Effect of backscattering in phase conjugation with weak scatterers

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An extension is presented of a recently developed theory (based on the first Born approximation) of cancellation of distortions by phase conjugation. The influence of backscattering of both the incident and the conjugate waves is considered. It is shown that, when backscattering is taken into account, distortions are not eliminated by phase conjugation, except when the conjugate wave is generated without a loss or a gain.

In a recent paper,[1] a theory of cancellation by phase conjugation of distortions produced by scatterers was developed on the basis of the first Born approximation. The analysis involved several approximations, one of which was the requirement that backscattering of both the incident wave and the conjugate wave be negligible.

In the present paper we discuss the influence of backscattering. We find that, even within the accuracy of the first Born approximation, backscattering prevents, in general, a complete cancellation of the distortion effects. Our analysis shows, however, that in the special case when there are no losses or gains in the process of phase conjugation, the conjugate wave cancels out exactly (within the accuracy of the first Born approximation) the distorting effects of the scatterer.

We begin by recalling some of the main formulas and results of Ref. 1 (cf. *Note Added in Proof*). Consider a monochromatic scalar wave $U^{(i)}(\mathbf{r})\exp(-i\omega t)$ incident from the half-space $\mathcal{R}^-$ (see Fig. 1) on a weak scatterer occupying a finite volume $\mathcal{V}$. When the wave interacts with the scatterer, a field[2] $U(\mathbf{r})$ is created that may be represented in the form

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r}) + U^{(s)}(\mathbf{r}), \tag{1}$$

where $U^{(s)}(\mathbf{r})$ is the scattered field. At a plane $z = z_1$ in the half-space $\mathcal{R}^+$, a "phase-conjugated" field distribution

$$V(\mathbf{r})|_{z=z_1} = \mu[U(\mathbf{r})]^*|_{z=z_1} \tag{2}$$

is then generated, where μ is a (generally complex) constant that takes into account losses ($|\mu| < 1$) or gains ($|\mu| > 1$) and the asterisk denotes the complex conjugate. The boundary condition expressed by Eq. (2) gives rise to a "conjugate wave"[3] that subsequently interacts with the scatterer and generates a new field $V(\mathbf{r})$ in the half-space $\mathcal{R}^-$.

It is known[4] that within the accuracy of the first Born approximation the scattered field $U^{(s)}(\mathbf{r})$ in the half-spaces $\mathcal{R}^+$ and $\mathcal{R}^-$ on either side of the scatterer (see Fig. 1) can be represented in the form

$$U^{(s)}(\mathbf{r}) = \iint_{|\kappa|<\infty} A^{(\pm)}(\boldsymbol{\kappa})\exp[i(\boldsymbol{\kappa}\cdot\boldsymbol{\rho} \pm wz)]\mathrm{d}^2\kappa, \tag{3}$$

where

$$A^{(\pm)}(\boldsymbol{\kappa}) = -\frac{i}{8\pi^2 w} \times \int_{\mathcal{V}} F(\mathbf{r}')U^{(i)}(\mathbf{r}')\exp[-i(\boldsymbol{\kappa}\cdot\boldsymbol{\rho}' \pm wz')]\mathrm{d}^3r'. \tag{4}$$

In Eq. (3), $\boldsymbol{\rho}$ is the transverse component (i.e., the component perpendicular to the z axis, considered as a two-dimensional vector) and z is the longitudinal component of the position vector $\mathbf{r}$,

$$w = +(k_o^2 - \kappa^2)^{1/2} \quad \text{if}\,|\kappa| \leqslant k_o, \tag{5a}$$

$$\;\;= +i(\kappa^2 - k_o^2)^{1/2} \quad \text{if}\,|\kappa| > k_o, \tag{5b}$$

where $k_o = \omega/c$ is the free-space wave number, c being the speed of light in vacuum. In Eq. (4),

$$F(\mathbf{r}) = -k_o^2[n^2(\mathbf{r}) - 1] \tag{6}$$

is the scattering potential associated with the distribution $n(\mathbf{r})$ of the refractive index throughout the scatterer. The upper

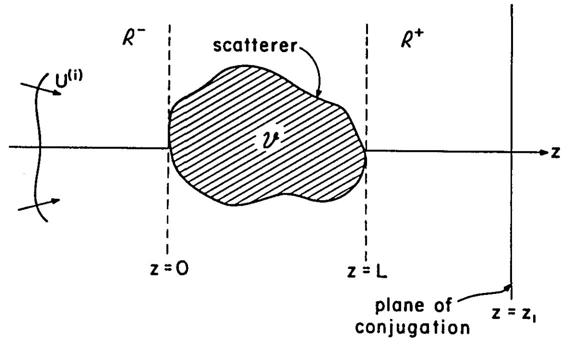

Fig. 1. Illustration of the notation.

signs (+) or the lower signs (−) are to be taken in Eqs. (3) and (4) according to whether the field point represented by $\mathbf{r} = (\rho, z)$ is situated in the half-space $\mathcal{R}^+$ or $\mathcal{R}^-$, respectively.

