf{r}, \mathbf{r}') = \frac{i}{2\pi}\iint_{|\kappa|\leq k} \frac{1}{w}\exp\{i[\kappa\cdot(\rho - \rho') + w(z - z')]\}\mathrm{d}^2\kappa, \tag{2.11a}$$

$$G_<^{(H)}(\mathbf{r}, \mathbf{r}') = \frac{i}{2\pi}\iint_{|\kappa|\leq k} \frac{1}{w}\exp\{i[\kappa\cdot(\rho - \rho') - w(z - z')]\}\mathrm{d}^2\kappa, \tag{2.11b}$$

and

$$G^{(I)}(\mathbf{r}, \mathbf{r}') = \frac{i}{2\pi}\iint_{|\kappa|> k} \frac{1}{w}\exp\{i[\kappa\cdot(\rho - \rho') + w|z - z'|]\}\mathrm{d}^2\kappa, \tag{2.12}$$

respectively. In the integrals on the right-hand sides of Eqs. (2.11) and (2.12), $\mathbf{r} = (\rho, z)$ and $\mathbf{r}' = (\rho', z')$ (with $\rho$ and $\rho'$ being two-dimensional vectors orthogonal to the $z$ axis), and

$$w = +(k^2 - \kappa^2)^{1/2} \quad \text{when } |\kappa| \leq k, \tag{2.13a}$$

$$= +i(\kappa^2 - k^2)^{1/2} \quad \text{when } |\kappa| > k. \tag{2.13b}$$

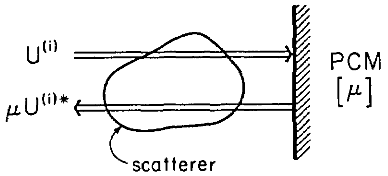


Fig. 2. Diagrammatic illustration of the zeroth-order term $U_c^{(0)}$ in iterative expansion (3.1) for the conjugate field $U_c$. $U^{(i)}$ denotes the incident wave, and $\mu U^{(i)*}$ denotes the wave that is generated by the PCM in the absence of the scatterer.

In terms of the four quantities listed as (1)–(4) above, the operator Green's function $\hat{G}_c(\mathbf{r}, \mathbf{r}')$ that appears in the basic integral equation (2.6) may then be expressed in the form

$$\hat{G}_c(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}, \mathbf{r}') + \mu G_>^{(H)*}(\mathbf{r}, \mathbf{r}')\hat{C}. \qquad (2.14)$$

Here the first term on the right-hand side represents the propagator for scattering in the absence of the PCM; the presence of the PCM in taken into account by the second term.

Making use of Eq. (2.9) and the relation

$$G_>^{(H)*}(\mathbf{r}, \mathbf{r}') = -G_<^{(H)}(\mathbf{r}, \mathbf{r}') \qquad (2.15)$$

that readily follows from Eqs. (2.11), we may write the operator Green's function $\hat{G}_c(\mathbf{r}, \mathbf{r}')$ given by Eq. (2.14) alternatively as

$$\hat{G}_c(\mathbf{r}, \mathbf{r}') = \Theta(z - z')(1 + \mu\hat{C})G_>^{(H)}(\mathbf{r}, \mathbf{r}') + \Theta(z' - z)G_<^{(H)}(\mathbf{r}, \mathbf{r}')(1 - \mu\hat{C}) + G^{(I)}(\mathbf{r}, \mathbf{r}'). \qquad (2.16)$$

This form of the operator Green's function $\hat{G}_c(\mathbf{r}, \mathbf{r}')$ will prove convenient later (Sections 6 and 7).

We note that the removal of the PCM is equivalent to letting $\mu \to 0$. We see from Eq. (2.14) [or Eq. (2.16)] that in this limit

$$\hat{G}_c(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}, \mathbf{r}') \qquad (\mu = 0), \qquad (2.17)$$

and, according to Eq. (2.7), one then also has

$$U_c^{(0)}(\mathbf{r}) = U^{(i)}(\mathbf{r}) \qquad (\mu = 0). \qquad (2.18)$$

Since, in the absence of the PCM, $U_c(\mathbf{r}) \to U(\mathbf{r})$, we see at once, on using Eqs. (2.17) and (2.18), that our integral equation (2.6) then correctly reduces to the usual integral equation (2.3).

## 3. ITERATIVE EXPANSION OF THE SOLUTION TO THE BASIC INTEGRAL EQUATION

The basic integral equation for the scattered field in the presence of a PCM [Eq. (2.6)] may be formally solved in the form of an iterative expansion[14]

$$U_c(\mathbf{r}) = \sum_{n=0}^{\infty} U_c^{(n)}(\mathbf{r}), \qquad (3.1)$$

where $U_c^{(0)}(\mathbf{r})$ is given by Eq. (2.7) and

$$U_c^{(n)}(\mathbf{r}) = -\frac{1}{4\pi}\int_{\mathcal{V}} \hat{G}_c(\mathbf{r}, \mathbf{r}')F(\mathbf{r}')U_c^{(n-1)}(\mathbf{r}')\mathrm{d}^3r' \qquad (n = 1, 2, 3, \ldots). \qquad (3.2)$$

We note that, in the absence of the scatterer, $F(\mathbf{r}) \equiv 0$ and all the terms $U_c^{(n)}(\mathbf{r})$ in series (3.1) then vanish, except the first one (i.e., the term $n = 0$). Hence in this case we have

$$U_c(\mathbf{r}) \to U_c^{(0)}(\mathbf{r}) = U^{(i)}(\mathbf{r}) + \mu U^{(i)*}(\mathbf{r}) = (1 + \mu\hat{C})U^{(i)}(\mathbf{r}), \qquad (3.3)$$

as expected. We emphasize that the result expressed by Eq. (3.3) holds only when there are no evanescent waves associated with the incident field[15] $U^{(i)}(\mathbf{r})$. The physical meaning of the right-hand side of Eq. (3.3) is illustrated schematically in Fig. 2. We give later (in Sections 4 and 5) similar diagrammatic representations for the next two terms ($n = 1$ and $n = 2$) of iterative expansion (3.1).

