Theory of phase conjugation with weak scatterers

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A theory relating to correction of distortions that may be achieved by phase conjugation is developed on the basis of the first Born approximation. It is shown that, to good accuracy, the effect of a distorting medium on an incident wave is eliminated by phase conjugation if the following conditions are satisfied: the incident field contains no evanescent components, the transmitting medium is a weak, nonabsorbing scatterer, and backscattering of the incident and of the conjugate wave and also the effects of scattered evanescent waves are negligible.

1. INTRODUCTION

Many ingenious applications have been proposed in recent years for utilizing waves that can be generated from a given wave by reversing its phase in a particular cross section. Such phase conjugation can be produced by nonlinear interaction that involves, for example, stimulated Brillouin scattering,1 three-wave mixing in crystals,2 or the so-called degenerate four-wave mixing.3 The potential usefulness of this new technique arises largely from the fact that it appears possible to correct, by means of phase conjugation, various kinds of distortion that might have been imparted on the original wave by the transmitting medium. Such distortions may be caused, for example, by aberrations of an optical system or by inhomogeneities in the refractive index of the transmitting medium (e.g., the atmosphere).

The possibility of correcting distortion effects by phase conjugation has been demonstrated under relatively simple conditions by a number of experiments.4 However, a satisfactory theory of this “healing” process that may be achieved by phase conjugation has not been developed so far.5

In this paper a theory relating to correction of distortions by phase conjugation is developed on the basis of the first Born approximation. We show that, to good accuracy, the effect of a distorting medium on an incident wave may be eliminated by phase conjugation if the following conditions are satisfied: the incident wave contains no evanescent components, the transmitting medium is a weak, nonabsorbing scatterer, and backscattering of the incident and of the conjugate wave and also the effects of scattered evanescent waves are negligible.

2. SCATTERED FIELD IN THE FIRST BORN APPROXIMATION

Let

\[ \mathbf{U}^{(i)}(x, y, z, t) = \mathbf{U}^{(i)}(x, y, z) \exp(-i\omega t), \]

with \((x, y, z)\) denoting the Cartesian coordinates of a typical field point and \(t\) denoting the time, be a monochromatic wavefield of (real) frequency \(\omega\), propagating into the half-space \(z > 0\). Suppose that this field is incident upon a weakly scattering object, which occupies a finite volume \(\mathcal{V}\), situated within a strip \(0 \leq z \leq L\) in free space (Fig. 1). Under very general conditions the space-dependent part \(U^{(i)}(\mathbf{r})\) may be expressed in the form of an angular spectrum of plane waves, i.e., in the form

\[ U^{(i)}(x, y, z) \]

\[ = \int \int_{-\infty}^{\infty} A^{(i)}(p, q)[\exp[ik_0(px + qy + mz)]]dpdq, \]

where

\[ k_0 = \frac{\omega}{c}, \]

c being the speed of light in free space, and

\[ m = \pm (1 - p^2 - q^2)^{1/2} \quad \text{when } p^2 + q^2 \leq 1, \]

\[ = \pm (p^2 + q^2 - 1)^{1/2} \quad \text{when } p^2 + q^2 > 1. \]

The values of \(m\) that are given by Eq. (2.4a) are associated with homogeneous plane waves propagating into the half-space \(z > 0\); those given by Eq. (2.4b) are associated with evanescent plane waves, whose amplitudes decay exponentially with increasing values of \(z\). We will assume that the incident field contains no evanescent components so that

![Fig. 1. Illustrating the notation relating to scattering of an incident field \(U^{(i)}(\mathbf{r})\). \(U^{(i)}(\mathbf{r})\) is the scattered field and \(U(\mathbf{r}) = U^{(i)}(\mathbf{r}) + U^{(s)}(\mathbf{r})\) is the total field.](image-url)
The scattered field $U^{(o)}(r)$ is the solution of Eq. (2.16) that sufficiently far away from the scatterer behaves as an outgoing spherical wave.

Assuming that $U(r)$ and its normal derivative are continuous across the surface of the scatterer, the differential Eq. (2.16) for the scattered field, together with the postulated outgoing behavior, may be recast, by the use of well-known Green's function techniques, into the integral equation

$$U^{(o)}(r) = -\frac{1}{4\pi} \int_V F(r')U(r')G(|r - r'|)d^3r'. \tag{2.18}$$

Here

$$G(R) = e^{iK_0R} R$$

is the outgoing free-space Green's function of the Helmholtz operator.