It was shown in Ref. 1 that, under the assumptions that

(a) the scatterer is nonabsorbing,
(b) the effects of evanescent waves are negligible, and
(c) backscattering can be ignored,

the conjugate field $V(\mathbf{r})$ [i.e., the field generated by the boundary condition given by Eq. (2) and propagated into the half-space $z < z_1$] is, after interacting with the scatterer, given by the expression

$$V(\mathbf{r}) = \mu[U^{(i)}(\mathbf{r})]^* + V^{(1)}(\mathbf{r}) + V^{(s)}(\mathbf{r}). \tag{7}$$

Here[5]

$$V^{(1)}(\mathbf{r}) = \mu \iint_{|\kappa|<k_0} [A^{(+)}(-\kappa)]^* \exp[i(\kappa \cdot \rho - wz)]\mathrm{d}^2\kappa, \tag{8}$$

where $A^{(+)}(-\kappa)$ is obtained from Eq. (4), and

$$V^{(s)}(\mathbf{r}) = \mu \iint_{|\kappa|<k_0} B^{(\pm)}(\kappa) \exp[i(\kappa \cdot \rho \pm wz)]\mathrm{d}^2\kappa, \tag{9}$$

with

$$B^{(\pm)}(\kappa) = -\frac{i}{8\pi^2 w} \times \int_{\mathcal{V}} F(\mathbf{r}')[U^{(i)}(\mathbf{r}')]^* \exp[-i(\kappa \cdot \rho' \pm wz')]\mathrm{d}^3 r'. \tag{10}$$

The choice of the upper or lower signs in Eqs. (9) and (10) is determined by the same rule as in connection with Eqs. (3) and (4).

As was already noted [assumption (c) above], the effects of backscattering on correction of distortions were neglected in the analysis of Ref. 1, i.e., the contribution

$$U_<^{(s)}(\mathbf{r}) = \iint_{|\kappa|<k_0} A^{(-)}(\kappa) \exp[i(\kappa \cdot \rho - wz)]\mathrm{d}^2\kappa \tag{11}$$

was neglected in the half-space $\mathcal{R}^-$ ($z < 0$) (see Fig. 2), and the contribution

$$V_>^{(s)}(\mathbf{r}) = \mu \iint_{|\kappa|<k_0} B^{(+)}(\kappa) \exp[i(\kappa \cdot \rho + wz)]\mathrm{d}^2\kappa \tag{12}$$

was neglected in the half-space $\mathcal{R}^+$ ($z > L$). In physical terms $U_<^{(s)}(\mathbf{r})$ represents that portion of the scattered field before conjugation that is propagated back into the half-space $\mathcal{R}^-$ from which the field $U^{(i)}(\mathbf{r})$ is incident. The contribution $V_>^{(s)}(\mathbf{r})$, on the other hand, represents that part of the scattered field that is created on interaction of the conjugate wave $V(\mathbf{r})$ with the scatterer and is then propagated back into the half-space $\mathcal{R}^+$. It is clear that, on reaching the plane $z = z_1$, this field is also conjugated and gives rise to an additional contribution to the field that interacts with the scatterer and eventually reaches the half-space $\mathcal{R}^-$.

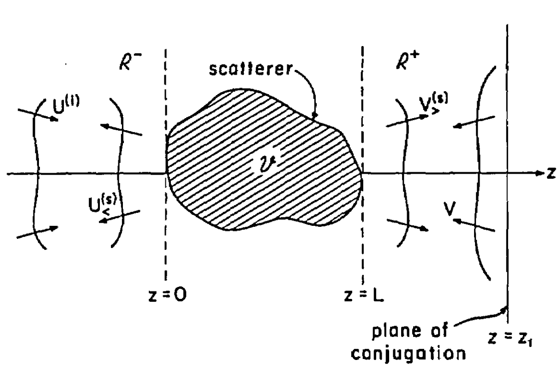

Fig. 2. Illustration of the backscattered-field contributions $U_<^{(s)}(\mathbf{r})$ in the half-space $\mathcal{R}^-$ and $V_>^{(s)}(\mathbf{r})$ in the half-space $\mathcal{R}^+$.

We also recall from Ref. 1 that, if the effects of backscattering are neglected [i.e., if the contributions $U_<^{(s)}(\mathbf{r})$ and $V_>^{(s)}(\mathbf{r})$ are omitted], then the last two terms on the right-hand side of Eq. (7) cancel out throughout the half-space $\mathcal{R}^-$, i.e.,

$$V^{(s)}(\mathbf{r}) = -V^{(1)}(\mathbf{r}) \qquad (\mathbf{r} \in \mathcal{R}^-). \tag{13}$$

Under these circumstances Eq. (7) reduces to

$$V(\mathbf{r}) = \mu[U^{(i)}(\mathbf{r})]^* \qquad (\mathbf{r} \in \mathcal{R}^-), \tag{14}$$

showing that the influence of the scatterer has been entirely eliminated by phase conjugation.