In applying the technique of optical phase conjugation to the correction of wave-front distortions suffered by a light wave on interaction with the scattering medium, one is usually interested in the field $U_c(\mathbf{r})$ in the half-space $\mathcal{R}^-$ only, i.e., on that side of the scatterer from which the wave $U^{(i)}(\mathbf{r})$ is incident. It follows from Eqs. (2.14) and (2.9) that, if the evanescent waves are neglected in the half-space $\mathcal{R}^-$, the operator Green's function $\hat{G}_c(\mathbf{r}_<, \mathbf{r}')$, with $\mathbf{r}_< \in \mathcal{R}^-$ and $\mathbf{r}' \in \mathcal{V}$, then reduces to [cf. also Eq. (2.16)]

$$\hat{G}_c^{(H)}(\mathbf{r}_<, \mathbf{r}') = G_<^{(H)}(\mathbf{r}_<, \mathbf{r}') + \mu G_>^{(H)*}(\mathbf{r}_<, \mathbf{r}')\hat{C}. \qquad (3.4)$$

Hence, in this approximation, each term $U_c^{(n)}(\mathbf{r}_<)$ in expansion (3.1) with $n > 1$ may be represented, throughout the half-space $\mathcal{R}^-$, in the following form:

$$U_c^{(n)}(\mathbf{r}_<) = -\frac{1}{4\pi}\int_{\mathcal{V}} \hat{G}_c^{(H)}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U_c^{(n-1)}(\mathbf{r}')\mathrm{d}^3r', \qquad (3.5)$$

where $\hat{G}_c^{(H)}(\mathbf{r}_<, \mathbf{r}')$ is given by Eq. (3.4). The field $U_c^{(n-1)}(\mathbf{r}')$ under the integral sign on the right-hand side of Eq. (3.5) must be calculated by the use of recursion formula (3.2), which is valid throughout the domain $z < z_1$. It should be noted that, although here the effects of the evanescent waves are neglected at the PCM and in the half-space $\mathcal{R}^-$, they are included inside the scattering medium.

## 4. DISTORTION CORRECTION IN THE FIRST BORN APPROXIMATION

We now analyze in some detail a few of the lowest-order terms of iterative expansion (3.1) in the special case when the observation point $\mathbf{r}_<$ is located in the half-space $\mathcal{R}^-$ and the evanescent waves are omitted in that half-space. If the scatterer is sufficiently weak and also thin enough, the total field $U_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ may then be adequately approximated by the first two terms in expansion (3.1), i.e., by

$$U_c(\mathbf{r}_<) \cong U_c^{(\mathrm{I})}(\mathbf{r}_<) = U_c^{(0)}(\mathbf{r}_<) + U_c^{(1)}(\mathbf{r}_<), \qquad (4.1)$$

where $U_c^{(0)}(\mathbf{r}_<)$ is given by Eq. (3.3) (with $\mathbf{r} = \mathbf{r}_<$) and $U_c^{(1)}(\mathbf{r}_<)$ is obtained from Eq. (3.5) with $n = 1$, viz.,

$$U_c^{(1)}(\mathbf{r}_<) = -\frac{1}{4\pi}\int_{\mathcal{V}} \hat{G}_c^{(H)}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U_c^{(0)}(\mathbf{r}')\mathrm{d}^3r'. \qquad (4.2)$$

This approximation represents the first Born approximation (superscript I) to the field $U_c(\mathbf{r}_<)$ in the presence of the PCM.

If we substitute from Eqs. (3.3) and (3.4) into Eq. (4.2), we obtain at once the formula

$$U_c{}^{(1)}(\mathbf{r}_<) = -\frac{1}{4\pi}\int_{\mathcal{V}} [G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}') + \mu G_>{}^{(H)*}(\mathbf{r}_<, \mathbf{r}')\hat{C}]\, F(\mathbf{r}')(1 + \mu\hat{C})U^{(i)}(\mathbf{r}')\mathrm{d}^3r'. \quad (4.3)$$

On expanding the product under the integral sign, we obtain for $U_c{}^{(1)}(\mathbf{r}_<)$ the expression

$$\begin{aligned} U_c{}^{(1)}(\mathbf{r}_<) = &-\frac{1}{4\pi}\int_{\mathcal{V}} G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r' \quad (a)\\ &-\frac{\mu}{4\pi}\int_{\mathcal{V}} G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')\mathrm{d}^2r' \quad (b)\\ &-\frac{\mu}{4\pi}\int_{\mathcal{V}} G_>{}^{(H)*}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')\mathrm{d}^3r' \quad (c)\\ &-\frac{|\mu|^2}{4\pi}\int_{\mathcal{V}} G_>{}^{(H)*}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r'. \quad (d) \end{aligned} \quad (4.4)$$

Here we made use of the fact that the scatterer was assumed to be nonabsorbing and that, consequently, the scattering potential $F(\mathbf{r})$ is a real function of position throughout the volume $\mathcal{V}$.

We note that each term on the right-hand side of Eq. (4.4) involves either the function $G_>{}^{(H)}(\mathbf{r}_<, \mathbf{r}')$ or the function $G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}')$, both of which are associated with the "homogeneous" part of the free-space Green's function $G(\mathbf{r}, \mathbf{r}')$ [cf. Eqs. (2.9) and (2.11)]. This situation is a direct consequence of the fact that we neglected the evanescent waves both at the PCM and in the half-space $\mathcal{R}^-$. With this neglection it is clear that, in the first Born approximation, the conjugate field $U_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ contains no effects of the evanescent waves that may have been created on scattering inside the volume $\mathcal{V}$.

In the approximation that the evanescent waves are neglected in the half-space $\mathcal{R}^-$, the quantity $G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}')$ represents the propagator from point $\mathbf{r}'$ inside the scattering volume $\mathcal{V}$ to point $\mathbf{r}_<$ in the half-space $\mathcal{R}^-$. Similarly, with the effects of the evanescent waves neglected at the PCM, the quantity $\mu G_>{}^{(H)*}(\mathbf{r}_<, \mathbf{r}')\hat{C}$ in Eq. (4.3), which gives rise to the last two terms in Eq. (4.4), may be interpreted as the propagator from $\mathbf{r}'$ to $\mathbf{r}_<$ via the PCM. Hence each of the four terms (a)–(d) in Eq. (4.4) specifies a well-defined elementary event associated with scattering in the presence of a PCM, namely, first-order scattering, which may or may not be followed or preceded by phase conjugation. These elementary events are illustrated diagrammatically in Fig. 3. The scattering taking place at a typical point within the volume $\mathcal{V}$ is denoted by a dot in each of the diagrams.