Now we assume that we are dealing with a weak scatterer, i.e., one that generates a scattered field whose absolute value is small compared with that of the incident field:

$$|U^{(o)}(r)| \ll |U^{(i)}(r)|. \tag{2.20}$$

Under these circumstances, we may replace the total field $U(r)$ with $U^{(i)}(r)$ in the integrand on the right-hand side of Eq. (2.18), and we then obtain the first Born approximation to the scattered field:

$$U^{(o)}(r) \approx -\frac{1}{4\pi} \int_V F(r')U^{(i)}(r')G(|r - r'|)d^3r'. \tag{2.21}$$

3. PHASE CONJUGATION WITH REFLECTION

A. Conjugate Field Incident upon the Scatterer

Let us suppose that at each point $(\rho_1, z_1)$ in a fixed plane $z = z_1$ located in the half-space $z > L$ (denoted by $\mathcal{R}^+$ in Fig. 3) a field distribution $\mu[U(\rho_1, z_1)]^*$ is generated, where $[U(\rho_1, z_1)]^*$ is, of course, the complex conjugate of $U(\rho_1, z_1)$ and $\mu$ is a (possibly complex) constant. The constant takes into account losses ($|\mu| < 1$) or gains ($|\mu| > 1$) that might occur in the generation of this conjugate distribution. According to Eq. (2.14) we have

$$U^{(o)}(r) = U^{(i)}(r) + U^{(o)}(r). \tag{2.14}$$

Then $U^{(o)}(r)$ represents the scattered field. Since the incident field obeys the Helmholtz equation

$$\nabla^2 U^{(i)}(r) + k_0^2 U^{(i)}(r) = 0, \tag{2.15}$$

it readily follows on substituting Eq. (2.15) from Eq. (2.13) and on using Eq. (2.14) that the scattered field obeys the equation

$$(\nabla^2 + k_0^2)U^{(o)}(r) = F(r)U(r), \tag{2.16}$$

where

$$F(r) = -k_0^2[n^2(r) - 1]. \tag{2.17}$$

We will call $F(r)$ the scattering potential, as is customary.

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We will call $F(r)$ the scattering potential, as is customary.
\[ \mu[U(\rho_1, z_1)]* = \mu[U^{(1)}(\rho_1, z_1)]* + \mu[U^{(2)}(\rho_1, z_1)]*. \]

Suppose that this distribution gives rise to a field that propagates back into the half-space \( z < z_1 \). We denote this new field by \( V^{(1)}(r) \). It is the field that is incident upon the scatterer after phase conjugation in the plane \( z = z_1 \), i.e., the field generated in the half-space \( z < z_1 \) by the distribution [Eq. (3.1)] in the plane \( z = z_1 \) when the scatterer is removed. We will express \( V^{(1)}(r) \) as the sum of two terms,

\[ V^{(1)}(r) = V^{(0)}(r) + V^{(1)}(r), \]

where \( V^{(0)}(r) \) and \( V^{(1)}(r) \) are contributions arising from the first and the second terms on the right-hand side of Eq. (3.1), respectively, i.e., the fields \( V^{(0)}(r) \) and \( V^{(1)}(r) \) obey the boundary conditions

\[ V^{(0)}(\rho_1, z_1) = \mu[U^{(1)}(\rho_1, z_1)]*, \quad (3.3a) \]

\[ V^{(1)}(\rho_1, z_1) = \mu[U^{(2)}(\rho_1, z_1)]*. \quad (3.3b) \]

in the plane \( z = z_1 \).

The contribution \( V^{(0)}(r) \) can be readily determined if we make use of Eq. (3.3a) and recall that the field \( U^{(0)}(r) \) was assumed to contain no evanescent components. It then follows at once from a recently established theorem [Theorem VI of Ref. 9] that

\[ V^{(0)}(r) = \mu[U^{(0)}(r)]*. \quad (3.4) \]