We will now turn our attention to the backscattered-field contributions $U_<^{(s)}(\mathbf{r})$ and $V_>^{(s)}(\mathbf{r})$ and examine their combined effects in the half-space $\mathcal{R}^-$. The expression for $U_<^{(s)}(\mathbf{r})$ in the half-space $\mathcal{R}^-$ is given by Eq. (11), with $A^{(-)}(\kappa)$ obtained from Eq. (4). To determine the contribution arising from $V_>^{(s)}(\mathbf{r})$ we note that, after this field reaches the plane $z = z_1$, it undergoes a phase conjugation there, i.e., it generates a boundary condition $\mu[V_>^{(s)}(\mathbf{r})]^*|_{z=z_1}$ in that plane. This boundary condition gives rise to a new field that propagates toward the scatterer. According to Theorem VI of Ref. 6, this field is just $\mu[V_>^{(s)}(\mathbf{r})]^*$ and may now be regarded as a new field incident upon the scatterer from the half-space $\mathcal{R}^+$. It is important to appreciate that $V_>^{(s)}(\mathbf{r})$ depends linearly on the scattering potential $F(\mathbf{r})$ [cf. Eqs. (12) and (10)], and hence the scattered field generated by $\mu[V_>^{(s)}(\mathbf{r})]^*$ may be neglected within the accuracy of the first Born approximation. Thus the total contribution in the half-space $\mathcal{R}^-$ that is due to the backscattering is given by

$$W^{(b\cdot s)}(\mathbf{r}) = U_<^{(s)}(\mathbf{r}) + \mu[V_>^{(s)}(\mathbf{r})]^* \qquad (\mathbf{r} \in \mathcal{R}^-). \tag{15}$$

Making use of Eqs. (11) and (12) and changing the integration variable from κ to $-\kappa$ in Eq. (12), we may express $W^{(b\cdot s)}(\mathbf{r})$ in the form

$$W^{(b\cdot s)}(\mathbf{r}) = \iint_{|\kappa|<k_0} A^{(b\cdot s)}(\kappa) \exp[i(\kappa \cdot \rho - wz)]\mathrm{d}^2\kappa, \tag{16}$$

where

$$A^{(b\cdot s)}(\kappa) = A^{(-)}(\kappa) + |\mu|^2[B^{(+)}(-\kappa)]^* \tag{17}$$

Since it was assumed that the scatterer is nonabsorbing, the scattering potential $F(\mathbf{r})$ is a real function of $\mathbf{r}$ throughout the volume $\mathcal{V}$. Hence we see at once from Eqs. (4) and (10) that

$$[B^{(+)}(-\kappa)]^* = -A^{(-)}(\kappa), \tag{18}$$

and consequently the spectral amplitude function $A^{(b\cdot s)}(\kappa)$ in angular spectrum representation (16) is given by

$$A^{(b\cdot s)}(\kappa) = (1 - |\mu|^2)A^{(-)}(\kappa), \quad (19)$$

where $A^{(-)}(\kappa)$ is obtained from Eq. (4). It then follows from Eqs. (16) and (19) that the total backscattered field in the half-space $\mathcal{R}^-$ is expressible in the form

$$W^{(b\cdot s)}(\mathbf{r}) = (1 - |\mu|^2) \times \iint_{|\kappa|<k_o} A^{(-)}(\kappa)\exp[i(\kappa \cdot \rho - wz)]d^2\kappa. \quad (20)$$

If we now use Eq. (14) and include the contributions from backscattering, it is clear that the total field $W(\mathbf{r})$ in the half-space $\mathcal{R}^-$ is given by

$$W(\mathbf{r}) = U^{(i)}(\mathbf{r}) + \mu[U^{(i)}(\mathbf{r})]^* + W^{(b\cdot s)}(\mathbf{r}) \quad (\mathbf{r} \in \mathcal{R}^-). \quad (21)$$

Unlike the first two terms on the right-hand side of Eq. (21), the last term, given by Eq. (20), contains influence of the scatterer because, according to the expression for the spectral amplitude $A^{(-)}(\kappa)$ obtained from Eq. (4), it depends on the scattering potential $F(\mathbf{r})$. Thus, *when the effect of backscattering is not negligible, the distortions introduced by the scatterer are (within the accuracy of the first Born approximation), in general, not eliminated by phase conjugation.*