In view of relation (2.15), the terms (b) and (c) in Eq. (4.4) are seen to cancel each other. The remaining two terms, viz., (a) and (d), represent the effects of backscattering[4] in the first Born approximation; the term (a) arises from the backscattering of the incident wave $U^{(i)}$, whereas the term (d) arises from the backscattering of the wave $\mu U^{(i)*}$. By making use of relation (2.15), Eq. (4.4) may be expressed in the form

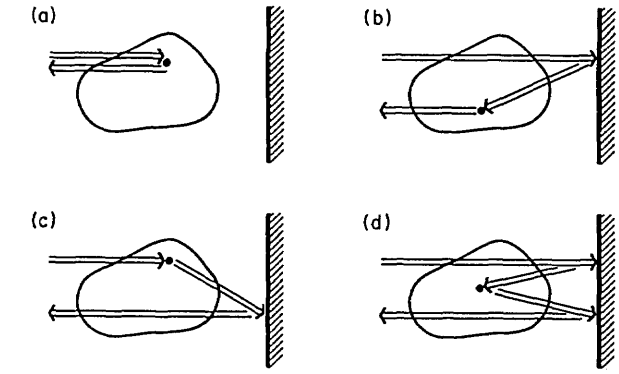


Fig. 3. Diagrammatic illustrations of the four contributions [(a)–(d)], given in Eq. (4.4), to the first-order term $U_c{}^{(1)}(\mathbf{r}_<)$ of iterative expansion (3.1).

$$U_c{}^{(1)}(\mathbf{r}_<) = (1 - |\mu|^2)U_{\mathrm{bs}}{}^{(1)}(\mathbf{r}_<), \quad (4.5)$$

where

$$U_{\mathrm{bs}}{}^{(1)}(\mathbf{r}_<) = -\frac{1}{4\pi}\int_{\mathcal{V}} G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r' \quad (4.6)$$

is the (first-order) backscattered contribution to the total field in the half-space $\mathcal{R}^-$ [represented by diagram (a) in Fig. 3]. We note that the complex "reflection coefficient" $\mu$ of the PCM enters expression (4.5) for the field $U_c{}^{(1)}(\mathbf{r}_<)$ only through its modulus.

According to Eqs. (4.1) and (4.5), the effect of backscattering prevents, in general, a complete cancellation of distortions by phase conjugation within the accuracy of the first Born approximation. However, distortion correction may be achieved in the first Born approximation if either

(1) $U_{\mathrm{bs}}{}^{(1)}(\mathbf{r}_<) = 0$, i.e., there is no backscattering into the half-space $\mathcal{R}^-$, or

(2) $|\mu| = 1$, i.e., there are no losses or gains in the process of phase conjugation.

Under the usual circumstances $|\mu| \ll 1$, and Eqs. (4.1) and (4.5) then yield

$$U_c{}^{(\mathrm{I})}(\mathbf{r}_<) \cong U_c{}^{(0)}(\mathbf{r}_<) + U_{\mathrm{bs}}{}^{(1)}(\mathbf{r}_<) \qquad (|\mu| \ll 1). \quad (4.7)$$

Hence, if the backscattered field $U_{\mathrm{bs}}{}^{(1)}(\mathbf{r}_<)$ is physically removed by some experimental means, the wave-front distortions resulting from the interaction of the incident field with a nonabsorbing scatterer can be corrected in the first Born approximation by the use of a PCM for which $|\mu| \ll 1$.

## 5. DISTORTION CORRECTION IN THE SECOND BORN APPROXIMATION

We now go one step beyond the first Born approximation by including the next higher term in iterative expansion (3.1), i.e., we approximate the field $U_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ by

$$U_c(\mathbf{r}_<) \cong U_c{}^{(\mathrm{II})}(\mathbf{r}_<) = U_c{}^{(0)}(\mathbf{r}_<) + U_c{}^{(1)}(\mathbf{r}_<) + U_c{}^{(2)}(\mathbf{r}_<). \quad (5.1)$$

The first two terms on the right-hand side of approximation (5.1) were discussed in Sections 3 and 4 above. The term $U_c{}^{(2)}(\mathbf{r}_<)$ is obtained from Eq. (3.5) with $n = 2$. Approximation (5.1) represents the second Born approximation (superscript II) to the conjugate field $U_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$.

Setting $n = 2$ in Eq. (3.5) and making use of Eq. (3.2) to find an expression for the field $U_c^{(1)}(\mathbf{r}')$ under the integral sign, we obtain the formula

$$U_c^{(2)}(\mathbf{r}_<) = \left(-\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} \hat{G}_c^{(H)}(\mathbf{r}_<, \mathbf{r}'') \times F(\mathbf{r}'')\hat{G}_c(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U_c^{(0)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r''. \quad (5.2)$$

We note that, unlike in the expression for $U_c^{(1)}(\mathbf{r}_<)$, the "total" free-space Green's function $G$ [contained in the operator Green's function $\hat{G}_c(\mathbf{r}'', \mathbf{r}')$ in the integrand—cf. Eq. (2.14)] appears in expression (5.2) for $U_c^{(2)}(\mathbf{r}_<)$. This fact implies that, within the scattering volume $\mathcal{V}$, contributions carried by both the homogeneous waves and the evanescent waves are now included [cf. Eq. (2.9)]. However, outside the scatterer the effects of the evanescent waves have again been neglected.

If we now substitute for the operator Green's function $\hat{G}_c(\mathbf{r}'', \mathbf{r}')$ from Eq. (2.14) and for the zeroth-order field $U_c^{(0)}(\mathbf{r}')$ from Eq. (3.3), Eq. (5.2) becomes

$$U_c^{(2)}(\mathbf{r}_<) = \left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} [G_<^{(H)}(\mathbf{r}_<, \mathbf{r}'') + \mu G_>^{(H)*}(\mathbf{r}_<, \mathbf{r}'')\hat{C}]F(\mathbf{r}'') \times [G(\mathbf{r}'', \mathbf{r}') + \mu G_>^{(H)*}(\mathbf{r}'', \mathbf{r}')\hat{C}]\, F(\mathbf{r}') \times (1 + \mu\hat{C})U^{(i)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r''. \quad (5.3)$$

On expanding the product under the integral signs on the right-hand side of Eq. (5.3), we obtain the following explicit expression for the second-order contribution $U_c^{(2)}(\mathbf{r}_<)$ to the total field in the half-space $\mathcal{R}^-$:

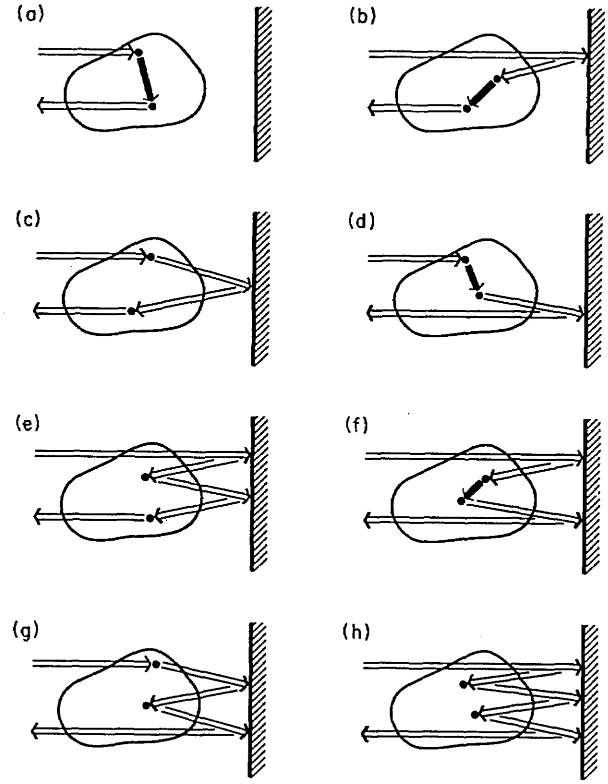


Fig. 4. Diagrammatic illustrations of the eight contributions [(a)–(h)], given in Eq. (5.4), to the second-order term $U_c^{(2)}(\mathbf{r}_<)$ of iterative expansion (3.1).

