Generalized Radon transform for tomographic measurement of short pulses

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Received March 24, 1994; revised manuscript received June 23, 1994

We propose a tomographic method for determining the Wigner function of a short pulse, which may be used with a wide class of optical systems.

1. INTRODUCTION

The Wigner distribution function, which was first introduced in the study of quantum statistical mechanics,[1] has many applications in optics. For example, an analog of the Wigner function and some of its variants play a central role in the theoretical foundations of radiometry.[2] In the analysis of nonstationary processes, the Wigner distribution has been called the master-form signal because all the measurable quantities pertaining to that process can readily be derived from it.[3] The Wigner distribution has also proved to be useful for characterizing ultrashort pulses.

A new method, called chronocyclic tomography, for measuring the Wigner function for short pulses was recently reported.[4] This method, which bears a close analogy to tomographic imaging methods, essentially relies on designing an optical system that can produce an output pulse that is a fractional Fourier transform[5–9] of the input pulse. The design of such optical systems, which consist of optical fibers and chirp modulators, is greatly facilitated by the use of the well-known analogy with beam propagation in paraxial optical systems that consist of thin lenses separated by free space.[10–13] However, in this context the class of optical systems that can produce a fractional Fourier transform is somewhat restricted. Therefore it would be advantageous if one were able to obtain the same information from a wider, more general, and more easily realizable class of optical systems. It is this problem that we address in the present paper.

2. PULSE PROPAGATION IN DISPERSIVE FIBERS

In this section, we briefly describe the analysis of pulse propagation in optical systems that consist of dispersive optical fibers and chirp modulators. There is a close analogy between pulses in such optical systems and the propagation of beams in paraxial optical systems that consist of thin lenses separated by free space. This analogy is discussed in detail in Refs. 10–13.

Consider a pulse propagating in the positive z direction in a one-dimensional dispersive medium, such as an optical fiber. Let $E(z, t)$ denote a cartesian component of the electric-field vector of the pulse at position z and time t. The Fourier transform of $E(z, t)$ defined by the formula

$$\tilde{E}(z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(z, t) \exp(i\omega t) \mathrm{d}t \,, \tag{1}$$

where ω is the frequency, will obey the differential equation

$$\left[\frac{\partial^2}{\partial z^2} + k(\omega)^2\right] \tilde{E}(z, \omega) = 0 \,. \tag{2}$$

In this equation, $k(\omega)$ is the wave number for propagation in the medium; $k(\omega)$ depends on the frequency in a manner appropriate to the particular medium. The solution to Eq. (2) is

$$\tilde{E}(z, \omega) = A(\omega) \exp[ik(\omega)z] + B(\omega) \exp[-ik(\omega)z] \,, \tag{3}$$

where $A(\omega)$ and $B(\omega)$ are functions that depend on the boundary conditions. The first term on the right-hand side of Eq. (3) represents a wave propagating in the positive z direction, while the second term represents a wave propagating in the negative z direction. We assume that there is no component of the field traveling in the negative z direction, and hence $B(\omega) \equiv 0$. Equation (1) and (3) then imply that

$$E(z, t) = \int_{-\infty}^{\infty} A(\omega) \exp[ik(\omega)z - i\omega t] \mathrm{d}\omega \,. \tag{4}$$

We assume that the function $A(\omega)$ is centered on some mean frequency ω_0 with effective width $\Delta\omega$. Thus we will make the substitution $A(\omega) = \alpha(\omega - \omega_0)$, where $\alpha(\omega)$ is a function of width $\Delta\omega$. Further, we assume that, for values of ω such that $|\omega - \omega_0| \lesssim \Delta\omega$, $k(\omega)$ may be approximated by the first three terms of its Taylor series, that is,

$$k(\omega) \cong k_0 + k_0'(\omega - \omega_0) + \frac{k_0''}{2} (\omega - \omega_0)^2 \,. \tag{5}$$

In relation (5), k_0, k_0', and k_0'' denote, respectively, the wave number and its first and second derivatives with respect to ω, all evaluated at $\omega = \omega_0$. We will neglect the effect of absorption of the medium; consequently, the parameters k_0, k_0', and k_0'' will all be real. On substituting from relation (5) into Eq. (4), we obtain the following expression for the electric-field component $E(z, t)$:

$$E(z, t) = u(z, t - k_0'z)\exp[i(k_0 z - \omega_0 t)]. \tag{6}$$

In this expression, $u(z, t)$ is the pulse profile function defined by the formula

$$u(z, t) = \int_{-\infty}^{\infty} \alpha(\omega)\exp\left(\frac{i}{2}\,\omega^2 k_0'' z - i\omega t\right)\mathrm{d}\omega\,. \tag{7}$$

From Eq. (6) we see that the phase velocity of the pulse in the dispersive medium is equal to ω_0/k_0 and that its group velocity is $(k_0')^{-1}$, as was to be expected.