The contribution \( V^{(1)}(r) \) to \( V^{(i)}(r) \) is somewhat more difficult to determine. We proceed as follows: We first expand \( U^{(s)}(r) \) into an angular spectrum of plane waves, using a technique employed previously in connection with an inverse-scattering problem that arises in holography. For this purpose we recall that the free-space Green’s function has the angular-spectrum representation

\[ G(|r - r'|) = \frac{i}{2\pi} \int_{-\infty}^{\infty} w \exp[i \cdot (\rho - \rho')] + w|z - z'|) d^3k, \quad (3.5) \]

where \( r = (\rho, z) \) and \( r' = (\rho', z') \). We substitute from Eq. (3.5) into Eq. (2.21), interchange the order of the integrations with respect to \( r' \) and \( \kappa \), and note that (see Fig. 2)

\[ |z - z'| = |z - z'| \quad \text{if} \ r \in \mathbb{R}^+, r' \in \mathbb{V}, \]

\[ = |z' - z| \quad \text{if} \ r \in \mathbb{R}^-, r' \in \mathbb{V}. \]

We then obtain the required angular-spectrum representation of the field \( U^{(s)}(r) \):

\[ U^{(s)}(r) = \int_{\kappa \in \mathbb{R}^3} A^{(s)}(\kappa) \exp[i(\kappa \cdot \rho + wz)] d^3k, \quad (3.7) \]

where the spectral amplitude function \( A^{(s)}(\kappa) \) is given by

\[ A^{(s)}(\kappa) = -\frac{i}{8\pi^2 w} \int_{\mathbb{V}} F(r') U^{(1)}(r') \exp[-i(\kappa \cdot \rho' + wz')] d^3r', \quad (3.8) \]

the upper signs being taken in Eqs. (3.7) and (3.8) when \( r \in \mathbb{R}^+ \) and the lower signs when \( r \in \mathbb{R}^- \). It should be noted that the integration in Eq. (3.7) extends over the whole \( \kappa \) plane, so that contributions of homogeneous wave \( (\kappa^2 < k_0^2) \) as well as of evanescent waves \( (\kappa^2 > k_0^2) \) generated in the interaction of the incident field \( U^{(1)}(r) \) with the scatterer are included. The evanescent waves may be neglected, to a good approximation, if the field point \( r \) is sufficiently far away from the scatterer. Alternatively, the contribution of the evanescent waves may also be omitted at any point \( r \) in the half-spaces \( \mathbb{R}^+ \) and \( \mathbb{R}^- \) if the effects of sufficiently small details of the structure of the scatterer are neglected. More precisely, we show in Appendix A that, if the evanescent waves are omitted, only those spatial Fourier components

\[ \tilde{F}(K) = \int_{\mathbb{V}} F(r') \exp(-i K \cdot r) d^3r \]

of the scattering potential are taken into account in calculating the scattered field for which

\[ |K| < 2k_0. \quad (3.9) \]

Since \( k_0 = 2\pi/\lambda_0 \), where \( \lambda_0 \) is the wavelength of the light, this condition implies, roughly speaking, that effects of details of the scatterer that are smaller in linear dimensions than about half of a wavelength are neglected. Such neglect seems to be consistent with the use of the first Born approximation, which, as is well known, applies in general only when the physical properties of the scatterer remain effectively constant over distances of the order of a few wavelengths. For these reasons, we will omit the evanescent waves on the right-hand side of Eq. (3.7), i.e., we approximate \( U^{(s)}(r) \) by the formula

\[ U^{(s)}(r) \approx \int_{\kappa \in \mathbb{R}^3} A^{(s)}(\kappa) \exp[i(\kappa \cdot \rho + wz)] d^3k. \]

In particular, this formula applies throughout the strip \( L < z < z_1 \) when the upper signs are taken (because the strip is located in the half-space \( \mathbb{R}^+ \)), so that one has at any point \( r \) in this strip

\[ U^{(s)}(r) \approx \int_{\kappa \in \mathbb{R}^3} A^{(s)}(\kappa) \exp[i(\kappa \cdot \rho + wz)] d^3k. \]

Now formula (3.12) contains no contributions from evanescent waves and hence, by a theorem of Sherman, the integral on the right-hand side of formula (3.12) represents, under fairly general conditions, a bounded solution of the Helmholtz equation that is valid throughout all space. If we recall Eq. (3.3b), it then follows at once from Theorem VI of Ref. 9 that the scattered field for which

\[ V^{(1)}(r) = \mu \int_{\kappa \in \mathbb{R}^3} A^{(s)}(-\kappa) \exp[i(\kappa \cdot \rho - wz)] d^3k, \]

where, according to Eq. (3.8),

\[ A^{(s)}(-\kappa) = -\frac{i}{8\pi^2 w} \int_{\mathbb{V}} F(r') U^{(1)}(r') \exp[-i(\kappa \cdot \rho' + wz')] d^3r'. \]