(5.4) contains the function $G_>^{(H)}$ or the function $G_<^{(H)}$ (depicted, as before, by hollow arrows in the diagrams). Some of the terms contain, in addition, the free-space Green's

$$\begin{aligned}
U_c^{(2)}(\mathbf{r}_<) = &+ \left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_<^{(H)}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' && \text{(a)}\\
&+ \mu\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_<^{(H)}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' && \text{(b)}\\
&+ \mu\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_<^{(H)}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G_>^{(H)*}(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')\mathrm{d}^3r'd^3r'' && \text{(c)}\\
&+ \mu\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_>^{(H)*}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G^*(\mathbf{r}'', r')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')d^3r'\mathrm{d}^3r'' && \text{(d)}\\
&+ |\mu|^2\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_<^{(H)}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G_>^{(H)*}(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' && \text{(e)}\\
&+ |\mu|^2\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_>^{(H)*}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G^*(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' && \text{(f)}\\
&+ |\mu|^2\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_>^{(H)*}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G_>^{(H)}(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' && \text{(g)}\\
&+ \mu|\mu|^2\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}}\int_{\mathcal{V}} G_>^{(H)*}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'')G_>^{(H)}(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r''. && \text{(h)} \quad (5.4)
\end{aligned}$$

Here we again made use of the fact that the scattering potential $F(\mathbf{r})$ is assumed to be real.

The eight terms (a)–(h) of Eq. (5.4) are illustrated in Fig. 4 in a diagrammatic form. Each diagram represents a single elementary event consisting of two scattering processes (denoted by dots, as before) and $m$ phase conjugations at the PCM, where $m = 0, 1, 2,$ or 3. We see that each term in Eq. (5.4) contains the function $G_>^{(H)}$ or the function $G_<^{(H)}$ (depicted, as before, by hollow arrows in the diagrams). Some of the terms contain, in addition, the free-space Green's function $G(\mathbf{r}'', \mathbf{r}')$ (depicted by a solid arrow), which, unlike $G_>^{(H)}$ and $G_<^{(H)}$, includes contributions from the evanescent waves; the appearance of $G(\mathbf{r}'', \mathbf{r}')$ in these terms allows for the possibility of a conversion of an evanescent wave into a homogeneous one, which subsequently contributes to the field $U_c^{(2)}(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ [diagrams (a), (b), (d), and (f) of Fig. 4].

If we make use of Eq. (2.15) and also of the relation

$$G^{(I)*}(\mathbf{r}, \mathbf{r}') = G^{(I)}(\mathbf{r}, \mathbf{r}') \tag{5.5}$$

that readily follows from Eq. (2.12), we find that many of the terms in Eq. (5.4) cancel one another. The formula (5.4) for the field $U_c{}^{(2)}(\mathbf{r}_<)$ may then be expressed in the following compact form:

$$U_c{}^{(2)}(\mathbf{r}_<) = (1 - |\mu|^2)[U_{bs}{}^{(2)}(\mathbf{r}_<) + {}'U^{(2)}(\mathbf{r}_<)], \tag{5.6}$$

where

$$U_{bs}{}^{(2)}(\mathbf{r}_<) = \left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}} \int_{\mathcal{V}} G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'') \times G(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' \tag{5.7}$$

is the (second-order) backscattered contribution to the conjugate field in the half-space $\mathcal{R}^-$ [diagram (a) in Fig. 4] and

$${}'U^{(2)}(\mathbf{r}_<) = -\mu\left(\frac{1}{4\pi}\right)^2 \int_{\mathcal{V}} \int_{\mathcal{V}} G_>{}^{(H)*}(\mathbf{r}_<, \mathbf{r}'')F(\mathbf{r}'') \times G_>{}^{(H)}(\mathbf{r}'', \mathbf{r}')F(\mathbf{r}')U^{(i)*}(\mathbf{r}')\mathrm{d}^3r'\mathrm{d}^3r'' \tag{5.8}$$

represents the (second-order) field contribution resulting from the sum of the three elementary events, each of which contains only a single phase conjugation at the PCM [diagrams (b)–(d) in Fig. 4]. It is seen that, in general, the field $U_c{}^{(2)}(\mathbf{r}_<)$ does not vanish throughout the half-space $\mathcal{R}^-$ and that it contains contributions of the evanescent waves created inside the scatterer through the backscattered term $U_{bs}{}^{(2)}(\mathbf{r}_<)$. However, in the special case when the conjugated waves are generated without losses or gains ($|\mu| = 1$), the second-order field contribution $U_c{}^{(2)}(\mathbf{r}_<)$ to the total field vanishes identically in the half-space $\mathcal{R}^-$ [cf. Eq. (5.6)].

If $|\mu| \ll 1$, as is usually the case, Eq. (5.6) yields

$$U_c{}^{(2)}(\mathbf{r}_<) \cong U_{bs}{}^{(2)}(\mathbf{r}_<) + {}'U^{(2)}(\mathbf{r}_<) \qquad (|\mu| \ll 1), \tag{5.9}$$

where the directly backscattered contribution $U_{bs}{}^{(2)}(\mathbf{r}_<)$ is independent of $\mu$ and the term ${}'U^{(2)}(\mathbf{r}_<)$, given by Eq. (5.8), is proportional to $\mu$. It can be seen, on comparing Eq. (5.8) with the expression for term (h) in Eq. (5.4), that the contribution ${}'U^{(2)}(\mathbf{r}_<)$ is equal to $-1/|\mu|^2$ times the contribution shown schematically in diagram (h) of Fig. 4.

## 6. DISTORTION CORRECTION IN THE ABSENCE OF LOSSES OR GAINS ON PHASE CONJUGATION

In view of the complexity of the physical processes implicit in the technique of distortion correction by phase conjugation, the theoretical analysis concerning the attainable degree of correction becomes prohibitive when one tries to carry out the calculations explicitly beyond the first few terms in the iterative expansion. However, it was found recently[5] that, in the special case when $|\mu| = 1$, i.e., when there are no losses or gains on phase conjugation at the PCM, the analysis can be carried out to all orders, and it leads to an interesting result. We now briefly discuss this case.