However, from Eq. (20) the remarkable fact is evident that when $|\mu| = 1$, i.e., *when the conjugate wave is generated without a gain or a loss,*

$$W^{(b\cdot s)}(\mathbf{r}) = 0 \quad (\mathbf{r} \in \mathcal{R}^-), \quad (22)$$

i.e., *the total backscattered field then vanishes (within the accuracy of the first Born approximation) throughout the half-space* $\mathcal{R}^-$. Under these circumstances Eq. (21) reduces to

$$W(\mathbf{r}) = U^{(i)}(\mathbf{r}) + \mu[U^{(i)}(\mathbf{r})]^* \quad (\mathbf{r} \in \mathcal{R}^-). \quad (23)$$

Thus in this special case ($|\mu| = 1$) distortions in the incident wave introduced by the scatterer are completely eliminated by phase conjugation.

If we make use of relation (18) in Eq. (11) and compare the resulting expression with Eq. (12), we find that the backscattered contributions $U_<^{(s)}(\mathbf{r})$ and $\mu[V_>^{(s)}(\mathbf{r})]^*$ are related throughout the half-space $\mathcal{R}^-$ by the formula

$$\mu[V_>^{(s)}(\mathbf{r})]^* = -|\mu|^2 U_<^{(s)}(\mathbf{r}) \quad (\mathbf{r} \in \mathcal{R}^-). \quad (24)$$

In practice $|\mu|$ is usually much smaller than unity; Eq. (24) then shows that the contribution generated by the backscattered conjugate wave {i.e., the contribution $\mu[V_>^{(s)}(\mathbf{r})]^*$} is then negligible compared with the contribution $U_<^{(s)}(\mathbf{r})$ that arises from the backscattering of the incident wave. Hence in this case Eq. (15) may be approximated by

$$W^{(b\cdot s)}(\mathbf{r}) \cong U_<^{(s)}(\mathbf{r}) \quad (\mathbf{r} \in \mathcal{R}^-), \quad (25)$$

and Eq. (21) reduces to

$$W(\mathbf{r}) \cong U^{(i)}(\mathbf{r}) + \mu[U^{(i)}(\mathbf{r})]^* + U_<^{(s)}(\mathbf{r}) \quad (\mathbf{r} \in \mathcal{R}^-). \quad (26)$$

Only the last term on the right-hand side of formula (26) depends on the scattering potential. Hence it follows (within the accuracy of the first Born approximation) that if the conjugate wave is generated with an appreciable loss ($|\mu| \ll 1$), distortion effects imparted upon the incident wave by the scatterer can be approximately eliminated by the technique of phase conjugation, provided that some experimental procedure is used that removes from the total field $W(\mathbf{r})$ in the half-space $\mathcal{R}^-$ the field $U_<^{(s)}(\mathbf{r})$ generated by backscattering of the incident field $U^{(i)}(\mathbf{r})$. It seems, however, unlikely that such removal can be experimentally achieved, except perhaps with very simple types of scatterers.

† *Note added in proof:* In Ref. 1 and in a note by E. Wolf and W. H. Carter, "Comments on the theory of phase conjugated waves," Opt. Commun. **40**, 397–400 (1982), the starting point of the analysis was the reduced scalar-wave equation for the fields. It was argued by A. Yariv, "Reply to the paper 'Comments on the theory of phase conjugated waves' by E. Wolf and W. H. Carter," Opt. Commun. **40**, 401 (1982), that the use of the scalar- rather than the vector-wave equation is essentially equivalent to the use of the paraxial equation. We do not subscribe to this view. Moreover, the scalar-wave equation was used by Wolf and Carter because the original argument of Yariv is based on that equation.

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2. From now on we omit the periodic time-dependent factor $e^{-i\omega t}$.
3. The term "conjugate wave" has to be interpreted here with caution because, in general, there will be additional contributions arising from backscattering toward the plane $z = z_1$.
4. E. Wolf, "Three-dimensional structure determination of semi-transparent objects from holographic data," Opt. Commun. **1**, 153–156 (1969).
5. The domain of integration in Eqs. (8) and (9) is $|\kappa| < k_o$ rather than $|\kappa| < \infty$ because the plane waves corresponding to $|\kappa| > k_o$ are evanescent [see Eq. (5b)], and such waves are omitted here in accordance with assumption (b) above.
6. E. Wolf, "Phase conjugacy and symmetries in spatially bandlimited wavefields containing no evanescent components," J. Opt. Soc. Am. **70**, 1311–1319 (1980). Equation (2.1) of this reference contains a misprint. $U^{(2)}(x, y, z)e^{i\omega t}$ should be replaced by $U^{(2)}(x, y, z) e^{-i\omega t}$. Also, Eq. (1.8) should read $A(u/k, v/k) = k^2\tilde{U}(u, v; z)e^{-i\omega t}$. These corrections do not affect any other equations or conclusions of that paper.