From Eq. (7) it follows that the pulse profile function $u(z, t)$ obeys the following partial differential equation:

$$i\,\frac{\partial u(z, t)}{\partial z} = \frac{k_0''}{2}\,\frac{\partial^2 u(z, t)}{\partial t^2}\,. \tag{8}$$

Apart from the constant factors, Eq. (8) is identical to the one-dimensional version of the Schrödinger equation or, alternatively, to the one-dimensional form of the paraxial wave equation. It is this latter fact that underlies the analogy between paraxial beam propagation and dispersive pulse propagation. Solutions to equations such as Eq. (8) are readily obtainable by the use of a propagation kernel.[14] The pulse profile $u_{\mathrm{out}}(t)$ that will emerge from the end of a length z of dispersive optical fiber is related to the profile $u_{\mathrm{in}}(t)$ of the pulse that was coupled into the fiber by the linear transform

$$u_{\mathrm{out}}(t) = \int_{-\infty}^{\infty} u_{\mathrm{in}}(t')K_0(t, t')\mathrm{d}t'\,, \tag{9}$$

where the propagation kernel $K_0(t, t')$ is given by the formula

$$K_0(t, t') = \frac{\exp(-i\pi/4)}{\sqrt{2\pi k_0'' z}}\,\exp\left[\frac{-i(t - t')^2}{2k_0'' z}\right]. \tag{10}$$

This kernel belongs to a class of more general kernels of the form[15]

$$K(t, t') = \frac{\exp(-i\pi/4)}{\sqrt{2\pi B}}\,\exp\left[\frac{-i(At'^2 - 2tt' + Dt^2)}{2B}\right], \tag{11}$$

where A, B, and D are elements of the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & k_0'' z \\ 0 & 1 \end{bmatrix}. \tag{12}$$

The parameter C in Eq. (12) is defined so that $AD - CB = 1$.

Another optical device also commonly employed in experiments with short pulses is the chirp modulator. The effect of such a device is to introduce a quadratic phase modulation into the pulse profile. Thus the pulse profile $u_{\mathrm{out}}(t)$ emerging from the chirp modulator is related to the profile function of the input pulse $u_{\mathrm{in}}(t)$ by the formula

$$u_{\mathrm{out}}(t) = u_{\mathrm{in}}(t)\exp(-it^2/2k_0'' f)\,, \tag{13}$$

where k_0'' characterizes the dispersion of the identical optical fibers on either side of the chirp modulator and f is a parameter that determines the strength of the chirp. Equation (13) can be rewritten in the form of the linear transform

$$u_{\mathrm{out}}(t) = \int_{-\infty}^{\infty} u_{\mathrm{in}}(t')K(t, t')\mathrm{d}t'\,, \tag{14}$$

where the kernel $K(t, t')$ is given by the formula

$$\begin{aligned} K(t, t') &= \delta(t - t')\exp(-it^2/2k_0'' f) \\ &= \lim_{\sigma\to 0^+} \frac{\exp(i\pi/4)}{\sqrt{2\pi\sigma}}\,\exp\left[\frac{i}{2\sigma}\,(t - t')^2 - \frac{it^2}{2k_0'' f}\right] \end{aligned} \tag{15}$$

(where $\delta(t)$ is the Dirac delta function). Evidently this kernel is of the form given by Eq. (11), with the parameters A, B, and D now given by the elements of the matrix

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \lim_{\sigma\to 0^+} \begin{bmatrix} 1 & \sigma \\ -1/k_0'' f & 1 - \sigma/k_0'' f \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1/k_0'' f & 1 \end{bmatrix}. \end{aligned} \tag{16}$$

Matrices (12) and (16) correspond to the *ABCD* matrices for optical beam propagation through a distance z in free space and to the *ABCD* matrices for passage of a beam through a thin lens, respectively. For this reason, chirp modulators are sometimes referred to as time lenses, where the parameter f is the focal length of the lens (see, for example, Ref. 11).