Let us examine a typical term $U_c{}^{(n)}(\mathbf{r})$ with $n \geq 1$ in the iterative expansion (3.1), making use of expression (2.16) for the Green's function $\hat{G}_c(\mathbf{r}, \mathbf{r}')$. With the field point $\mathbf{r} = \mathbf{r}_<$ situated in the half-space $\mathcal{R}^-$ and with the evanescent waves omitted in that half-space, an expression for the term $U_c{}^{(n)}(\mathbf{r}_<)$ is given by Eq. (3.5). One finds that, irrespective of the exact value of the complex "reflectance" $\mu$ of the PCM, the $n$th-order field contribution $U_c{}^{(n)}(\mathbf{r}_<)$ may be expressed in the form (cf. Ref. 5)

$$U_c{}^{(n)}(\mathbf{r}_<) = \left(-\frac{1}{4\pi}\right)^n \int_{\mathcal{V}} \mathrm{d}^3r_1F(\mathbf{r}_1) \ldots \int_{\mathcal{V}} \mathrm{d}^3r_nF(\mathbf{r}_n) \times G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}_1)(1 - \mu\hat{C})\hat{P}_{1,2} \ldots \hat{P}_{n-1,n}(1 + \mu\hat{C})U^{(i)}(\mathbf{r}_n), \tag{6.1}$$

where

$$\hat{P}_{j,j+1} \equiv \hat{G}_c(\mathbf{r}_j, \mathbf{r}_{j+1}) = (1 + \mu\hat{C})A_{j,j+1} + B_{j,j+1}(1 - \mu\hat{C}) + G^{(I)}(\mathbf{r}_j, \mathbf{r}_{j+1}), \tag{6.2}$$

with

$$A_{j,k} = \Theta(z_j - z_k)G_>{}^{(H)}(\mathbf{r}_j, \mathbf{r}_k) \tag{6.3a}$$

and

$$B_{j,k} = \Theta(z_k - z_j)G_<{}^{(H)}(\mathbf{r}_j, \mathbf{r}_k). \tag{6.3b}$$

We recall that the expressions (6.1)–(6.3) are applicable only when the scatterer is nonabsorbing. It can be shown that the $n$th-order term $U_c{}^{(n)}(\mathbf{r}_<)$ represents the sum of contributions from $2^{n+1}$ elementary events. Each of these events involves $n$ scattering processes and $m$ phase conjugations, with $m$ taking on the values $0, 1, 2, \ldots, n + 1$; the number of events containing $m$ phase conjugations is given by the binomial coefficient $\binom{n+1}{m}$.

It was further shown in Ref. 5 that, in the special case when $|\mu| = 1$, the following identity holds:

$$(1 - \mu\hat{C})\hat{P}_{1,2}\hat{P}_{2,3} \ldots \hat{P}_{n-1,n} = \hat{Q}_{1,2}\hat{Q}_{2,3} \ldots \hat{Q}_{n-1,n}(1 - \mu\hat{C}), \tag{6.4}$$

where

$$\hat{Q}_{j,j+1} = (1 - \mu\hat{C})B_{j,j+1} + G^{(I)}(\mathbf{r}_j, \mathbf{r}_{j+1}). \tag{6.5}$$

In deriving this identity, use was made of relation (5.5) and of the operator identity

$$(1 - \mu\hat{C})(1 + \mu\hat{C}) = (1 - |\mu|^2), \tag{6.6}$$

which follows readily from definition (2.8) of the complex-conjugation operator $\hat{C}$.

On substituting from Eq. (6.4) into Eq. (6.1), we obtain the following expression for $U_c{}^{(n)}(\mathbf{r}_<)$, valid when $|\mu| = 1$:

$$U_c{}^{(n)}(\mathbf{r}_<) = \left(-\frac{1}{4\pi}\right)^n \int_{\mathcal{V}} \mathrm{d}^3r_1F(\mathbf{r}_1) \ldots \int_{\mathcal{V}} \mathrm{d}^3r_nF(\mathbf{r}_n) \times G_<{}^{(H)}(\mathbf{r}_<, \mathbf{r}_1)\hat{Q}_{1,2} \ldots \hat{Q}_{n-1,n} \times (1 - \mu\hat{C})(1 + \mu\hat{C})U^{(i)}(\mathbf{r}_n) \quad (n = 1, 2, 3, \ldots). \tag{6.7}$$

If we make use once more of identity (6.6) and recall that $|\mu|$ is now assumed to be unity, we see at once from Eq. (6.7) that

$$U_c{}^{(n)}(\mathbf{r}_<) = 0, \qquad n = 1, 2, 3, \ldots. \tag{6.8}$$

Hence, with $\mathbf{r} = \mathbf{r}_<$, the right-hand side of Eq. (3.1) for the field $U_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ reduces, in this case, to the single term $U_c{}^{(0)}(\mathbf{r}_<)$ given by Eq. (3.3), i.e.,

$$U_c(\mathbf{r}_<) = U^{(i)}(\mathbf{r}_<) + e^{i\phi}U^{(i)*}(\mathbf{r}_<) \qquad (|\mu| = 1), \tag{6.9}$$

where $\phi$ is the argument (phase) of $\mu$. This result implies that, *if there are no losses or gains on phase conjugation at the PCM* (i.e., if $|\mu| = 1$), *the field $U_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ does not depend at all on the scatterer, and, consequently, distortions introduced by the scatterer on the incident field $U^{(i)}$ are now completely eliminated by the technique of phase conjugation*, it being assumed, as before, that the scatterer is nonabsorbing and that the effects of the evanescent waves outside the scatterer are negligible.

## 7. ELECTROMAGNETIC THEORY OF DISTORTION CORRECTION BY PHASE CONJUGATION

Until now we represented the (monochromatic) optical field by a scalar function of position, and hence we completely ignored its polarization properties. We now briefly discuss some generalizations of our main results, which take the polarization properties into account. For this purpose we must first know what changes are introduced into the state of an electromagnetic field on interaction with a PCM.

Let $\mathbf{E}^{(i)}(\mathbf{r})$ be the electric field vector [with the time-dependent part $\exp(-i\omega t)$ omitted] of an electromagnetic field incident in free space (i.e., in the absence of the scatterer) upon an infinite PCM located in the plane $z = z_1$. The response of the PCM will depend on the experimental technique that is used to generate the conjugate field.[6] We consider only the case when the PCM produces, in the plane of the mirror, the transformation

$$\mathbf{E}^{(i)}(\mathbf{r})|_{z=z_1} \rightarrow \mu \mathbf{E}^{(i)*}(\mathbf{r})|_{z=z_1}, \tag{7.1}$$

where $\mu$ is a constant that represents the "reflectivity" of the PCM. Formula (7.1) implies that the state of polarization of light is completely reversed on interaction with the PCM. The assumption of constant "reflection coefficient" $\mu$ implies that the response of the PCM to a plane wave is independent of both the angle of incidence and the state of polarization of the wave. This, of course, is an idealization. However, we wish to mention that experimental techniques have been developed, for example, by Zel'dovich and co-workers[16] for producing PCM's that give rise to complete reversal of the state of polarization, at least in the paraxial regime.