We conclude from Eqs. (3.2) and (3.4) that the conjugate wave incident upon the scatterer may be represented in the
V(r) = V(0)(r) + V(i)(r) \tag{3.16}

will be generated, where V(0) represents the scattered field. Within the accuracy of the first Born approximation, V(0)(r) is given by an expression of the form of Eq. (2.21) but with U replaced with V(0), i.e.,

V(0)(r) \approx \frac{-1}{4\pi} \int_{\mathcal{V}} F(r') V(0)(r') G(|r - r'|) d^3r' \tag{3.17}

or, using Eq. (3.2),

V(0)(r) \approx -\frac{1}{4\pi} \int_{\mathcal{V}} F(r') [V(0)(r')] + V(1)(r') G(|r - r'|) d^3r'. \tag{3.18}

Now, according to Eq. (3.4), V(0)(r') is independent of the scattering potential. On the other hand, V(1)(r') is linear in the potential, as is seen from Eqs. (3.13) and (3.14). Hence, within the accuracy of the first Born approximation, we may neglect the term V(1)(r') under the integral sign on the right-hand side of formula (3.18) and, if we use Eq. (3.4), we obtain the following expression for V(0)(r):

V(0)(r) \approx \frac{-i}{\pi} \int_{\mathcal{V}} F(r') \left[ U(r') \right]^{*} G(|r - r'|) d^3r'. \tag{3.19}

By following the same procedure as we employed earlier in connection with the field U(r), we may readily transform expression (3.19) into an angular-spectrum form. We need only to substitute for the Green’s function from Eq. (3.5) and interchange the orders of the integration with respect to r and k. We then obtain the required angular-spectrum representation of V(0)(r):

V(0)(r) \approx \mu \int_{-\infty}^{\infty} B^{(\pm)}(k) \exp[i(k \cdot \rho - wz)] d^3k, \tag{3.20}

where

B^{(\pm)}(k) = -\frac{i}{8\pi^2 k} \int_{\mathcal{V}} F(r') \left[ U(r') \right]^{*} \exp[-i(k \cdot \rho' \pm wz')] d^3r'. \tag{3.21}

with the positive signs being taken in Eqs. (3.20) and (3.21) when the field point r is situated in the strip L < z < z_1 (denoted by \mathcal{R}^+ in Fig. 2) and the negative signs when the field point is in the half-space z < 0 (denoted by \mathcal{R}^-). For the same reasons as explained below Eq. (3.8), we will neglect the effect of evanescent waves, i.e., we replace the domain of integration on the right-hand side of formula (3.20) with the interior of the circle \kappa^2 = h_0^2. It then follows that a good approximation for the scattered field V(0)(r), valid at all points in the half-space \mathcal{R}^-, is

V(0)(r) \approx \mu \int_{z_1}^{\infty} B^{(-)}(k) \exp[i(k \cdot \rho - wz)] d^3k, \tag{3.22}

where B^{(-)}(k) is, of course, given by Eq. (3.21), taken with the lower signs.

Let us compare the angular-spectrum representation [formula (3.22)] for V(0)(r) with the angular-spectrum representation [Eq. (3.13)] for V(1)(r). The angular-spectrum amplitude function B^{(-)}(k) [Eq. (3.21)] of V(0)(r) is seen to resemble closely the angular-spectrum amplitude [A^{(+)}(-k)]^*, given by Eq. (3.14), of V(1)(r). In fact, in the special case when

\left[F(r')\right]^{*} = F(r), \tag{3.23}

i.e., when the scattering potential F(r) is real (nonabsorbing scatterer), they are related as follows:

\left[A^{(+)}(-k)\right]^{*} = -B^{(-)}(k). \tag{3.24}

Hence, under these circumstances,

V(0)(r) = -V(1)(r), \quad r \in \mathcal{R}^- \tag{3.25}

Now the total field V(r) is, according to Eqs. (3.16) and (3.2), given by

V(r) = V(0)(r) + V(1)(r) + V(i)(r). \tag{3.26}

In view of Eq. (3.25), the last two terms on the right-hand side of Eq. (3.26) cancel each other out at every point r in the half-space \mathcal{R}^-,. The remaining term V(0)(r) is given by Eq. (3.4). Hence we finally obtain the following simple expression for V(r):

V(r) = \mu \left[U(r)\right]^{*}, \quad r \in \mathcal{R}^- \tag{3.27}

Equation (3.27) shows that throughout the half-space \mathcal{R}^- the field obtained by scattering the conjugate field V(i) is proportional to the complex conjugate of the original incident field U, i.e., the effect of the scatterer is completely canceled out.