It can be shown that the change in the profile of an optical pulse as it propagates through any lossless optical system that consists of a combination of chirp modulators and lengths of dispersive optical fiber is given by Eq. (14) when the kernel $K(t, t')$ is of the general form given by Eq. (11). For an arbitrary system, the values of A, B, C, and D can be calculated by multiplying together, in appropriate order, the matrices corresponding to the various optical components of the system. The use of *ABCD* matrices to characterize optical-pulse-propagation systems is discussed in more detail in Ref. 12.

3. RADON TRANSFORMS AND CHRONCYCLIC TOMOGRAPHY

The Wigner function associated with the pulse profile $u_{\mathrm{in}}(t)$ is defined by the formula

$$W_{\mathrm{in}}(t, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_{\mathrm{in}}^{*}\left(t + \frac{\tau}{2}\right)u_{\mathrm{in}}\left(t - \frac{\tau}{2}\right)\exp(i\omega\tau)\mathrm{d}\tau\,. \tag{17}$$

This representation of optical pulses has many useful applications. Most measurable quantities connected with the optical field can be derived directly from this function. For this reason, this function has been called the master-form signal.[3] In particular, the intensity profile can be deduced from the Wigner function by the formula

$$I_{\rm in}(t) = \int_0^\infty W_{\rm in}(t, \omega){\rm d}\omega\,. \tag{18}$$

One can deduce from Eqs. (11), (14), and (17), after some straightforward calculation, that the Wigner function of the output of an optical system that is characterized by an *ABCD* matrix is related to the Wigner function of the input by the expression[16,17]

$$W_{\rm out}(t, \omega) = W_{\rm in}(Dt - B\omega, A\omega - Ct)\,. \tag{19}$$

It follows from Eqs. (18) and (19) that the intensity profile of the output pulse is given by the formula

$$I_{\rm out}(t) = \int_0^\infty W_{\rm in}(Dt - B\omega, A\omega - Ct){\rm d}\omega\,. \tag{20}$$

Recently, a special type of optical system, called a fractional Fourier transform system, has received considerable attention.[5–9] Such a system has a kernel of the form

$$K_\theta{}^{\rm FracFT}(t, t') = \frac{\exp(-i\pi/4)}{\sqrt{2\pi\tau^2 \sin\theta}} \times \exp\left\{\frac{i}{2}\,\frac{[(\cos\theta)t^2 + (\cos\theta)t'^2 - 2tt']}{\tau^2 \sin\theta}\right\}, \tag{21}$$

where θ and τ are real constants. Comparing Eqs. (11) and (21), we see that the *ABCD* matrix for a fractional Fourier transform system is given by the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\theta & \tau^2 \sin\theta \\ -\sin\theta/\tau^2 & \cos\theta \end{bmatrix}. \tag{22}$$

We can see the importance of fractional Fourier transform systems at once by applying Eq. (22) to Eq. (20). It then follows that the intensity profile of the output pulse from a fractional Fourier transform system has the form

$$I_{\rm out}{}^{\rm FracFT}(t) \equiv \Lambda_\theta(t) = \int_0^\infty W_{\rm in}[t\cos\theta - (\omega\tau^2)\sin\theta,\ \omega\cos\theta + (t/\tau^2)\sin\theta]{\rm d}\omega\,. \tag{23}$$

This integral is equivalent to a Radon transform.[18] Methods for inverting the Radon transform $\Lambda_\theta(t)$ are well known from tomographic imaging (see, for example, Refs. 19 and 20). Thus, by measuring the intensity profile of the output pulse from fractional Fourier transform systems [for which the parameter θ takes values in the range (0 to π)], one can, by inversion of the Radon transform, obtain the Wigner distribution function of the input pulse. This method, called chronocyclic tomography, was recently proposed for use in connection with short pulses.[4] Practical difficulties nevertheless exist in realizing the fractional Fourier transform for pulse propagation systems. In particular, it is difficult to obtain chirp modulators with the required range of values for the focal length. It is, therefore, of considerable interest to investigate whether it is possible to determine the Wigner distribution of an optical pulse from measurements obtained by the use of more general and more easily realizable optical systems than the fractional Fourier transform systems.