One can show by an argument similar to that leading to Theorem VI of Ref. 15 that, if the distribution $\mathbf{E}^{(i)}(\mathbf{r})|_{z=z_1}$ contains no evanescent contributions, the distribution $\mu\mathbf{E}^{(i)*}(\mathbf{r})|_{z=z_1}$ on the right-hand side of formula (7.1) will generate, in the absence of the scatterer, an electric field $\mu\mathbf{E}^{(i)*}(\mathbf{r})$ throughout the domain $z < z_1$.

Suppose now that the scatterer is present in the half-space $z < z_1$. For simplicity we assume that the scatterer is linear, time independent, isotropic, nonabsorbing, nonmagnetic, and spatially nondispersive and that it occupies a finite volume $\mathcal{V}$. It then follows from Maxwell's equations that the electric field $\mathbf{E}(\mathbf{r})$ satisfies (in the Gaussian system of units) the differential equation[17]

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 n^2(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0, \tag{7.2}$$

where

$$n^2(\mathbf{r}) = 1 + 4\pi\chi(\mathbf{r}) \tag{7.3}$$

and $\chi(\mathbf{r})$ is the dielectric susceptibility, defined by the formula

$$\mathbf{P}(\mathbf{r}) = \chi(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{7.4}$$

with $\mathbf{P}(\mathbf{r})$ denoting the induced polarization. We may rewrite Eq. (7.2) in the form

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = -F(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{7.5}$$

where

$$F(\mathbf{r}) = -k^2[n^2(\mathbf{r}) - 1] = -4\pi k^2\chi(\mathbf{r}) \tag{7.6}$$

is the scattering potential associated with the distribution $\chi(\mathbf{r})$ of the dielectric susceptibility throughout the volume $\mathcal{V}$. Since we assumed that the scatterer is nonabsorbing, the scattering potential $F(\mathbf{r})$ is a real function of position.

It is convenient to introduce the dyadic Green's function $\tilde{G}(\mathbf{r}, \mathbf{r}')$, defined as the outgoing free-space solution to the differential equation[18]

$$\nabla \times \nabla \times \tilde{G}(\mathbf{r}, \mathbf{r}') - k^2\tilde{G}(\mathbf{r}, \mathbf{r}') = 4\pi\delta^{(3)}(\mathbf{r} - \mathbf{r}')\tilde{I}, \tag{7.7}$$

where $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ is the three-dimensional Dirac delta function and $\tilde{I}$ denotes the unit dyadic. It is well known that $\tilde{G}(\mathbf{r}, \mathbf{r}')$ is related to the scalar Green's function $G(\mathbf{r}, \mathbf{r}')$ [Eq. (2.4)] by the formula

$$\tilde{G}(\mathbf{r}, \mathbf{r}') = \left(\tilde{I} + \frac{1}{k^2}\nabla\nabla\right) G(\mathbf{r}, \mathbf{r}'). \tag{7.8}$$

Let us first consider the case when an electric field $\mathbf{E}^{(i)}(\mathbf{r})$ is incident upon the scatterer $\mathcal{V}$ in the absence of the PCM. When the field interacts with the scatterer, a new field $\mathbf{E}(\mathbf{r})$ is generated that can be shown, on using Eqs. (7.5) and (7.7), to satisfy the integral equation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(i)}(\mathbf{r}) - \frac{1}{4\pi}\int_{\mathcal{V}} \tilde{G}(\mathbf{r}, \mathbf{r}') \cdot F(\mathbf{r}')\mathbf{E}(\mathbf{r}')\mathrm{d}^3r'. \tag{7.9}$$

This equation represents an electromagnetic analog to the usual integral equation (2.3) for scattering of scalar waves.

Next, suppose that the electric field $\mathbf{E}^{(i)}(\mathbf{r})$ is incident upon the scatterer with the PCM being present. We denote by $\mathbf{E}_c(\mathbf{r})$ the total electric field that is now generated. In analogy with the scalar case, we refer to $\mathbf{E}_c(\mathbf{r})$ as the conjugate electric field [cf. Ref. 9]. By arguments similar to those used in deriving the corresponding integral equation (2.6) for the conjugate scalar field $U_c(\mathbf{r})$, one can show that the conjugate electric field $\mathbf{E}_c(\mathbf{r})$ satisfies the integral equation

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}_c^{(0)}(\mathbf{r}) - \frac{1}{4\pi}\int_{\mathcal{V}} \tilde{G}_c(\mathbf{r}, \mathbf{r}') \cdot F(\mathbf{r}')\mathbf{E}_c(\mathbf{r}')\mathrm{d}^3r', \tag{7.10}$$

where $\tilde{G}_c(\mathbf{r}, \mathbf{r}')$ is a dyadic operator Green's function, to be discussed shortly, which takes into account the presence of the PCM. The field $\mathbf{E}_c^{(0)}(\mathbf{r})$ is related to the incident field $\mathbf{E}^{(i)}(\mathbf{r})$ by the formula

$$\begin{aligned}\mathbf{E}_c^{(0)}(\mathbf{r}) &= \mathbf{E}^{(i)}(\mathbf{r}) + \mu\mathbf{E}^{(i)*}(\mathbf{r}) \\ &= (1 + \mu\hat{C})\mathbf{E}^{(i)}(\mathbf{r}),\end{aligned} \tag{7.11}$$

where $\hat{C}$ denotes, as before, the complex-conjugation operator defined by Eq. (2.8). Formula (7.11) is strictly analogous to Eq. (3.3), and it represents the total electric field in the absence of the scatterer [$F(\mathbf{r}) \equiv 0$] but with the PCM present.