It should be noted that we neglect backscattering of both the incident and the conjugate wave. Thus we obtained the above result under the following assumptions:

1. The incident field contains no evanescent components. This condition is to an excellent approximation satisfied by, but not restricted to, incident fields that are “beamlike”.
2. The scatterer is a weak scatterer in the sense that the scattered field may be described, to a good approximation, by the first Born approximation.
3. The scatterer is nonabsorbing.
4. Backscattering of both the incident and the conjugate wave is negligible.
5. The effect of scattered evanescent waves is negligible. This will be the case when the plane z = z_1 of conjugation and the field point r in the half-space \mathcal{R}^- (see Fig. 2) at which the total conjugate field V(r) is considered are sufficiently far away from the scattering object. Alternatively, this condition is satisfied (without any restriction on the distances) if the Fourier components \hat{F}(K) of the scattering potential F(r) for which \abs{K} > 2k_0 = 4\pi/h_0 do not significantly contribute to the scattered field. It seems likely (although we have not verified this conjecture) that this requirement is, in fact, necessary for the validity of the first Born approximation.
Conditions (1) and (5) will almost certainly be satisfied in most cases of practical interest.

APPENDIX A: APPROXIMATION ARISING FROM THE NEGLIGENCE OF EVANESCENT WAVES IN THE SCATTERED FIELD

We have shown that, within the accuracy of the first Born approximation, the spectral amplitudes $A^{(\pm)}(\mathbf{r})$ in the angular representation [Eq. (3.7)] of the scattered field $U^{(\omega)}(\mathbf{r})$ are given by

$$A^{(\pm)}(\mathbf{k}) = -\frac{i}{8\pi^2w} \int_V F(\mathbf{r}')U^{(\omega)}(\mathbf{r}')\exp[-i(\mathbf{k} \cdot \mathbf{r}' \pm w\mathbf{z}')]d^3r',$$  \hspace{1cm} (A1)

where, according to Eqs. (2.8) and (2.3) and the second expression in Eq. (2.7),

$$w = +(k_0^2 - k^2)^{1/2} \quad \text{when} \quad k^2 \leq k_0^2, \hspace{1cm} (A2a)$$

$$w = +(k^2 - k_0^2)^{1/2} \quad \text{when} \quad k^2 > k_0^2. \hspace{1cm} (A2b)$$

We recall that the values of $w$ that are given by Eq. (A2a) are associated with homogeneous plane waves and those given by Eq. (A2b) are associated with evanescent plane waves. Now, according to Eq. (2.11), the incident field $U^{(\omega)}(\mathbf{r}')$ has the angular-spectrum representation

$$U^{(\omega)}(\mathbf{r}') = \int_{\mathbf{k}' \in \mathbb{K}} A^{(\omega)}(\mathbf{k}')\exp[i(\mathbf{k}' \cdot \mathbf{r}' + w'\mathbf{z}')]d^3\mathbf{k}',$$  \hspace{1cm} (A3)

where

$$w' = +(k_0^2 - k^2)^{1/2}. \hspace{1cm} (A4)$$

On substituting from Eq. (A3) into Eq. (A1) we obtain for $A^{(\pm)}(\mathbf{k})$ the expression

$$A^{(\pm)}(\mathbf{k}) = -i \frac{1}{8\pi^2w} \int_{V} d^3r' F(\mathbf{r}') \int_{\mathbf{k}' \in \mathbb{K}} A^{(\omega)}(\mathbf{k}') \times \exp[-i[(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}' + (\pm w - w')\mathbf{z}']]d^3\mathbf{k}'. \hspace{1cm} (A5)$$

Let us set

$$\mathbf{k}^\pm = (\mathbf{k}, \pm w), \hspace{1cm} \mathbf{k}' = (\mathbf{k}', w'). \hspace{1cm} (A6)$$