4. GENERALIZED RADON TRANSFORM

Let us consider a general optical system characterized by the parameters A, B, C, and D. We will assume that these four parameters are functions of some variable f, for example, a focal length of some time lens that forms a part of the system. From Eq. (20) we see that the intensity profile of the output pulse of such a system is then given by the formula

$$\Lambda(t, f) \equiv \int_0^\infty W_{\rm in}[D(f)t - B(f)\omega, A(f)\omega - C(f)t]{\rm d}\omega\,. \tag{24}$$

Because of the analogy with Eq. (23), we refer to the function $\Lambda(t, f)$ as a generalized Radon transform of the Wigner distribution function of the input pulse.

The inversion of this generalized Radon transform can be obtained as follows. Consider the one-dimensional Fourier transform of $\Lambda(t, f)$ with respect to the variable t. This function is related to the Wigner function of the input pulse by the formula

$$\frac{1}{(2\pi)^2}\int_{-\infty}^\infty \Lambda(t, f)\exp(-i\xi t){\rm d}t = \tilde{W}_{\rm in}[A(f)\xi, B(f)\xi], \tag{25}$$

where $\tilde{W}_{\rm in}(K_1, K_2)$ is the two-dimensional Fourier transform of the Wigner function

$$\tilde{W}_{\rm in}(K_1, K_2) \equiv \frac{1}{(2\pi)^2}\iint_{-\infty}^\infty \tilde{W}_{\rm in}(t, \omega) \times \exp[i(K_1 t + K_2\omega)]{\rm d}t{\rm d}\omega\,. \tag{26}$$

Equation (25) demonstrates that, providing that $A(f)$ and $B(f)$ obey certain constraints, the generalized Radon transform contains complete information about the Wigner function. In particular, if the optical system is such that the ratio $B(f)/A(f)$ can take any possible value in the range $(-\infty, \infty)$, then it is possible to invert the Fourier transform in Eq. (26). Formally this inversion is given by the formula

$$W_{\rm in}(t, \omega) = \iint_{-\infty}^\infty \tilde{W}_{\rm in}[A(f)\xi, B(f)\xi] \times \exp\{i\xi[A(f)t + B(f)\omega]\}{\rm d}[A(f)\xi]{\rm d}[B(f)\xi]\,. \tag{27}$$

On substituting from Eq. (25) and introducing the parameters ξ and f as the variables of integration, Eq. (27) can be rewritten in the form

$$W_{\rm in}(t, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty {\rm d}t' \int_{-\infty}^\infty {\rm d}\xi \int_{f_1}^{f_2} {\rm d}f \times \left|A(f)\frac{{\rm d}B(f)}{{\rm d}f} - B(f)\frac{{\rm d}A(f)}{{\rm d}f}\right| \times |\xi|\Lambda(t', f)\exp\{i\xi[A(f)t + B(f)\omega - t']\}, \tag{28}$$

where the limits of integration f_1 and f_2 are chosen so that the ratio $B(f)/A(f)$ takes on all values in the range $(-\infty, \infty)$ as f varies from f_1 to f_2.

As mentioned before, this requirement on $A(f)$ and $B(f)$ places a restriction on the optical systems that can be employed to realize a generalized Radon transform.