The dyadic operator Green's function $\tilde{G}_c(\mathbf{r}, \mathbf{r}')$ may be constructed in a manner similar to that employed in the derivation[5,11] of the operator Green's function $\hat{G}_c(\mathbf{r}, \mathbf{r}')$ [see also Ref. 19], and it can be expressed in the following analogous form [cf. Eq. (2.16)]:

$$\tilde{G}_c(\mathbf{r}, \mathbf{r}') = \Theta(z - z')(1 + \mu\hat{C})\tilde{G}_>{}^{(H)}(\mathbf{r}, \mathbf{r}') + \Theta(z' - z)\tilde{G}_<{}^{(H)}(\mathbf{r}, \mathbf{r}')(1 - \mu\hat{C}) + \tilde{G}^{(I)}(\mathbf{r}, \mathbf{r}'). \quad (7.12)$$

Here $\Theta(\zeta)$ is again the unit step function [Eq. (2.10)]

$$\tilde{G}_>{}^{(H)}(\mathbf{r}, \mathbf{r}') = \left(\tilde{I} + \frac{1}{k^2}\nabla\nabla\right) G_>{}^{(H)}(\mathbf{r}, \mathbf{r}'), \quad (7.13a)$$

$$\tilde{G}_<{}^{(H)}(\mathbf{r}, \mathbf{r}') = \left(\tilde{I} + \frac{1}{k^2}\nabla\nabla\right) G_<{}^{(H)}(\mathbf{r}, \mathbf{r}'), \quad (7.13b)$$

and

$$\tilde{G}^{(I)}(\mathbf{r}, \mathbf{r}') = \left(\tilde{I} + \frac{1}{k^2}\nabla\nabla\right) G^{(I)}(\mathbf{r}, \mathbf{r}'), \quad (7.14)$$

where the functions $G_>{}^{(H)}(\mathbf{r}, \mathbf{r}')$, $G_<{}^{(H)}(\mathbf{r}, \mathbf{r}')$, and $G^{(I)}(\mathbf{r}, \mathbf{r}')$ are given by Eqs. (2.11a), (2.11b), and (2.12), respectively. We emphasize that the vectorial integral equation (7.10) is valid only for PCM's that give rise to a complete reversal of polarization in the plane of the mirror, as implied by Eq. (7.1). The incident field was again assumed to contain no evanescent components, and the effects of the evanescent waves have been neglected at the PCM.

Just as in the scalar case, integral equation (7.10) may be formally solved by iteration. We express the conjugate electric field $E_c(\mathbf{r})$ in the form

$$\mathbf{E}_c(\mathbf{r}) = \sum_{n=0}^{\infty} \mathbf{E}_c{}^{(n)}(\mathbf{r}), \quad (7.15)$$

where the zeroth-order term $\mathbf{E}_c{}^{(0)}(\mathbf{r})$ is given by Eq. (7.11) and the higher-order terms are obtained from the recursion formula

$$\mathbf{E}_c{}^{(n)}(\mathbf{r}) = -\frac{1}{4\pi}\int_{\mathcal{V}} \tilde{G}_c(\mathbf{r}, \mathbf{r}') \cdot F(\mathbf{r}')\mathbf{E}_c{}^{(n-1)}(\mathbf{r}')\mathrm{d}^3r' \quad (n = 1, 2, 3, \ldots). \quad (7.16)$$

Because the mathematical structures of Eqs. (7.11) and (7.16) and of the dyadic operator Green's function (7.12) are similar to the structures of the corresponding formulas for the scalar case [Eqs. (3.3), (3.2), and (2.16), respectively], one may expect that many of the results that we obtained for the conjugate scalar field $U_c(\mathbf{r})$ will have strict analogs for the conjugate electric field $\mathbf{E}_c(\mathbf{r})$. In particular, one can show that, if $|\mu| = 1$, i.e., if the conjugate electric field is generated without losses or gains, and if, in addition, the effects of the evanescent waves are negligible at the PCM and at the detector in the half-space $\mathcal{R}^-$, then (with the detector located at $\mathbf{r}_< \in \mathcal{R}^-$)

$$\mathbf{E}_c{}^{(n)}(\mathbf{r}_<) = 0 \qquad \text{for } n = 1, 2, 3, \ldots \qquad (|\mu| = 1). \quad (7.17)$$

Using Eqs. (7.11), (7.15), and (7.17), it follows at once that (with $\mathbf{r}_< \in \mathcal{R}^-$)

$$\mathbf{E}_c(\mathbf{r}_<) = \mathbf{E}^{(i)}(\mathbf{r}_<) + e^{i\phi}\mathbf{E}^{(i)*}(\mathbf{r}_<) \qquad (|\mu| = 1), \quad (7.18)$$

where $\phi$ denotes the phase of the complex "reflectance" $\mu$ of the PCM.

Equation (7.18) represents an electromagnetic generalization of the corresponding result derived in Section 6 on the basis of scalar wave theory [cf. Eq. (6.9)]. It implies that, within the framework of the electromagnetic theory, a complete correction of distortions is obtained if the following conditions are satisfied:

(1) The phase conjugation at the PCM gives rise to a complete reversal of polarization [see formula (7.1)].

(2) There are no losses or gains on phase conjugation (i.e., $|\mu| = 1$).

(3) The scatterer is linear, time independent, isotropic, nonmagnetic, and spatially nondispersive and occupies a finite volume.

(4) The effects of the evanescent waves are negligible at the PCM and in the region of the half-space $\mathcal{R}^-$ where the detector is situated.

If $|\mu| \neq 1$ and the scatterer is sufficiently weak, the electric field $\mathbf{E}_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ will, to a good approximation, be given by the first few terms of iterative expansion (7.15). For example, within the accuracy of the first Born approximation, one may deduce from our integral equation (7.10) that

$$\mathbf{E}_c(\mathbf{r}_<) \cong \mathbf{E}_c{}^{(0)}(\mathbf{r}_<) + \mathbf{E}_c{}^{(1)}(\mathbf{r}_<), \quad (7.19)$$

where $\mathbf{E}_c{}^{(0)}(\mathbf{r}_<)$ is given by Eq. (7.11) and

$$\mathbf{E}_c{}^{(1)}(\mathbf{r}_<) = -(1 - |\mu|^2)\frac{1}{4\pi}\int \left(\tilde{I} + \frac{1}{k^2}\nabla\nabla\right) G^{(H)}(\mathbf{r}_<, \mathbf{r}') \cdot F(\mathbf{r}')\mathbf{E}^{(i)}(\mathbf{r}')\mathrm{d}^3r'. \quad (7.20)$$

The term $\mathbf{E}_c{}^{(1)}(\mathbf{r}_<)$ represents the contribution to the conjugate electric field $\mathbf{E}_c(\mathbf{r}_<)$ in the half-space $\mathcal{R}^-$ that arises from backscattering, in the first Born approximation [cf. Eqs. (4.5) and (4.6)]. If the first-order backscattering is negligible, then Eq. (7.19) gives $\mathbf{E}_c(\mathbf{r}_<) \cong \mathbf{E}_c{}^{(0)}(\mathbf{r}_<)$, showing, in view of Eq. (7.11), that in this approximation the scatterer has no effect on the conjugated electric field in the half-space $\mathcal{R}^-$, i.e., that the distortions in the incident field produced by a nonabsorbing scatterer are then eliminated by phase conjugation.