Then, since $\mathbf{r} = (\rho, z), \mathbf{r}' = (\rho', z')$, the expression appearing in the exponent on the right-hand side of Eq. (A5) becomes

$$(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}' + (\pm w - w')z' = (\mathbf{k}^\pm \cdot \mathbf{r}') \hspace{1cm} (A7)$$

where

$$\mathbf{K}^\pm = \mathbf{k}^\pm - \mathbf{k}'. \hspace{1cm} (A8)$$

If we substitute from Eq. (A7) into Eq. (A6) and interchange the order of the integrations, we obtain for $A^{(\pm)}(\mathbf{k})$ the expression

$$A^{(\pm)}(\mathbf{k}) = -i \frac{1}{8\pi^2w} \int_{\mathbb{K}} d^3\mathbf{k}' A^{(\omega)}(\mathbf{k}') \times \int_{V} F(\mathbf{r}')\exp(-i\mathbf{K}^\pm \cdot \mathbf{r}')d^3\mathbf{r}'. \hspace{1cm} (A9)$$

To interpret the integral over the volume $V$, we note that, according to Eqs. (A8), (A6), (A2), and (A4),

$$\mathbf{K}^\pm = [\mathbf{k} - \mathbf{k}', \pm(k_0^2 - k^2)^{1/2} - (k_0^2 - k^2)^{1/2}] \hspace{1cm} \text{when} \quad k^2 \leq k_0^2 \hspace{1cm} (A10a)$$

$$\mathbf{K}^\pm = [\mathbf{k} - \mathbf{k}', \pm i(k^2 - k_0^2)^{1/2} - (k_0^2 - k^2)^{1/2}] \hspace{1cm} \text{when} \quad k^2 > k_0^2 \hspace{1cm} (A10b)$$

and recall that $k^2 \leq k_0^2$ [because the incident field $U^{(0)}(\mathbf{r}')$ contains no evanescent waves [cf. Eq. (A3)]. Hence we see that

$$\mathbf{K}^\pm \text{ is a real vector} \quad \text{when} \quad k^2 \leq k_0^2, \hspace{1cm} (A11a)$$

$$\mathbf{K}^\pm \text{ is a complex vector} \quad \text{when} \quad k^2 > k_0^2. \hspace{1cm} (A11b)$$

It follows that when $k^2 \leq k_0^2$ the integral over $V$ in Eq. (A9) is precisely the Fourier inverse $F(\mathbf{K}^\pm)$ [cf. Eq. (3.9)] of the scattering potential $F(\mathbf{r})$, and Eq. (A9) then reduces to

$$A^{(\pm)}(\mathbf{k}) = -\frac{i}{8\pi^2w} \int_{\mathbb{K}} d^3\mathbf{k}' A^{(\omega)}(\mathbf{k}') F(\mathbf{K}^\pm) \hspace{1cm} (A12)$$

Further, since $F(\mathbf{K})$ is the three-dimensional Fourier transform of a function $F(\mathbf{r})$ that vanishes outside a finite domain $V$ in space, it follows by a well-known theorem \(^1\) that, provided only that $F(\mathbf{r})$ is continuous in $V$, $F(\mathbf{K}) = F(K_x, K_y, K_z)$ is the boundary value on the three real axes $K_x, K_y, K_z$ of an entire analytic function of three complex variables. Consequently, when $k^2 > k_0^2$, Eq. (A12) still holds, but $F(\mathbf{K}^\pm)$ is then no longer the Fourier transform of $F(\mathbf{r})$ but must be interpreted as representing its analytic continuation. Hence we see that, within the accuracy of the first Born approximation, each homogeneous plane wave $(k^2 \leq k_0^2)$ in the angular-spectrum representation of the scattered field $U^{(0)}(\mathbf{r})$ is associated with certain Fourier components of the scattering potential and that each evanescent wave in this representation is associated with components in the analytic continuation of the Fourier transform of the potential.

Let us consider the homogeneous waves more closely. Although each homogeneous plane wave is associated with certain Fourier components $F(\mathbf{K})$ of the scattering potential, the converse is not true, i.e., there are Fourier components of the scattering potential that do not couple to homogeneous waves of the angular spectrum. It is true to determine which Fourier components couple to the homogeneous waves. It follows at once from Eq. (A8) that when the vector $\mathbf{K}^\pm$ is real,

$$|\mathbf{K}^\pm| = |\mathbf{k}^\pm - \mathbf{k}'| \leq |\mathbf{k}^\pm| + |\mathbf{k}'|. \hspace{1cm} (A13)$$

Now, according to Eqs. (A6), (A2a), and (A4), $|\mathbf{k}^\pm| = k_0, |\mathbf{k}'| = k_0$ and hence

$$|\mathbf{K}^\pm| \leq 2k_0 = \frac{4\pi}{\lambda_0}, \hspace{1cm} (A14)$$

where $\lambda_0 = 2\pi/k_0$ is the wavelength of the light.