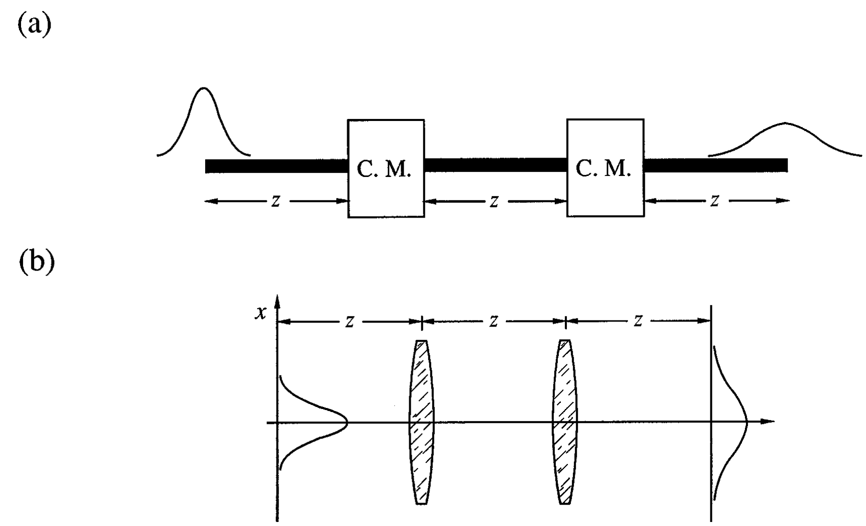

Fig. 1. Optical system used in the example of the generalized Radon transform discussed in the text. (a) The optical system for transmission of short pulses is shown symbolically with the thick lines representing lengths of a dispersive optical fiber and the chirp modulators (or time lenses) shown as blocks marked C.M. (b) The equivalent paraxial optical system for beam propagation is shown with a pair of lenses of focal length f that are separated from each other and from the input and the output planes by the distance z, as shown. The lengthening of the time width of the optical pulse as it propagates through the system in (a) is mathematically analogous to the broadening of an optical beam as it is diffracted through the system in (b).

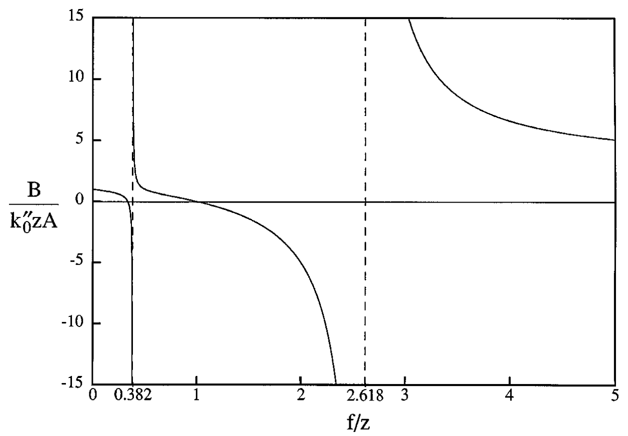

Fig. 2. Variation of the ratio $B(f)/A(f)$, with the focal length f for the optical system shown in Fig. 1.

Consider, for example, the optical system as shown in Fig. 1. For this system, the $ABCD$ matrix is

$$\begin{bmatrix} A(f) & B(f) \\ C(f) & D(f) \end{bmatrix} = \begin{bmatrix} (z/f)^2 - 3z/f + 1 & k_0{}''z[(z/f)^2 - 4z/f + 3] \\ (z/f - 2)/k_0{}''f & (z/f)^2 - 3z/f + 1 \end{bmatrix}. \tag{29}$$

The variation of the ratio $B(f)/A(f)$ with the focal length f is plotted in Fig. 2. It is straightforward to see that, as required, the ratio $B(f)/A(f)$ takes all possible values in the range $(-\infty, \infty)$ as f varies from $(3 - \sqrt{5})z/2$ to $(3 + \sqrt{5})z/2$. The inversion formula is therefore given by the expression

$$W_{\mathrm{in}}(t, \omega) = \frac{k_0{}''}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0.382z}^{2.618z} \mathrm{d}f \left(\frac{z}{f}\right)^2 \times \left[\left(\frac{z}{f}\right)^2 - 4\left(\frac{z}{f}\right) + 5\right] \times |\xi| \Lambda(t', f) \exp\{i\xi[A(f)t + B(f)\omega - t']\}, \tag{30}$$

where $A(f) = (z/f)^2 - 3z/f + 1$ and $B(f) = k_0{}''z[(z/f)^2 - 4z/f + 3]$. This example illustrates that it is theoretically possible to obtain the Wigner distribution tomographically from a simple, easily manipulated optical system.

The chirp modulation of optical pulses is commonly performed by an electro-optic device. For such devices, the focal length f is given by the formula

$$f = (k_0{}''\Phi_0\overline{\omega}^2)^{-1}, \tag{31}$$

where Φ_0 is the amplitude of the phase modulation and $\overline{\omega}$ is the electrical modulation frequency. If we employ lengths of dispersive fiber for which $k_0{}''z = 10^{-21}s^2$ and we assume that $\overline{\omega} = 10$ GHz, then the above arguments show that a complete reconstruction of the Wigner distribution of the pulse can be obtained (by the use of the system shown in Fig. 1), if it is possible to vary Φ_0 between the values 3.82 and 26.18 rad. Numerical implementation of the inverse of Eq. (30) should not pose a serious challenge, as one can adapt existing algorithms developed to invert radon transforms.