## 8. DISCUSSION

In this paper we have developed a systematic approach, both within the framework of the scalar wave theory and within the framework of Maxwell's electromagnetic theory, to the problem of determining the degree of distortion correction that may be achieved by the technique of optical phase conjugation. This approach is based on new integral equations for scattering of monochromatic waves in the presence of a phase-conjugate mirror (PCM). We treated only situations in which the distorting medium is linear, time independent, and nonabsorbing, even though the basic integral equations may be shown to apply also to absorbing media. It was found that, if the PCM produces in its plane a complete reversal of polarization, essentially the same conclusions follow from the scalar wave theory and from the electromagnetic theory.

In the derivation of the basic integral equations [Eqs. (2.6) and (7.10)], we assumed that the effects of the evanescent waves are negligible at the PCM. We also assumed that the

PCM is infinite in extent and that it is characterized by a constant "reflectivity" $\mu$. In practice, the PCM will be finite, and this may be expected to lead to a reduction in its ability to produce a field that would compensate for high-spatial-frequency components of the distortions imparted on the incident wave by the scatterer. Moreover, $\mu$ may be a function of position across the PCM and will, in general, also depend on the angles of incidence of the plane-wave components that are present in the angular spectrum representation of the field. These more-complex situations could be treated by appropriately modifying our integral equations, but such modifications are not discussed in the present paper.

Perhaps the most striking feature of the present approach is that the contributions that arise from elementary events consisting of scattering and phase-conjugation processes of all orders appear explicitly in the expressions for the conjugated fields. Such contributions must, in general, be included when the wave fields are monochromatic. The situation may be different when pulsed waves are used or when the macroscopic properties of the scatterer vary with time; however, the analysis of such cases would require a separate investigation.

Finally, we wish to mention that, although we assumed throughout this paper that the scatterer is "deterministic," our integral equations may be used even when the scattering medium is random in nature. In such a case the scattering potential and the conjugate field become random processes, and our integral equations then refer to single realizations. Quantities of physical interest, such as field correlations, can then be obtained from the solutions of these equations by taking appropriate ensemble averages.[20]


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6. Several schemes, such as those based on three-wave mixing [A. Yariv, "Three-dimensional pictorial transmission in optical fibers," Appl. Phys. Lett. **28**, 88–89 (1976)], four-wave mixing [R. W. Hellwarth, "Generation of time-reversed wave fronts by nonlinear refraction," J. Opt. Soc. Am. **67**, 1–3 (1977)], stimulated Brillouin scattering (Zel'dovich *et al.*[2]), etc. have been proposed and used for the generation of the conjugated field. The polarization properties of the conjugated field will, in general, be different from those of the incident (probe) field. A complete reversal of polarization can be achieved by suitably arranging the experimental geometry and by choosing the polarization properties of the various interacting waves appropriately [see, for example, Ref. 16 below and J. F. Lam, D. G. Steel, R. A. McFarlane, and R. C. Lind, "Atomic coherence effects in resonant degenerate four-wave mixing," Appl. Phys. Lett. **38**, 977–979 (1981)].
7. This integral equation is derived, in the context of time-independent quantum-mechanical potential scattering, for example in P. Roman, *Advanced Quantum Theory* (Addison-Wesley, Reading, Mass., 1965), Sec. 3.2.
8. The concept of a PCM is a convenient idealization that describes the effect of a true physical device located beyond the plane $z = z_1$, by means of which a field distribution $U$ is replaced by a new field distribution $\mu U^*$ in that plane. The transformation $U \rightarrow \mu U^*$ is usually achieved by nonlinear optical interactions, such as stimulated scattering processes or optical parametric interactions [see Ref. 1].
9. The term "conjugate" (or "conjugated") field is somewhat ambigous and must be interpreted with caution. The field $U_c(\mathbf{r})$ is generated as a result of the interaction of the incident field $U^{(i)}(\mathbf{r})$ with the scattering medium in the presence of the PCM. In general, it will include, in addition to $U^{(i)}(\mathbf{r})$, also contributions (usually ignored) arising from backscattering of the incident field and from successive conjugations of waves backscattered onto the PCM [see, for example, Figs. 3(a), 3(d), 4(a), and 4(h)].
10. For a discussion of some of the complications that arise when the evanescent waves are taken into account, see E. Wolf and W. H. Carter, "Comments on the theory of phase-conjugated waves," Opt. Commun. **40**, 397–400 (1982). The approximation resulting from the neglection of the evanescent waves in the scattered field is examined, within the accuracy of the first Born approximation, in App. A of Ref. 3.
11. A. T. Friberg, "Integral equation of the scattered field in the presence of a phase-conjugate mirror" J. Opt. Soc. Am. (to be published.)
12. Throughout this paper, a caret above a symbol denotes an operator.
13. See, for example, A. Baños, *Dipole Radiation in the Presence of a Conducting Half-Space* (Pergamon, Oxford, 1966), Eq. (2.19).
14. We will not discuss here the conditions under which series (3.1) will converge, a subject that would require a separate investigation.
15. E. Wolf, "Phase conjugacy and symmetries in spatially bandlimited wavefields containing no evanescent components," J. Opt. Soc. Am. **70**, 1311–1319 (1980). Equation (2.1) of this reference contains a misprint. $U^{(2)}(x, y, z)e^{i\omega t}$ should be replaced with $U^{(2)}(x, y, z)e^{-i\omega t}$. Also, Eq. (1.8) should read $A(u/k, v/k) = k^2\tilde{U}(u, v; z)e^{-iwz}$. These corrections do not affect any other equations or conclusions of that paper.
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17. Alternatively, by using the Maxwell equation $\nabla \cdot \mathbf{D} = 0$ (where

$\mathbf{D} \equiv \mathbf{E} + 4\pi\mathbf{P}$ is the electric displacement vector), the constitutive relation $\mathbf{D} = n^2\mathbf{E}$ and the vector identity $\nabla \cdot (n^2\mathbf{E}) = n^2\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla n^2$, Eq. (7.2) may be rewritten in the familiar form

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2n^2(\mathbf{r})\mathbf{E}(\mathbf{r}) + \nabla\{\mathbf{E}(\mathbf{r}) \cdot \nabla \log[n^2(\mathbf{r})]\} = 0.$$

However, form (7.2) is more convenient for the present purposes.

18. See, for example, C.-T. Tai, *Dyadic Green's Functions in Electromagnetic Theory* (Intext, Scranton, Pa., 1971), Sec. 14.
19. G. S. Agarwal, "Dipole radiation in the presence of a phase conjugate mirror," Opt. Commun. 42, 205–207 (1982).
20. In this connection, see V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation* (National Technical Information Service, Springfield, Va., 1971); E. Wolf, "New theory of partial coherence in the space-frequency domain. Part I: spectra and cross spectra of steady-state sources," J. Opt. Soc. Am. **72**, 343–351 (1982).