The result that we just derived is a generalization of a result derived in Ref. 10 for the special case when the incident field $U^{(0)}(\mathbf{r})$ consists of a single homogeneous plane wave and has the following physical interpretation: if we denote by $\Delta x, \Delta y, \Delta z$ the spatial periods associated with the spatial frequency vector $\mathbf{K} = (K_x, K_y, K_z)$, viz.,

$$K_x = \frac{2\pi}{\Delta x}, \hspace{1cm} K_y = \frac{2\pi}{\Delta y}, \hspace{1cm} K_z = \frac{2\pi}{\Delta z}, \hspace{1cm} (A15)$$

inequality (A14) then implies that
\[
\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \leq \frac{2}{\lambda_0}.
\]  
(A16)

If we denote the left-hand side of formula (A16) by \(1/d^2\), formula (A16) is equivalent to the inequality

\[ d \geq \lambda_0/2. \]  
(A17)

Now the quantity \(d\), which has the dimension of length, may be taken to be a rough measure of the linear dimension of the periodic component of the scattering potential that is associated with the spatial Fourier component that is labeled by \(K\). Hence inequality (A17) implies that the neglect of the evanescent waves in the angular-spectrum representation of the scattered field \(U(r)\) is equivalent to neglect of contributions to the scattered field that arise from details of the scatterer that, roughly speaking, are smaller than about half a wavelength in linear dimensions.

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**REFERENCES**


7. From now on we omit the time-dependent factor \(e^{-i\omega t}\), and the term "field" or "wave" refers to their space-dependent parts.

8. See, for example, D. N. Pattanayak and E. Wolf, "Scattering states and bound states as solutions of the Schrödinger equation with nonlocal boundary conditions," Phys. Rev. D 13, 913–923 (1976), Sec. II.

9. E. Wolf, "Phase conjugacy and symmetries in spatially bandlimited wavefields containing no evanescent components," J. Opt. Soc. Am. 70, 1311–1319 (1980). Equation (2.1) of this reference contains a misprint. \(U(x, y, z)e^{i\omega t}\) should be replaced by \(U(x, y, z)e^{-i\omega t}\). Also, Eq. (1.8) should read \(A(u, v) = k^2U(u, v) = U(x, y, z)e^{-i\omega t}\). These corrections do not affect any other equations or conclusions of that paper.


12. In more picturesque language, condition (3.10) is equivalent to the neglect of spatial Fourier components of the scattering potential for which the representative \(K\) vectors lie outside Weyl's limiting sphere. [See, for example, R. W. James, The Optical Principles of the Diffraction of X-Rays (Bell, London, 1948), pp. 14–15, or H. Lipson and C. A. Taylor, "X-ray crystal-structure determination as a branch of physical optics," Progress in Optics, E. Wolf, ed. (North-Holland, Amsterdam, 1966), Vol. 5, pp. 287–350, Sec. 4.


14. It is important to appreciate that when \(x < L\) the integral on the right-hand side of formula (3.12) does not represent, even approximately, the scattered field that arises from the interaction of the incident field \(U^{(0)}\) with the scatterer. This can be most easily seen by examining the asymptotic behavior of the integral in formula (3.12), in the half-space \(z < 0\), at large distances from the origin. One finds that the integral represents an incoming wave in that half-space, whereas the scattered field must clearly behave there as an outgoing wave.

15. This result is, essentially, a three-dimensional analog of a well-known theorem that the Fourier transform of a continuous function that vanishes outside a finite interval is a boundary value of an entire analytic function. This theorem follows at once from a well-known result in the theory of analytic functions defined by definite integrals [cf. E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable (Oxford U. Press, London, 1935), Sec. 5.2]. The multidimensional form of the theorem is the well-known Plancherel–Pólya theorem [cf. B. A. Fuks, Introduction to the Theory of Analytic Functions of Several Complex Variables (American Mathematical Society, Providence, R.I., 1963), pp. 353 et seq.].