Recently, several methods were proposed for reconstructing two-dimensional field correlation functions in some plane from measurements of the intensity of the field radiated by secondary sources.[21–23] Because of the analogy between paraxial propagation and dispersive pulse propagation, the method that we propose in this paper can be extended to two-dimensional paraxial optical systems.

ACKNOWLEDGMENTS

The authors thank Dave Fischer, Greg Forbes, Ian Walmsley, Weijian Wang, and Emil Wolf for useful discussions and helpful comments. Some of the work described in this paper was performed when D. F. V. James was visiting the School of Physics at the University of Hyderabad. He expresses his thanks for the hospitality extended to him there by the faculty, students, and staff.

This research was supported by the National Science Foundation under grant INT-9100685, by the U.S. Army Research Office under the University Research Initiative Program, and by the National Science Foundation Center for Electronic Imaging Systems at the University of Rochester.

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REFERENCES AND NOTES

1. E. Wigner, "On the quantum corrections for thermodynamic equilibrium," Phys. Rev. **40,** 749–759 (1932).
2. A. Walther, "Radiometry and coherence," J. Opt. Soc. Am. **58,** 1256–1259 (1968).
3. H. O. Bartelt, K.-H. Brenner, and A. W. Lohmann, "The Wigner distribution function and its optical production," Opt. Commun. **32,** 32–38 (1980).
4. M. Beck, M. G. Raymer, I. A. Walmsley, and V. Wong, "Chronocyclic tomography for measuring the amplitude and phase structure of optical pulses," Opt. Lett. **18,** 2041–2043 (1993).
5. Z. Namias, "The fractional Fourier transform and its applications in quantum mechanics," J. Inst. Math. Appl. **25,** 241–265 (1980).
6. A. C. McBride and F. H. Kerr, "On Namias's fractional Fourier transforms," IMA J. Appl. Math. **39,** 159–175 (1987).
7. D. Mendelovic and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: 1," J. Opt. Soc. Am. A **10,** 1875–1881 (1993).
8. A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," J. Opt. Soc. Am. A **10,** 2181–2186 (1993).
9. G. S. Agarwal and R. Simon, "A simple realization of fractional Fourier transforms and the relation to the harmonic oscillator Green's function," Opt. Commun. **110,** 23–26 (1994).
10. E. B. Treacy, "Optical pulse compression with diffraction gratings," IEEE J. Quantum Electron. **QE-5,** 454–458 (1969).
11. B. H. Kolner and M. Nazarathy, "Temporal imaging with time lenses," Opt. Lett. **14,** 630–632 (1989).
12. S. P. Dijaili, A. Dienes, and J. S. Smith, "*ABCD* matrices for dispersive pulse propagation," IEEE J. Quantum Electron. **QE-26,** 1158–1164 (1990).
13. A. E. Siegman, *Lasers* (University Science, Mill Valley, Calif., 1989), Chap. 9.
14. E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961), Sec. 8.
15. The general properties of linear transforms with kernels of the type given by Eq. (11) is discussed in detail in D. F. V. James and G. S. Agarwal, "Generalized Fresnel transform in optics," to be submitted Opt. Commun.
16. G. S. Agarwal, "Wigner-function description of quantum noise in interferometers," J. Mod. Opt. **34,** 909–921 (1987).
17. A. T. Friberg, "Propagation of a generalized radiance in paraxial optical systems," Appl. Opt. **30,** 2443–2446 (1991).
18. H. H. Barrett, "The Radon transform and its applications," in *Progress in Optics*, E. Wolf, ed. (North-Holland, Amsterdam, 1984), Vol. XXI, pp. 217–286.
19. G. T. Herman, *Image Reconstruction from Projections* (Academic, New York, 1980), Chap. 6.
20. A. C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging* (IEEE, New York, 1988), Chap. 3.
21. F. Gori, M. Santarsiero, and G. Guattari, "Coherence and the spatial distribution of intensity," J. Opt. Soc. Am. A **10,** 673–679 (1993).
22. M. W. Kowarz, "Noninterferometric reconstruction of optical-field correlations," Phys. Rev. E **49,** 890–893 (1994).
23. M. G. Raymer, M. Beck, and D. F. McAlister, "Complex wavefield reconstruction using phase-space tomography," Phys. Rev. Lett. **72,** 1137–1140 (1994).